ON $\gamma$-WEAK SOLUTIONS OF ELLIPTIC EQUATIONS

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Abstract. We build on the 'alternative' approach to the Dirichlet problem in weighted Sobolev spaces (cf. A. Kufner & J. Rakosnik, [11]) to formulate sufficient conditions for interior regularity of $\gamma$-weak solutions of the Dirichlet problem for second order elliptic PDEs with Lipschitz coefficients. Next, we prove a maximum principle (for $\gamma$-weak subsolutions). Last, we generalize (from power type weights to a larger class of weights) a trace theorem by J. Necas, [14].

1. REVIEW OF A. KUFNER’S THEORY AND STATEMENT OF MAIN RESULTS

Let $\Omega \subset \mathbb{R}^N$ be a domain with boundary $\partial \Omega$. Let $\gamma(x)$ be a weight, i.e. a Lebesgue measurable function on $\Omega$, and $\gamma > 0$ a.e. on $\Omega$. The weighted Sobolev space $W^{1,2}(\Omega, \gamma)$ consists of all $u \in W^1(\Omega)$ so that:

$$
\|u\|_{W^{1,2}(\Omega, \gamma)} \equiv \left( \sum_{|\alpha| \leq 1} \left( \int_{\Omega} |D^\alpha u(x)|^2 \gamma(x) dx \right) \right)^{1/2} \quad < \infty
$$

(1)

where $W^1(\Omega)$ denotes the space of all weakly differentiable functions on $\Omega$. We request (together with [11], p. 187) that:

$$
\gamma \in L^1_{loc}(\Omega), \quad \gamma^{-1} \in L^1_{loc}(\Omega)
$$

(2)

Consequently $C_0^\infty(\Omega) \subset W^{1,2}(\Omega, \gamma)$ and $W^{1,2}(\Omega, \gamma)$ is complete (in the norm (1)).

A large amount of literature was dedicated in the last thirty years to the study of the structure of weighted Sobolev spaces (cf. R.A. Adams, [1], A. Avantaggiati, [2], A. Kufner, [10], M. Troisi, [17], etc.) as well as to their applications in the theory of PDEs (cf. P. Bolley & J. Camus, [4], V. Benci & D. Fortunato, [3], B. Hanouzet, [9], A. Kufner & A.M. Sandig, [12], A. Kufner & J. Voldrich, [13], J. Necas, [14], etc.). The present paper falls into this second category. Let:

$$
(Lu)(x) \equiv \sum_{|\alpha|, |\beta| \leq 1} (-1)^{|\alpha|} D^\alpha (a_{\alpha\beta}(x) D^\beta u(x))
$$

(3)

$x \in \Omega$, be a second order linear differential operator with $a_{\alpha\beta} \in L^\infty(\Omega)$, for any $|\alpha| \leq 1$, $|\beta| \leq 1$. Assume $\gamma \in C^1(\Omega)$. Let $W^{1,2}_0(\Omega, \gamma)$ denote the closure of $C_0^\infty(\Omega)$ in $W^{1,2}(\Omega, \gamma)$ (with respect to the norm (1)). Let:

$$
u_0 \in W^{1,2}(\Omega, \gamma)
$$
and:

\[ F \in [W^{1,2}_0(\Omega, \gamma)]^* \]

be given. Then \( u \in W^{1,2}(\Omega, \gamma) \) is a \( \gamma \)-weak solution of the Dirichlet problem for \( L \) if:

\[ u - u_0 \in W^{1,2}_0(\Omega, \gamma) \tag{4} \]
\[ a_\gamma(u, \phi) = F(\phi) \tag{5} \]

for any \( \phi \in W^{1,2}_0(\Omega, \gamma) \). Here \( a_\gamma \) is given by:

\[ a_\gamma(u, v) = \sum_{|\alpha| = |\beta| - 1} \int_\Omega a_{\alpha\beta}(x) D^\beta u(x) D^\alpha v(x) \gamma(x) \, dx \tag{6} \]

A weight \( \gamma \in C^1(\Omega) \) is said to possess the property \( (P_1) \) (cf. [11], p. 188) if there is \( C^* > 0 \) so that:

\[ |\nabla \gamma(x)| \leq C^* \gamma(x) \tag{7} \]
a.e. on \( \Omega \). If \( L \) is elliptic, i.e.

\[ \sum_{|\alpha| = |\beta| = 1} a_{\alpha\beta}(x) \xi_\alpha \xi_\beta \geq C_0 |\xi|^2 \tag{8} \]

for some \( C_0 > 0 \) and any \( \xi, (\xi_\alpha)_{|\alpha| = 1} \in \mathbb{R}^N \), and \( a_{\alpha\beta} \) are essentially bounded then there is \( C > 0 \) so that for any weight \( \gamma \) with the property \( (P_1) \) and with \( C > C^* \), and for any \( u_0 \in W^{1,2}(\Omega, \gamma) \) and any \( F \in [W^{1,2}_0(\Omega, \gamma)]^* \) there is a unique \( \gamma \)-weak solution of the Dirichlet problem (4)-(5) for \( L \). Cf. Theor. 40.11 in [12], p. 192. The weaker ellipticity assumption:

\[ a(\phi, \phi) \geq C_0 \|\phi\|_{W^{1,2}(\Omega)}^2 \tag{9} \]

for any \( \phi \in W^{1,2}_0(\Omega) \) is actually sufficient, where \( a(u, v) \) is given by (6) for \( \gamma \equiv 1 \). It is our purpose in the present paper to formulate sufficient conditions under which a given \( \gamma \)-weak solution \( u \in W^{1,2}(\Omega, \gamma) \) is regular at the interior of \( \Omega \).

A weight \( \gamma(x) \) is termed admissible if:

i) for any \( s = 1, \ldots, N \) and any \( h \in \mathbb{R} \) with \( \Omega_{s,h} \neq \emptyset \), there is a constant \( 1 \leq C(h) < \infty \) so that:

\[ C(h)^{-1} \gamma(x) \leq \gamma(x + he_s) \leq C(h) \gamma(x) \tag{10} \]

for any \( x \in \Omega_{s,h} \) (except for a set of measure zero),

ii) \( \lim_{h \to 0} |h|^{-1}(C(h) - 1) \) exists \( (< \infty) \), and

iii) for any \( \delta > 0 \) fixed one has:

\[ \sup_{|h| \leq \delta} C(h) < \infty, \sup_{|h| \leq \delta} \frac{C(h) - 1}{|h|} < \infty \tag{11} \]

Here \( e_s \) is the \( s \)-th vector in the canonical basis of \( \mathbb{R}^N \) and \( \Omega_{s,h} = \{ x \in \Omega: x + he_s \in \Omega \} \).

Besides from restricting ourselves to the case of admissible weights, we assume \( a_{00} = \)
0, \alpha_0 = 0, a_\alpha_0 = 0, i.e. (3) is given by $Lu \equiv -\sum_{|\alpha| = |\beta| = 1} D^\alpha (a_{\alpha\beta} D^\beta u)$. We obtain the following:

**Theorem 1.** Let $u \in W^{1,2}_0(\Omega, \gamma)$ be a $\gamma$-weak solution of the Dirichlet problem (4) - (5) for the elliptic operator $L$, where $u_0 \equiv 0$ and $F \in [W^{1,2}_0(\Omega, \gamma)]^*$ is given by $F(\phi) = \int_{\Omega} \sum_{|\alpha| = 1} f_\alpha D^\alpha \phi - f \phi \, dx$, with $f_\alpha, f \in L^2(\Omega, \gamma^{-1})$, $|\alpha| = 1$. If i) $\gamma$ is admissible, ii) $a_{\alpha\beta}$ are Lipschitz in $\Omega$, $|\alpha| = |\beta| = 1$, and iii) $f_\alpha \in W^{1,2}_0(\Omega, \gamma^{-1})$, $|\alpha| = 1$, then $u \in W^{2,2}_0(\Omega, \gamma)$ and for any $\Omega' \subset \subset \Omega$ there is a constant $Q = Q(\Omega, \Omega', \gamma) > 0$ so that:

$$
\int_{\Omega'} |D^\alpha u|^2 \gamma \, dx \leq Q \int_{\Omega} (|\nabla u|^2 \gamma + |\nabla F|^2 \gamma^{-1} + |f|^2 \gamma^{-1}) \, dx
$$

(12) for any $|\alpha| = 2$. Here $F = (f_\alpha)|_{\alpha=1}$.

The proof of our **Theorem 1** relies on the **Lemmas 1 and 2** (cf our Section 3). **Lemma 2** is a straightforward version (for weighted spaces and norms) of Prop. 3.4 in [7], p. 47 (a proof is included for the sake of completeness). The main estimate (12) is performed in Section 4.

Let us fix $u_0 \in W^{1,2}(\Omega, \gamma)$ and a continuous linear functional:

$$
F \in [W^{1,2}_0(\Omega, \gamma)]^*
$$

Originally (cf. [10], p. 129) a function $u \in W^{1,2}(\Omega, \gamma)$ was termed weak solution of the Dirichlet problem for $L$ if (4) holds while (5) is replaced by:

$$
a(u, \phi) = F(\phi)
$$

(13) for any $\phi \in C_0^\infty(\Omega)$. Moreover, a theorem of existence and uniqueness of such weak solutions in $W^{1,2}(\Omega, \gamma)$ of the Dirichlet problem (4), (13) still holds (cf. Theor. 15.2 in [10], p. 144) when:

$$
\gamma(x) = s(d_M(x))
$$

(14) where $s(t)$ is a continuous positive function defined for $t > 0$ and such that either $\lim_{t \to 0} s(t) = 0$ or $\lim_{t \to 0} s(t) = \infty$ and $d_M(x) = \text{dist}(x, M)$ for some real $m$-dimensional ($0 \leq m \leq N - 1$) manifold $M \subset \partial \Omega$. It is noteworthy that while the classical (unweighted) solution of the Dirichlet problem (cf. Theor. 8.3 in [8], p. 181) relies on the Lax-Milgram theorem, the proof of Theor. 15.2 in [10] requests a generalization (cf. [15], Ch. 6, Sect. 3.1) of the Lax-Milgram theorem (cf. also Lemma 13.6 in [10], p. 131). As pointed out in [11], the advantage of the ‘alternative’ approach (4)-(5) to the Dirichlet problem for $L$ in $W^{1,2}(\Omega, \gamma)$ (based on the use of the bilinear form $a_{\gamma}(u, v)$ rather than $a(u, v)$) comes from the fact that J. Necas’ generalized Lax-Milgram theorem is not needed anylonger (the proof of Theor. 40.11 in [12], p 192, follows from Theor. 39.5, p. 179, alone). Once $a(u, v)$ is replaced by $a_{\gamma}(u, v)$ one may say that $u \in W^{1,2}(\Omega, \gamma)$ satisfies (by definition) $Lu \leq 0 (= 0, \leq 0)$ in $\Omega$ if $a_{\gamma}(u, \phi) \leq 0 (= 0, \geq 0)$ for any nonnegative $\phi \in W^{1,2}_0(\Omega, \gamma)$. It is a natural question to ask whether one may establish a (weak) maximum principle for $L$ (cf. Theor. 8.1 in [8], p. 179, for the unweighted case).

Let $\gamma \in C^0(\Omega), \gamma > 0$ in $\Omega$. Set:

$$
I(x_0) = \{x \in \Omega : |x - x_0|^2 < \gamma(x_0)\}$$
for $x_0 \in \Omega$. We say $\gamma$ is admissible in the sense of M. Troisi if the weight $\rho = \gamma^{1/2}$ satisfies the properties I)-II) in [17], p. 50. In particular $I(x_0)$ has the cone property for any $x_0 \in \Omega$, and there is a constant $A_0 > 0$ so that:

$$A_0^{-1} \rho(x_0) \leq \rho(x) \leq A_0 \rho(x_0) \tag{15}$$

for any $x_0 \in \Omega$ and any $x \in I(x_0)$. We may state the following:

**Theorem 2.** Let $N$ be an odd integer $\geq 3$. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain and $\gamma \in C^1(\Omega) \cap C^0(\bar{\Omega})$ an admissible (in the sense of M. Troisi) weight. Assume that $a^{ij}, b^i, c^i$ and $d$ satisfy (51) up to (53) and $a^{ij} \in L^\infty(\Omega)$. If $u \in W^{1,2}(\Omega, \gamma)$ is such that $Lu \geq 0$ in $\Omega$ then:

$$\sup_{\Omega} u \leq \sup_{\partial \Omega} u^+ \tag{16}$$

In a forthcoming paper we shall study the boundary regularity of $\gamma$-weak solutions of the Dirichlet problem (4)-(5) via a trace theorem of J. Necas (established for $\gamma(x) = d_M(x)^\epsilon$, $M = \partial \Omega$, $\epsilon \geq 0$, cf. [14], p. 309) generalized versions of which are discussed in Section 7 (cf. Theorems 4 and 5).

The authors are grateful to the referee who's remarks led to an improvement of Theorem 2.

2. ADMISSIBLE WEIGHTS

Let $f(x)$ be a function on $\Omega$ and $1 \leq s \leq N$, $h \in \mathbb{R}$. Set $\tau_{s,h} f(x) = \frac{1}{h} (f(x + he_s) - f(x))$, for any $x \in \Omega_{s,h}$. Let also $\Omega_h = \{ x \in \Omega : dist(x, \partial \Omega) > |h| \}$. Then $\Omega_h \subset \Omega_{s,h}$ (because, for any $x \in \Omega_h$, $B(x, |h|) \subset \Omega$ and $x + he_s \in \partial B(x, |h|)$). Next, if $\Omega' \subset \subset \Omega$ and $0 < \delta < \frac{1}{2} dist(\Omega', \partial \Omega)$ then $\Omega_h \neq \emptyset$ for any $|h| \leq \delta$ (indeed $\Omega' \subset \subset \Omega_h$). Throughout $B(x, r)$ denotes the ball of center $x$ and radius $r > 0$.

Let $\gamma(x)$ be an admissible weight on $\Omega$. Then:

$$|\tau_{s,h} \gamma| \leq \frac{C(h) - 1}{|h|} \gamma \tag{17}$$

a.e. in $\Omega$.

**Example 1.** Let $\gamma(x)$ be given by (14) for $s(t) = \exp(\epsilon t), t > 0, \epsilon \in \mathbb{R}$, (cf. [11], p. 187). If $\epsilon > 0$ then $\gamma$ is admissible. Indeed, we may take $C(h) = \exp(\epsilon |h|) \geq 1$ and then $\lim_{|h| \to 0} |h|^{-1} (C(h) - 1) = \epsilon$, $\sup_{|h| \leq \delta} C(h) = \exp(\epsilon |h|) < \infty$, $\sup_{|h| \leq \delta} |h|^{-1} (C(h) - 1) = \delta^{-1} (\exp(\epsilon \delta) - 1) < \infty$.

We end Section 2 by listing some properties of the operator $\tau_{s,h}$ i.e. 1) if $f \in W^{1,2}(\Omega, \gamma)$ then $\tau_{s,h} f \in W^{1,2}(\Omega_h, \gamma)$, 2) if $supp(f) \subset \Omega_h$ (or $supp(g) \subset \Omega_h$) then $\int_{\Omega} g \tau_{s,h} f dx = - \int_{\Omega} g f \tau_{s,h} g dx$, 3) $\tau_{s,h} (fg) = f(x + he_s) \tau_{s,h} g + g(x) \tau_{s,h} f$. Only 1) needs to be justified (cf. [7], p. 46, for 2)–3)). If $\xi \in \mathbb{R}^N$ let $T_\xi : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be given by $T_\xi(x) = x + \xi$. We may perform the following estimate:

$$\|\tau_{s,h} f\|_{L^2(\Omega_h, \gamma)}^2 \leq 2|h|^{-2} \|f\|_{L^2(\Omega, \gamma)}^2 + I_{s,h}$$
where:
\[
I_{s,h} = 2|h|^{-2} \int_{T_{h_s}(\Omega_h)} |f(x)|^2 \gamma(x - h e_s) dx
\]

Note that \(I_{s,h}\) may be estimated by using (10) in the hypothesis of admissibility. Indeed:
\[
I_{s,h} \leq 2|h|^{-2} C(h) \|f\|_{L^2(\Omega, \gamma)}^2 < \infty
\]
so that \(\tau_{s,h} f \in L^2(\Omega_h, \gamma)\). The estimate:
\[
\|\tau_{s,h} f\|_{L^2(\Omega_h, \gamma)} \leq \sqrt{2|h|^{-1}} (1 + C(h))^{1/2} \|f\|_{L^2(\Omega, \gamma)}
\]

together with the identity \(\tau_{s,h} D^\alpha f = D^\alpha \tau_{s,h} f\) complete the proof.

3. TWO LEMMAS

We shall need the following:

**Lemma 1.** Let \(\gamma(x)\) be an admissible weight. Then for any \(\Omega' \subset \subset \Omega\) there is a constant \(K = K(\Omega, \Omega', \gamma) > 0\) so that for any \(v \in W^{1,2}(\Omega, \gamma)\) and any \(|h| < \frac{1}{2} \text{dist}(\Omega', \partial \Omega)\) we have:
\[
\|\tau_{s,h} v\|_{L^2(\Omega', \gamma)} \leq K \|\frac{\partial v}{\partial x_s}\|_{L^2(\Omega, \gamma)}
\]

(18)

**Proof.** Let \(\Omega' = C \times (a, b) \subset \Omega\), where \(C \subset \mathbb{R}^{N-1}\) is a cube. We may assume w.l.o.g. that \(s = N\). Set \(x' = (x_1, \ldots, x_{N-1})\). Then:
\[
\int_{\Omega'} |\tau_{N,h} v(x)|^2 \gamma(x) dx = \int_C \int_a^b \int_{x_N}^{x_N + h} |\tau_{N,h} v(x', x_N)|^2 \gamma(x', x_N) dx_N dx' =
\]
\[
= \int_C \int_a^b \int_{x_N}^{x_N + h} \left| \frac{\partial v}{\partial x_N}(x', t) \right|^2 \gamma(x', x_N) dx_N dx'
\]

On the other hand:
\[
\left| \frac{1}{h} \int_{x_N}^{x_N + h} \frac{\partial v}{\partial x_N}(x', t) dt \right|^2 = \left| \frac{1}{h} \int_0^h \frac{\partial v}{\partial x_N}(x', t + t_N) dt \right|^2 \leq
\]
\[
\leq \frac{1}{h} \int_0^h \left| \frac{\partial v}{\partial x_N}(x', t + t_N) \right|^2 dt
\]

(one applies the Hölder inequality at the last step). Therefore:
\[
\int_a^b \left| \frac{1}{h} \int_{x_N}^{x_N + h} \frac{\partial v}{\partial x_N}(x', t) \right|^2 \gamma(x', x_N) dx_N \leq \frac{1}{h} F(h)
\]
where:
\[
F(h) = \int_0^h dt \int_a^b \left| \frac{\partial \nu}{\partial x_N}(x', t + x_N) \right|^2 \gamma(x', x_N)dx_N
\]

Set \( h_0 = \frac{1}{2} \text{dist}(\Omega', \partial \Omega) \). By the mean value theorem (applied to \( F \)) there is \( \xi \), between \( 0 \) and \( h \) so that:
\[
\frac{1}{h} F(h) = \int_a^b \left| \frac{\partial \nu}{\partial x_N}(x', \xi + x_N) \right|^2 \gamma(x', x_N)dx_N =
\]
\[
= \int_{a+\xi}^{b+\xi} \left| \frac{\partial \nu}{\partial x_N}(x', x_N) \right|^2 \gamma(x', x_N - \xi)dx_N \leq
\]
\[
\leq C(\xi) \int_{a+\xi}^{b+\xi} \left| \frac{\partial \nu}{\partial x_N}(x) \right|^2 \gamma(x)dx_N \leq
\]
\[
\leq K \int_{a-h_0}^{b+h_0} \left| \frac{\partial \nu}{\partial x_N}(x) \right|^2 \gamma(x)dx_N
\]
(noting that \((x', x_N - \xi) = x - \xi e_N\) one has applied \((10)\) followed by \(iii\) in the definition of admissibility) where \( K = \sup_{|\xi| \leq h_0} C(\xi) \). Finally:
\[
\int_{\Omega'} |\tau_{N,h} v(x)|^2 \gamma(x)dx \leq K \int_{C \times (a-h_0, b+h_0)} \left| \frac{\partial \nu}{\partial x_N}(x) \right|^2 \gamma(x)dx \leq
\]
\[
\leq K \int_{\Omega} \left| \frac{\partial \nu}{\partial x_N} \right|^2 \gamma dx = K \left\| \frac{\partial \nu}{\partial x_N} \right\|_{L^2(\Omega, \gamma)}^2
\]

So \((18)\) is proved when \( \Omega' \) is a cube. In general, it is sufficient to cover \( \Omega' \) with a finite number of cubes, Q.E.D.

**Lemma 2.** Let \( v \in L^2(\Omega, \gamma) \) and assume that there is \( K > 0 \) and \( h_0 > 0 \) so that:
\[
\|\tau_{s,h} v\|_{L^2(\Omega_h, \gamma)} \leq K
\]  
(19)

for any \( |h| < h_0 \). Then \( \partial v / \partial x_s \in L^2(\Omega, \gamma) \) and:
\[
\left\| \frac{\partial \nu}{\partial x_s} \right\|_{L^2(\Omega, \gamma)} \leq K
\]  
(20)

**Proof.** Let \((h_j)_{j \geq 1}\) be a sequence of real numbers so that \( h_j \to 0 \) as \( j \to \infty \). Set \( g_j = \tau_{s,h} v, j \geq 1 \). Let \( \Omega' \subset \subset \Omega \). As \( h_j \to 0 \) there is \( j_0 = j(\Omega', h_0) \geq 1 \) so that \( |h_j| < \min \left( h_0, \frac{1}{2} \text{dist} (\Omega', \partial \Omega) \right) \) for any \( j \geq j_0 \). So on one hand:
\[
\|\tau_{s,h} v\|_{L^2(\Omega_{h_j}, \gamma)} \leq K
\]  
(21)

for any \( j \geq j_0 \), and on the other:
\[
\Omega' \subset \Omega_{h_j}
\]  
(22)
for any $j \geq j_0$. Using (21)-(22) we have:

\[ \|g_j\|_{L^2(\Omega', \gamma)}^2 = \int_{\Omega'} |g_j|^2 \gamma dx \leq \int_{\Omega_{h_j}} |g_j|^2 \gamma dx \leq K^2 \]

so that:

\[ (K^{-1} g_j)_{j \geq j_0} \]

is a sequence in the closed unit ball:

\[ \overline{B}(0, 1) \subset L^2(\Omega', \gamma) \]

But $L^2(\Omega', \gamma)$ is reflexive (any Hilbert space is reflexive) so that $\overline{B}(0, 1)$ is weakly sequentially compact. Consequently, there is a subsequence $(h_{n_j})$ of $(h_j)$ so that:

\[ g_{n_j} \rightharpoonup g \]

as $j \to \infty$ for some $g \in L^2(\Omega', \gamma)$ (weak convergence). Note that:

\[ g = \partial v / \partial x_s \]

Indeed:

\[ \int_{\Omega'} g \phi dx = \lim_{j \to \infty} \int_{\Omega'} g_{n_j} \phi dx = \lim_{j \to \infty} \int_{\Omega'} (\tau_{s,h_{n_j}} v) \phi dx = \]

\[ = - \lim_{j \to \infty} \int_{\Omega'} v_{\tau_{s,-h_{n_j}}} \phi dx = - \int_{\Omega'} v \partial \phi / \partial x_s dx \]

for any $\phi \in C_0^\infty(\Omega')$. Thus:

\[ \partial v / \partial x_s \in L^2_{loc}(\Omega, \gamma) \]

and:

\[ \lim_{h \to 0} \tau_{s,h} v = \partial v / \partial x_s \]

weakly in $L^2(\Omega', \gamma)$. Finally:

\[ \left\| \frac{\partial v}{\partial x_s} \right\|_{L^2(\Omega', \gamma)} \leq \liminf_{h \to 0} \| \tau_{s,h} v \|_{L^2(\Omega, \gamma)} \leq \]

\[ \leq \liminf_{h \to 0} \| \tau_{s,h} v \|_{L^2(\Omega, \gamma)} \leq K \]

and thus (20) holds, Q.E.D.

If $a_{\alpha, \beta}, f_\alpha$ have locally integrable derivatives then a $\gamma$-weak solution $u \in C^2(\Omega)$ is also a classical solution of the equation $\gamma \sum_{|\alpha|=|\beta|=1} D^\alpha (a_{\alpha, \beta} D^\beta u) = \sum_{|\alpha|=1} D^\alpha f_\alpha + f$ in $\Omega$, where $f, f_\alpha \in L^2(\Omega, \gamma^{-1})$. We end Section 3 with a formal argument justifying our assumption $u_0 \equiv 0$ in the Dirichlet problem (4)-(5). Indeed, if $w = u - u_0$ then $\gamma \sum_{|\alpha|=|\beta|=1} D^\alpha (a_{\alpha, \beta} D^\beta w) = \sum_{|\alpha|=1} D^\alpha F_\alpha + F$
where \( F_\alpha = f_\alpha - \sum_{|\beta|=1} a_{\alpha\beta} \cdot D^\beta u_0 \cdot \gamma \) and \( F = f + \sum_{|\alpha|=|\beta|=1} a_{\alpha\beta} \cdot D^\beta u_0 \cdot D^\alpha \gamma \). All we need to show is that \( F, F_\alpha \in L^2(\Omega, \gamma^{-1}) \). Indeed:

\[
\|F\|_{L^2(\Omega, \gamma^{-1})}^2 = \int_\Omega |F|^2 \gamma^{-1} dx \leq 2 \int_\Omega \left( |f|^2 + \sum_{|\alpha|=|\beta|=1} a_{\alpha\beta} \cdot D^\beta u_0 \cdot D^\alpha \gamma \right) \gamma^{-1} dx
\]

Set \( M = \max_{|\alpha|=|\beta|=1} \|a_{\alpha\beta}\|_\infty \). Using (7), i.e. \( |D^\alpha \gamma| \leq C^* \gamma \) a.e. in \( \Omega \), we may perform the estimate:

\[
\sum_{|\alpha|=|\beta|=1} a_{\alpha\beta} \cdot D^\beta u_0 \cdot D^\alpha \gamma \leq 2(MNC^*)^2 \gamma^2 \sum_{|\beta|=1} |D^\beta u_0|^2
\]

Therefore:

\[
\|F\|_{L^2(\Omega, \gamma^{-1})}^2 \leq 2 \int_\Omega |f|^2 \gamma^{-1} dx + 4(MNC^*)^2 \sum_{|\beta|=1} \int_\Omega |D^\beta u_0|^2 \gamma dx \leq 2\|f\|_{L^2(\Omega, \gamma^{-1})}^2 + 4(MNC^*)^2 \|u_0\|_{W^{1,2}(\Omega, \gamma)}^2 < \infty
\]

Finally, we leave it to the reader to verify that \( \|F_\alpha\|_{L^2(\Omega, \gamma^{-1})}^2 \leq 2\|f\|_{L^2(\Omega, \gamma^{-1})}^2 + 4M^2 \|u_0\|_{W^{1,2}(\Omega, \gamma)}^2 < \infty \).

4. THE MAIN ESTIMATE

At this point we may prove our Theorem 1. To this end, let \( \eta \in C^\infty_0(\Omega), 0 \leq \eta \leq 1 \), and \( |h| < \frac{1}{4} \) dist (\( \text{supp}(\eta), \partial \Omega \)) be fixed data. Set \( \phi = \tau_{s,-h}(\eta^2, \tau_{s,h}u) \). Note that \( \phi \in W^{1,2}(\Omega_h, \gamma) \) (and actually \( \phi \in W^{1,2}_0(\Omega, \gamma) \) because \( \text{supp}(\phi) \subseteq \text{supp}(\eta) \subset \Omega_h \)). Then (5) may be written:

\[
\sum_{|\alpha|=1} \int_\Omega \left( \sum_{|\beta|=1} a_{\alpha\beta} \cdot D^\beta u \cdot \gamma - f_\alpha \right) \cdot \tau_{s,-h}(\eta^2 \tau_{s,h} D^\alpha u + 2\eta \cdot D^\alpha \eta \cdot \tau_{s,h}u) dx =
\]

\[
= - \int_\Omega (f + \sum_{|\alpha|=|\beta|=1} a_{\alpha\beta} \cdot D^\beta u \cdot D^\alpha \gamma) \tau_{s,-h}(\eta^2 \tau_{s,h}u) dx
\]

Let us use the properties 2)-3) of the operator \( \tau_{s,h} \) (cf. Section 2) so that to obtain:

\[
\sum_{|\alpha|=1} \int_\Omega \left\{ \sum_{|\beta|=1} [a_{\alpha\beta}(x + he_s)(\gamma(x + he_s) \tau_{s,h} D^\beta u + \right.
\]
\[ + D^\beta u \cdot \tau_{s,h} + D^\beta u \cdot \gamma \cdot \tau_{s,h} a_{\alpha,\beta} \]
\[- \tau_{s,h} f_{\alpha} \{ \eta^2 \tau_{s,h} D^\alpha u + 2 \eta \cdot D^\alpha \eta \cdot \tau_{s,h} u \} dx = \]
\[= \int_\Omega (f + \sum_{|\alpha|=|\beta|=1} a_{\alpha,\beta} \cdot D^\beta u \cdot D^\alpha \gamma) \cdot \tau_{s,-h}(\eta^2 \tau_{s,h} u) dx \quad (23) \]

Consider the following integrals:

\[ I_1 = - \sum_{|\alpha|=|\beta|=1} 2 \int_\Omega \eta \cdot \tau_{s,h} D^\beta u \cdot a_{\alpha,\beta}(x + he_s) \cdot D^\alpha \eta \cdot \tau_{s,h} u \cdot \gamma(x + he_s) dx \]

\[ I_2 = - \sum_{|\alpha|=1} \int_\Omega \eta^2 \cdot \tau_{s,h} D^\alpha u \cdot \left( \sum_{|\beta|=1} D^\beta u \cdot \tau_{s,h} a_{\alpha,\beta} \cdot \gamma - \tau_{s,h} f_{\alpha} \right) dx \]

\[ I_3 = - \sum_{|\alpha|=1} 2 \int_\Omega \eta \cdot D^\alpha \eta \cdot \tau_{s,h} u \cdot \left( \sum_{|\beta|=1} u \cdot \tau_{s,h} a_{\alpha,\beta} \cdot \gamma - \tau_{s,h} f_{\alpha} \right) dx \]

\[ I_4 = \int_\Omega \left( f + \sum_{|\alpha|=|\beta|=1} a_{\alpha,\beta} \cdot D^\beta u \cdot D^\alpha \gamma \right) \tau_{s,-h}(\eta^2 \tau_{s,h} u) dx \]

\[ I_5 = - \sum_{|\alpha|=1} \int_\Omega \eta^2 \cdot \tau_{s,h} D^\alpha u \cdot \sum_{|\beta|=1} a_{\alpha,\beta}(x + he_s) \cdot D^\beta u \cdot \tau_{s,h} \gamma dx \]

\[ I_6 = - \sum_{|\alpha|=1} 2 \int_\Omega \eta \cdot D^\alpha \eta \cdot \tau_{s,h} u \cdot \sum_{|\beta|=1} a_{\alpha,\beta}(x + he_s) \cdot D^\beta u \cdot \tau_{s,h} \gamma dx \]

With these notations the identity (23) becomes:

\[ \sum_{|\alpha|=|\beta|=1} \int_\Omega a_{\alpha,\beta}(x + he_s) \cdot \tau_{s,h} D^\alpha u \cdot \tau_{s,h} D^\beta u \cdot \gamma(x + he_s) \cdot \eta^2 dx = \sum_{j=1}^6 I_j \quad (24) \]

By the ellipticity assumption:

\[ \sum_{|\alpha|=|\beta|=1} \int_\Omega a_{\alpha,\beta}(x + he_s) \cdot \tau_{s,h} D^\alpha u \cdot \tau_{s,h} D^\beta u \cdot \gamma(x + he_s) \cdot \eta^2 dx \geq \]

\[ \geq C_0 \int_\Omega \eta^2 \gamma(x + he_s) |\tau_{s,h} \nabla u|^2 dx \]
so that (24) yields:

$$C_0 \int_{\Omega} \eta^2(x + h\varepsilon_s)|\tau_{s,h} \nabla u|^2 dx \leq \sum_{j=1}^{6} |I_j|$$

(25)

It remains to estimate the integrals $I_j$, $1 \leq j \leq 6$. Set:

$$L = \max_{|\alpha| = |\beta| = 1} \| \partial_{a_{\alpha \beta}} / \partial x_\alpha \|_\infty$$

We obtain:

$$|I_1| \leq 2MN \int_{\Omega} \eta|\tau_{s,h} \nabla u| |\nabla \eta||\tau_{s,h} u|\gamma(x + h\varepsilon_s) dx$$

(26)

$$|I_2| \leq N^{1/2} \int_{\Omega} \eta^2|\tau_{s,h} \nabla u|(NL|\nabla u|\gamma + |\tau_{s,h} F|) dx$$

(27)

$$|I_3| \leq 2N^{1/2} \int_{\Omega} \eta|\nabla \eta||\tau_{s,h} u|(NL|\nabla u|\gamma + |\tau_{s,h} F|) dx$$

(28)

Let us use $2ab \leq \varepsilon a^2 + \varepsilon^{-1} b^2$, $\varepsilon > 0$, to further estimate the integrals $I_j$, $j \in \{1, 2, 3\}$. For instance, set $a = \eta|\tau_{s,h} \nabla u|\gamma(x + h\varepsilon_s)^{1/2}$ and $b = M|\tau_{s,h} u| |\nabla \eta|\gamma (x + h\varepsilon_s)^{-1/2}$. Then (26) furnishes:

$$|I_1| \leq Ne \int_{\Omega} \eta^2|\tau_{s,h} \nabla u|^2 \gamma(x + h\varepsilon_s) dx +$$

$$+ \frac{M^2 N}{\varepsilon} \int_{\Omega} |\tau_{s,h} u|^2 |\nabla \eta|^2 \gamma(x + h\varepsilon_s) dx$$

(29)

Similarly, to estimate $I_2$ (respectively $I_3$) let us choose:

$$a = |\tau_{s,h} \nabla u|\gamma(x + h\varepsilon_s)^{1/2}$$

and:

$$b = (NL|\nabla u|\gamma + |\tau_{s,h} F|)\gamma(x + h\varepsilon_s)^{-1/2}$$

(respectively:

$$a = |\nabla u||\tau_{s,h} u|\gamma(x + h\varepsilon_s)^{1/2}$$

$$b = \eta(NL|\nabla u|\gamma + |\tau_{s,h} F|)\gamma(x + h\varepsilon_s)^{-1/2}$$

and $\varepsilon = 1$). We obtain:

$$|I_2| \leq N^{1/2} \int_{\Omega} \frac{\varepsilon}{2} \eta^2|\tau_{s,h} \nabla u|^2 \gamma(x + h\varepsilon_s) dx +$$

$$+ \frac{N^{1/2}}{2\varepsilon} \int_{\Omega} \eta^2(NL|\nabla u|\gamma + |\tau_{s,h} F|)^2 \gamma(x + h\varepsilon_s)^{-1} dx$$

(30)

$$|I_3| \leq N^{1/2} \int_{\Omega} |\nabla \eta|^2|\tau_{s,h} u|^2 \gamma(x + h\varepsilon_s) dx +$$
\[ + \mathcal{N}^{1/2} \int_{\Omega} \eta^2 (NL|\nabla u|^2 + |\tau_{s,h} F|^2)^2 \gamma(x + he_s)^{-1} dx \]

(31)

The integral \( I_4 \) is somewhat trickier. Since:

\[
\tau_{s,-h}(\eta^2 \tau_{s,h}u) = \eta(x - he_s)\tau_{s,-h}(\eta \tau_{s,h}u) + \eta \cdot \tau_{s,h}u \cdot \tau_{s,-h}\eta
\]

it follows that:

\[
|I_4| \leq J_1 + J_2
\]

(32)

where:

\[
J_1 = \int_{\Omega} |\tau_{s,-h}(\eta \cdot \tau_{s,h}u)| \cdot \left| f + \sum_{|\alpha|=|\beta|=1} a_{\alpha\beta} \cdot D^\beta u \cdot D^\alpha \gamma \right| dx
\]

\[
J_2 = \int_{\Omega} \eta \cdot |\tau_{s,h}u| \cdot |\tau_{s,-h}\eta| \cdot |f + \sum_{|\alpha|=|\beta|=1} a_{\alpha\beta} \cdot D^\beta u \cdot D^\alpha \gamma| dx
\]

To estimate \( J_1 \) (respectively \( J_2 \)) choose:

\[
a = |\tau_{s,-h}(\eta \cdot \tau_{s,h}u)| \gamma(x)^{1/2}
\]

\[
b = \left| f + \sum_{|\alpha|=|\beta|=1} a_{\alpha\beta} \cdot D^\beta u \cdot D^\alpha \gamma \right| \gamma(x)^{-1/2}
\]

(respectively:

\[
a = \eta|\tau_{s,h}u| \cdot |\tau_{s,-h}\eta| \gamma(x)^{1/2}
\]

and \( \epsilon = 1 \)) and use again \( 2ab \leq \epsilon a^2 + b^2 \) / \( \epsilon \) so that to obtain:

\[
J_1 \leq \frac{\epsilon}{2} \int_{\Omega} |\tau_{s,-h}(\eta \cdot \tau_{s,h}u)^2 \gamma dx +
\]

\[
+ \frac{1}{2\epsilon} \int_{\Omega} \left| f + \sum_{|\alpha|=|\beta|=1} a_{\alpha\beta} \cdot D^\beta u \cdot D^\alpha \gamma \right|^2 \gamma^{-1} dx
\]

(33)

\[
J_2 \leq \frac{1}{2} \int_{\Omega} \eta^2 |\tau_{s,-h}\eta|^2 \gamma dx +
\]

\[
+ \frac{1}{2} \int_{\Omega} \left| f + \sum_{|\alpha|=|\beta|=1} a_{\alpha\beta} D^\beta u \cdot D^\alpha \gamma \right|^2 \gamma^{-1} dx
\]

(34)
Next we use (32)-(33) and our Lemma 1 with \( v = \eta \tau_{s,h} u \) so that to perform the following estimates:

\[
|I_4| \leq \frac{\varepsilon}{2} \int_{\Omega} |\tau_{s,-h}(\eta \tau_{s,h} u)|^2 \gamma dx +
\]

\[
+ \frac{1}{2} (1 + \frac{1}{\varepsilon}) \int_{\Omega} |f + \sum_{|\alpha|=|\beta|=1} a_{\alpha \beta} D^\beta u \cdot D^\alpha \gamma|^2 \gamma^{-1} dx +
\]

\[
+ \frac{1}{2} \int_{\Omega} |\tau_{s,-h} \eta|^2 |\tau_{s,h} u|^2 \gamma dx \leq \frac{K \varepsilon}{2} \int_{\Omega} |\nabla(\eta \tau_{s,h} u)|^2 \gamma dx +
\]

\[
+(1 + \frac{1}{\varepsilon}) \int_{\Omega} |f|^2 + \left| \sum_{|\alpha|=|\beta|=1} a_{\alpha \beta} D^\beta u \cdot D^\alpha \gamma \right|^2 \gamma^{-1} dx +
\]

\[
+ \frac{a_0^2}{2} \int_{\Omega} |\tau_{s,h} u|^2 \gamma dx
\]

where \( a_0 = \sup |\nabla \eta| \) (for the last integral one has used \( \eta^2 \leq 1 \) and:

\[
|\tau_{s,-h} \eta(x)| = \frac{1}{|h|} |\eta(x - he_s) - \eta(x)| \leq \frac{1}{|h|} \sup_{y \in [x, x - he_s]} |\nabla \eta(y)||x - he_s - x| \leq \sup |\nabla \eta|
\]

where \([x, y] = \{tx + (1 - t)y : 0 \leq t \leq 1 \}\)). On the other hand (by (7)) we have:

\[
\left| \sum_{|\alpha|=|\beta|=1} a_{\alpha \beta} D^\beta u \cdot D^\alpha \gamma \right| \leq MN^2 C^* |\nabla u| \gamma \tag{35}
\]

a.e. on \( \Omega \). Using again Lemma 1 for \( v = u \) and (35) it follows:

\[
|I_4| \leq \frac{K a_0^2}{2} + M^2 N^4 (C^*)^2 (1 + \frac{1}{\varepsilon}) \int_{\Omega} |\nabla u|^2 \gamma dx +
\]

\[
+(1 + \frac{1}{\varepsilon}) \int_{\Omega} |f|^2 \gamma^{-1} dx + \frac{K \varepsilon}{2} \int_{\Omega} |\nabla(\eta \tau_{s,h} u)|^2 \gamma dx \tag{36}
\]

for some \( K = K(\Omega, supp(\eta), \gamma) > 0 \) (furnished by Lemma 1). We need to estimate the last integral in (36). To this end, note that:

\[
|\nabla(\eta \tau_{s,h} u)|^2 = \sum_{|\alpha|=1} |D^\alpha(\eta \tau_{s,h} u)|^2 \leq
\]

\[
\leq \sum_{|\alpha|=1} (|D^\alpha \eta||\tau_{s,h} u| + \eta|D^\alpha \tau_{s,h} u|)^2 \leq
\]
\[
\leq 2 \sum_{|\alpha|=1} (|D^\alpha \eta|^2 |\tau_{s,h} u|^2 + \eta^2 |D^\alpha \tau_{s,h} u|^2) =
\]
\[
= 2(\nabla \eta^2 |\tau_{s,h} u|^2 + \eta^2 \nabla |\tau_{s,h} u|^2)
\]
Consequently:
\[
\int_{\Omega} |\nabla (\eta \tau_{s,h} u)|^2 \gamma dx \leq
\]
\[
\leq 2a_0^2 \int_{\Omega} |\tau_{s,h} u|^2 \gamma dx + 2 \int_{\Omega} \eta^2 |\tau_{s,h} \nabla u|^2 \gamma dx \leq
\]
\[
\leq 2a_0^2 K \int_{\Omega} |\nabla u|^2 \gamma dx + 2 \int_{\Omega} \eta^2 |\tau_{s,h} \nabla u|^2 \gamma dx
\]
and our estimate (36) becomes:
\[
|I_4| \leq K \varepsilon \int_{\Omega} \eta^2 |\tau_{s,h} \nabla u|^2 \gamma dx + (1 + \frac{1}{\varepsilon}) \int_{\Omega} |f|^2 \gamma^{-1} dx +
\]
\[
+ (a_0^2 \varepsilon K^2 + \frac{a_0^2}{2} K + M^2 N^4 (C^*)^2 (1 + \frac{1}{\varepsilon}) \int_{\Omega} |\nabla u|^2 \gamma dx
\]
(37)

Finally, one has to estimate the integrals \(I_5, I_6\). Note that \(I_5, I_6\) have no counterpart in the classical (unweighted) theory (cf. [7], p. 50-51) i.e. \(I_5 = I_6 = 0\) for \(\gamma \equiv 1\). Using (17) we obtain:
\[
|I_5| \leq MN \frac{C(h) - 1}{|h|} \int_{\Omega} \eta^2 |\tau_{s,h} \nabla u| |\nabla u| \gamma dx
\]
\[
|I_6| \leq 2MN \frac{C(h) - 1}{|h|} \int_{\Omega} \eta |\nabla \eta||\tau_{s,h} u||\nabla u| \gamma dx
\]
To further estimate \(I_5\) (respectively \(I_6\)) choose:
\[
a = |\tau_{s,h} \nabla u| \gamma^{1/2}, \quad b = |\nabla u| \gamma^{1/2}
\]
(respectively:
\[
a = \eta |\nabla \eta||\nabla u| \gamma^{1/2}, \quad b = |\tau_{s,h} u| \gamma^{1/2}
\]
and \(\varepsilon = 1\). We obtain:
\[
|I_5| \leq MNA \frac{\varepsilon}{2} \int_{\Omega} \eta^2 |\tau_{s,h} \nabla u|^2 \gamma dx +
\]
\[
+ MNA \frac{1}{2 \varepsilon} \int_{\Omega} |\nabla u|^2 \gamma dx
\]
(38)
\[
|I_6| \leq MNA(K + a_0^2) \int_{\Omega} |\nabla u|^2 \gamma dx
\]
(39)
where:
\[
A = \sup_{|h| \leq h_0} \frac{C(h) - 1}{|h|}
\]
and:
\[ h_0 = \frac{1}{4} \text{dist}(\text{supp}(\eta), \partial \Omega) \]

(in (39) one has made use of Lemma 1 for \( v = u \)). Set:
\[ B = \sup_{|h| \leq h_0} C(h) \]

Then (29)-(31) yield:
\[
|I_1| \leq NB \frac{\epsilon}{2} \int_{\Omega} \eta^2 |\tau_{s,h} \nabla u|^2 \gamma dx +
M^2 N a_0^2 KB \frac{1}{\epsilon} \int_{\Omega} |\nabla u|^2 \gamma dx \tag{40}
\]
\[
|I_2| \leq N^{1/2} B \frac{\epsilon}{2} \int_{\Omega} \eta^2 |\tau_{s,h} \nabla u|^2 \gamma dx +
N^{1/2} B \frac{\epsilon}{2} \int_{\Omega} \eta^2 (NL|\nabla u| \gamma + |\tau_{s,h} F|^2 \gamma^{-1} dx \tag{41}
\]
\[
|I_3| \leq N^{1/2} B K a_0^2 \int_{\Omega} |\nabla u|^2 \gamma dx +
N^{1/2} B \int_{\Omega} \eta^2 (NL|\nabla u| \gamma + |\tau_{s,h} F|^2 \gamma^{-1} dx \tag{42}
\]

On the other hand, by Lemma 1 for the weight \( \gamma^{-1} \) (indeed, it is easily seen (by (10)) that \( \gamma \) admissible yields \( \gamma^{-1} \) admissible, as well):
\[
\int_{\Omega} \eta^2 (NL|\nabla u| \gamma + |\tau_{s,h} F|^2 \gamma^{-1} dx \leq
\]
\[
\leq 2 \int_{\Omega} \eta^2 (N^2 L^2 |\nabla u|^2 \gamma + |\tau_{s,h} F|^2 \gamma^{-1} dx \leq
\]
\[
\leq 2 N^2 L^2 \int_{\Omega} |\nabla u|^2 \gamma dx + 2K \int_{\Omega} |\nabla F|^2 \gamma^{-1} dx
\]

(as \( \eta^2 \leq 1 \) and \( (a + b)^2 \leq 2(a^2 + b^2) \)) with the corresponding modification of (41)-(42).

Recollecting the information in (37)-(42) our main estimate (25) becomes:
\[
(C_0 B^{-1} - \epsilon C) \int_{\Omega} \eta^2 |\tau_{s,h} \nabla u|^2 \gamma dx \leq
\]
\[
\leq C_1(\epsilon) \int_{\Omega} |\nabla u|^2 \gamma dx + C_2(\epsilon) \int_{\Omega} |\nabla F|^2 \gamma^{-1} dx + C_3(\epsilon) \int_{\Omega} |f|^2 \gamma^{-1} dx \tag{43}
\]

for some \( C_i(\epsilon) > 0, i = 1, 2, 3, \) (e.g. \( C_3(\epsilon) = 1 + 1/\epsilon \)). Let now \( \Omega' \subset \subset \Omega \) and \( \eta = 1 \) on \( \Omega' \). Take \( \epsilon = \epsilon_0 \) where:
\[
0 < \epsilon_0 < \frac{C_0}{BC}
\]

and set:
\[
Q = \max_{i=1}^{3} \frac{C_i(\epsilon_0)}{C_0 B^{-1} - \epsilon_0 C}
\]

Finally, we may use Lemma 2 for \( v = D^3 u \) so that to obtain (12), Q.E.D.
5. A WEIGHTED ESTIMATE

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\gamma \in C^0(\overline{\Omega})$, $\gamma < 0$ in $\Omega$, a weight. Assume that $\gamma$ is admissible, in the sense of M. Troisi. Then (13) in [17], p. 50, holds with $\rho = \gamma^{1/2}$. Precisely, there are two positive constants $A_1, A_2$ so that:

$$A_1 \rho(x)^N \leq \int_\Omega \varphi(x, x_0)dx_0 \leq A_2 \rho(x)^N \quad (44)$$

for any $x \in \Omega$. Here $\varphi(\cdot, x_0)$ denotes the characteristic function of $I(x_0)$. The constant $A_2$ depends only on $N$, while $A_1$ depends on $N$ and on the characteristic cone $C$ of $J(x_0)$, where:

$$J(x_0) = T_{\gamma, x_0}(I(x_0))$$

and the transformation $\xi = T_{\gamma, x_0}(x)$ is defined by:

$$x - x_0 = \rho(x_0)(\xi - x_0) \quad (45)$$

We recall (cf. [17], p. 50) that $C$ does not depend upon $x_0$. The aim of the present section is to establish the following:

**Theorem 3.** Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\gamma \in C^0(\overline{\Omega})$, $\gamma > 0$ in $\Omega$, an admissible (in the sense of M. Troisi) weight. Let $j, m \in \mathbb{Z}$ such that $0 \leq j < m$. Let $q, r \in [1, +\infty]$. Assume that:

$$\frac{1}{p} = \frac{j}{N} + a \left( \frac{1}{r} - \frac{m}{N} \right) + \frac{1}{1}$$

$$a \in \left[ \frac{j}{m}, 1 \right]$$

$$m - j - \frac{N}{r} \not\in \{ \ldots, -2, -1 \}$$

(i.e. $m - j - N / r$ is not a negative integer). Then:

$$|D^j \phi|_{\rho, s+N/p} \leq C |D^m \phi|_{r, \mu}^{\mu-a} \phi|_{q, \mu_0}^{1-a} \quad (46)$$

for any $\phi \in C^\infty_0(\Omega)$ and any $s \in \mathbb{R}$. The constant $C$ in (46) is of the form:

$$C = C_0(A_1^{-1}A_2)^{(1)} / \rho A_0^{(1)} \rho s+N/p \rho a+\mu-N/p \rho 1-a \rho$$

where $C_0$ is a positive constant depending only on the characteristic cone $C$ and on $m, j, r, p$. Also:

$$\mu_0 = s - \left( \frac{j + N}{q} \right) + \frac{2N}{p}$$

$$\mu = s + \frac{1-a}{a} \left( \frac{j + N}{q} - \frac{N}{p} \right) + \frac{N}{p}$$
As to the weighted norms in (46) we adopt the following notation. We set:

$$|u|_{p, \alpha} = \|\rho^\alpha u\|_{L^p(\Omega)}$$

for any $0 < p \leq +\infty, \alpha \in \mathbb{R}$. Also $D^j \phi$ stands for $D^\sigma \phi$ for any fixed multiindex $\sigma$ with $|\sigma| = j$. To prove Theorem 3, we use Theorem 9.3 in [5] so that tp get:

$$\|D^j \phi\|_{L^p(\Omega)} \leq C_0 \|D^m \phi\|_{L^q(\Omega)}^a \|\phi\|_{L^r(\Omega)}^{1-a} \tag{47}$$

where $\phi$ is thought of as a function of $\xi = T_{\gamma, x_0}(x)$. Under the change of variable (45) the inequality (47) becomes:

$$\rho(x_0)^{j-N/p} \|D^j \phi\|_{L^p(\Omega)} \leq \rho(x_0)^{a(m-N/r)-(1-a)N/q} \|D^m \phi\|_{L^q(\Omega)}^a \|\phi\|_{L^r(\Omega)}^{1-a}$$

or (because of $-j + N/p + a(m-N/r) = 1-a$):

$$\|D^j \phi\|_{L^p(\Omega)} \leq C_0 \|D^m \phi\|_{L^q(\Omega)}^a \|\phi\|_{L^r(\Omega)}^{1-a}$$

hence (for any $v \geq 1, s \in \mathbb{R}$):

$$\int \rho(x_0)^{sv} \|D^j \phi\|_{L^v(\Omega)}^v \, dx_0 \leq \int \rho(x_0)^{sv} \|D^m \phi\|_{L^v(\Omega)}^v \|\phi\|_{L^r(\Omega)}^{1-a} \, dx_0$$

Then (by Hölder's inequality):

$$\int \rho(x_0)^{sv} \|D^j \phi\|_{L^v(\Omega)}^v \, dx_0 \leq C_0^v \left( \int \rho(x_0)^{(s+(1-a)(j-N/q-N/p))/v} \|D^m \phi\|_{L^v(\Omega)}^v \, dx_0 \right)^{\frac{1}{v}} \cdot \left( \int \rho(x_0)^{(1-s)(j-N/q+N/p)/v} \|\phi\|_{L^r(\Omega)}^{1-a} \, dx_0 \right)^{1-a-v} \tag{48}$$

We shall need the Lemmas 1.1 and 1.2 in [17], p. 52. That is:

**Lemma 3.** Let $p > 0$ and $s \in \mathbb{R}$. For any function $v$ so that $(x_0 \mapsto \rho(x_0)^s \|v\|_{L^p(\Omega)}) \in L^p(\Omega)$ we have:

$$\int (\rho(x_0)^s \|v\|_{L^p(\Omega)})^p \, dx_0 \geq A_0^{-|s|} A_1 |v|_{p,s+N/p}^p$$

**Lemma 4.** Let $\nu = p > 0$ and $s \in \mathbb{R}$. For any function $v$ so that $\rho^{s+N/p} v \in L^p(\Omega)$ we have:

$$\int (\rho(x_0)^s \|v\|_{L^p(\Omega)})^\nu \, dx_0 \leq A_0^{|s|} A_2 |v|_{p,s+N/p}^{\nu}$$
\[ \nu = p \text{ in (48) and use Lemmas 3 and 4 so that to obtain:} \]
\[ A_0^{-|\nu|p} |D^j \phi|_{p,s+N/p}^p \leq \]
\[ \leq C_0 A_0^{s+(1-a)(j+N/q-N/p)} a^{(a_p)p} A_2^{1-a} |\phi|_{q,s-j-N/q+2N/p}^{(1-a)p} \]
\[ \cdot A_0^{s-j-N/q+N/p} A_2^{1-a} |\phi|_{q,s-j-N/q+2N/p}^{1-a} \]
which is clearly equivalent to (46). Taking \( a = 1, j = 0, m = 1 \) and \( r = 2 \) we obtain the following:

**Corollary 1.** Let \( \Omega \subset R^N \) be a bounded domain and \( \gamma \in C^0(\Omega), \gamma > 0 \) in \( \Omega \), an admissible (in the sense of M. Troisi) weight. If \( N \) is an odd integer \( \geq 3 \) then:
\[ \| \phi \|_{L^q(\Omega, \gamma^{r/2})} \leq C(A_1^{-1} A_2)^{1/q} A_0^{4-N/N} \| D^1 \phi \|_{L^2(\Omega, \gamma)} \]  
(49)
for any \( \phi \in C_0^\infty(\Omega) \), where \( q = 2N/(N-2) \) and \( C \) is a positive constant depending on \( N \) and \( \gamma \).

**6. A MAXIMUM PRINCIPLE**

We adopt the Einstein summation convention (as to upper and lower indices). Let \( \gamma \in C^0(\Omega) \cap C^1(\Omega), \gamma > 0 \) in \( \Omega \), be an admissible weight and rewrite (6) as:
\[ a_{\gamma}(u, \phi) = \int_{\Omega} \{(a^{ij} D_i u + b^i u)D_j(\gamma \phi) - (c^i D_i u + du)\gamma \phi\} dx \]  
(50)
where \( a^{ij}, b^i, c^i, d, (i, j = 1, \ldots, N) \) are measurable functions on \( \Omega \). Then \( u \in W^1(\Omega) \) satisfies \( Lu \geq 0 \) (by definition) if \( a^{ij} D_i u + b^i u, c^i D_i u + du \) are locally integrable and \( a_{\gamma}(u, \phi) \leq 0 \) for any \( \phi \in W_0^{1,2}(\Omega, \gamma), \phi \geq 0 \). We assume (8), i.e. in the new notations:
\[ \sum_{i \leq j \leq N} a^{ij}(x) \xi_i \xi_j \geq C_0 \xi_i^2 \]  
(51)
for any \( x \in \Omega, \xi \in R^N \) and:
\[ C_0^{-2} \sum_{i=1}^N (|b^i(x)|^2 + |c^i(x)|^2 + C_0^{-1}|d(x)| \leq \nu^2 \]  
(52)
for some \( \nu > 0 \). Finally, we request that:
\[ \int_{\Omega} (d \gamma \phi - b^i D_i(\gamma \phi)) dx \leq 0 \]  
(53)
for any \( \phi \in C_0^\infty(\Omega), \phi \geq 0 \).
Let $u \in W^{1,2}(\Omega, \gamma)$. Then $u \leq 0$ on $\partial \Omega$ (by definition) if:

$$u^+ = \max\{u, 0\} \in W^{1,2}_0(\Omega, \gamma)$$

Next, we set:

$$\sup_{\partial \Omega} u = \inf\{k \in \mathbb{R} : u - k \leq 0 \text{ on } \partial \Omega\}$$

Note that (53) continues to hold for any nonnegative $\phi \in W^{1,1}_0(\Omega, \gamma)$, assuming that $b^i$ and $d$ are bounded. Indeed, let $\phi \in W^{1,1}_0(\Omega, \gamma)$ and $\{\phi_k\}_{k \geq 1} \subset C_0^\infty(\Omega)$ so that $||\phi_k - \phi||_{W^{1,1}(\Omega, \gamma)} \to 0$ as $k \to \infty$ and $\phi_k \geq 0$ for any $k \geq 1$. Then:

$$\int_{\Omega} \{d\phi_k \gamma - b^i D_i (\gamma \phi_k)\} dx \leq 0$$

and we may perform the following estimates:

$$\int_{\Omega} (d\phi \gamma - b^i D_i (\gamma \phi)) dx \leq$$

$$\leq \left| \int_{\Omega} \{d(\phi - \phi_k) \gamma - b^i D_i ((\phi - \phi_k) \gamma)\} dx \right| \leq$$

$$\leq M \int_{\Omega} \left| (\phi - \phi_k) \gamma + \sum_{i=1}^N |D_i (\phi - \phi_k) \gamma + \sum_{i=1}^N |\phi - \phi_k| |D_i \gamma| \right| dx \leq$$

$$\leq M ||\phi - \phi_k||_{W^{1,1}(\Omega, \gamma)} + M \int_{\Omega} \sum_{i=1}^N |\phi - \phi_k| C^* \gamma dx \leq$$

$$\leq M (1 + NC^*) ||\phi - \phi_k||_{W^{1,1}(\Omega, \gamma)}$$

for some $M > 0$ with $|b^i| \leq M$, $|d| \leq M$, $1 \leq i \leq N$. Note that we made use of (7) as well.

At this point we may prove Theorem 2. Let $u \in W^{1,2}(\Omega, \gamma)$ and $\phi \in W^{1,2}_0(\Omega, \gamma)$. Then $u \phi \in W^{1,1}_0(\Omega, \gamma)$ and $\nabla (u \phi) = \phi \nabla u + u \nabla \phi$. Indeed, as $u \in W^{1,2}(\Omega, \gamma)$ it follows that $u \in L^1_{loc}(\Omega)$ (by definition). Let $\rho \in C^\infty(\mathbb{R}^N)$, $\rho \geq 0$, $\int \rho dx = 1$, and $\rho = 0$ outside $B(0, 1)$, be some mollifier and:

$$u_h(x) = h^{-N} \int_{\Omega} \rho\left(\frac{x - y}{h}\right) u(y) dy, \quad h > 0, \quad x \in \Omega_h$$

the regularization of $u$. Also, consider $\{\phi_k\}_{k \geq 1} \subset C_0^\infty(\Omega)$ so that $\phi_k \to \phi$ in the norm (1) as $k \to \infty$. Now, on one hand $u \phi \in W^{1,1}(\Omega, \gamma)$ as a consequence of:

$$||u \phi||_{W^{1,1}(\Omega, \gamma)} \leq (2N + 1)||u||_{W^{1,2}(\Omega, \gamma)} ||\phi||_{W^{1,2}(\Omega, \gamma)}$$

and on the other hand:

$$||u_h \phi_k - u \phi||_{W^{1,1}(\Omega, \gamma)} \to 0 \quad (54)$$
as \( h \to 0, k \to \infty \), as a consequence of the following considerations:

\[
\|u_h \phi_k - u \phi\|_{W^{1,1}(\Omega, \gamma)} =
\]

\[
= \int_{\Omega} |u_h \phi_k - u \phi| \gamma \, dx + \int_{\Omega} \sum_{i=1}^{N} |D_i(u_h \phi_k - u \phi)| \gamma \, dx
\]

These integrals are estimated separately. Firstly:

\[
\int_{\Omega} |u_h \phi_k - u \phi| \gamma \, dx \leq
\]

\[
\leq M_k \int_{\Gamma_k} |u_h - u| \gamma \, dx + \|u\|_{W^{1,2}(\Omega, \gamma)} \|\phi_k - \Phi\|_{W^{1,2}(\Omega, \gamma)}
\]

(55)

where \( 0 < M_k = \sup_{\Gamma_k} |\phi_k| \) and \( \Gamma_k = \text{supp}(\phi_k), k \geq 1 \). We need:

**Lemma 5.** If \( u \in L^p_{\text{loc}}(\Omega, \gamma) \) then \( u_h \to u \) in \( L^p_{\text{loc}}(\Omega, \gamma) \), i.e. \( \|u_h - u\|_{L^p(\Omega', \gamma)} \to 0 \) as \( h \to 0 \), for any \( \Omega' \subset \subset \Omega \).

**Proof.** Note that:

\[
|u_h(x)|^p \leq \int_{|z| \leq 1} \rho(z)|u(x - hz)|^p \, dz
\]

(56)

**Step 1.** For any \( \Omega' \subset \subset \Omega, 0 < \delta < \frac{1}{2} \text{dist}(\Omega', \partial \Omega) \) set:

\[
M = M(\gamma, \Omega', \delta) = \inf \{ \gamma(y) : y \in B_\delta(\Omega') \}
\]

where \( B_\delta(\Omega') = \{ x \in \Omega : \text{dist}(x, \Omega') < \delta \} \). Then:

i) \( M > 0 \)

ii) For any \( 0 < h < \min(\delta, M) \) we have \( u_h \in L^p(\Omega', \gamma) \) and:

\[
\|u_h\|_{L^p(\Omega', \gamma)} \leq A_0^{2/p} \|u\|_{L^p(B_\delta(\Omega'), \gamma)}
\]

(57)

**Proof.** Assume \( M = 0 \). There is a sequence \( \{y_n\} \) in \( B_\gamma(\Omega') \) with \( \lim_{n \to \infty} \gamma(y_n) = 0 \). As \( \{y_n\} \) is bounded, we may extract a subsequence \( \{y'_n\} \) so that \( y'_n \to y_0 \) as \( n \to \infty \), for some \( y_0 \in B_\delta(\Omega') \subset \Omega \). Thus \( \gamma(y_0) = 0 \) for some \( y_0 \in \Omega \), a contradiction.

To prove ii) in **Step 1**, let \( \Omega'_z = T_{-hz}(\Omega') \subset \Omega, |z| \leq 1 \). As

\[
|y + hz - y| = h|z| \leq h < M
\]

\[
\Omega'_z \subset B_h(\Omega') \subset B_\delta(\Omega')
\]

it follows that \( y + hz \in l(y) \) for any \( y \in \Omega'_z \). Thus (by (15)):

\[
\gamma(y + hz) \leq A_0^2 \gamma(y)
\]
for any \( y \in \Omega'_z \). Thus (by (56)) we have the following estimates:

\[
\int_{\Omega'} |u_h(x)|^p \gamma(x) dx \leq \\
\leq \int_{\Omega'} \gamma(x) \left[ \int_{|z| \leq 1} \rho(z) |u(x - Hz)|^p dz \right] dx = \\
= \int_{|z| \leq 1} \rho(z) \left[ \int_{\Omega'} |u(x - Hz)|^p \gamma(x) dx \right] dz
\]

and:

\[
\int_{\Omega'} |u(x - Hz)|^p \gamma(x) dx = \int_{\Omega'} |u(y)|^p \gamma(y + Hz) dy \leq \\
\leq A_0^3 \int_{\Omega'_z} |u(y)|^p \gamma(y) dy \leq A_0^2 \int_{B_h(\Omega')} |u(y)|^p \gamma(y) dy
\]

because \( \Omega'_z \subset B_h(\Omega') \). Indeed, if \( x \in \Omega'_z \) then \( x = y - Hz \) for some \( y \in \Omega' \). Thus \( \text{dist}(x, \Omega') = \inf_{\xi \in \Omega'} |x - \xi| \leq |x - y| = h |z| < h \). Finally:

\[
\int_{\Omega'} |u_h(x)|^p \gamma(x) dx \leq \\
\leq \int_{|z| \leq 1} \rho(z) \left[ A_0^3 \int_{B_h(\Omega')} |u(y)|^p \gamma(y) dy \right] dz = \\
= A_0^3 \int_{B_h(\Omega')} |u(y)|^p \gamma(y) dy = A_0^2 \left( \|u\|_{L^p(B_h(\Omega'), \gamma)} \right)^p
\]

and (57) is completely proved.

**Step 2.** For any \( \Omega' \subset \subset \Omega \), \( u_h \to u \) in \( L^p(\Omega', \gamma) \) as \( h \to 0 \).

Note that \( C^0(\Omega) \subset L^p_{\text{loc}}(\Omega, \gamma) \) by (2). Indeed:

\[
\int_{\Omega'} |w|^p \gamma dx \leq \sup_{\Omega'} |w|^p \int_{\Omega'} \gamma dx < \infty
\]

Let \( w \in C^0(\Omega) \). Then we may apply (57) for the function \( u - w \in L^p_{\text{loc}}(\Omega, \gamma) \), that is:

\[
\|u_h - w_h\|_{L^p(\Omega', \gamma)} \leq A_0^{2/p} \|u - w\|_{L^p(B_h(\Omega'), \gamma)} \tag{58}
\]

for any \( w \in C^0(\Omega) \). Set \( \Omega'' = B_h(\Omega') \). Let \( \varepsilon > 0 \) arbitrary. By a classical result in measure theory (cf. e.g. [16], p. 213) there is \( w \in C^0(\Omega) \) so that:

\[
\|u - w\|_{L^p(\Omega', \gamma)} \leq \frac{1}{n} \tag{59}
\]
Using (58)-(59) we have:
\[
\|u - u_h\|_{L^p(\Omega', \gamma)} \leq
\]
\[
\leq \|u - w\|_{L^p(\Omega', \gamma)} + \|w - w_h\|_{L^p(\Omega', \gamma)} + \|w_h - u_h\|_{L^p(\Omega', \gamma)} \leq
\]
\[
\leq \epsilon (1 + A_0^{1/p}) + \|w - w_h\|_{L^p(\Omega', \gamma)}
\]

But \(w_h\) converges uniformly to \(w\) (as \(w \in C^0(\Omega), cf. [8], p. 147\) on any relatively compact subset of \(\Omega\). Thus for any \(\epsilon > 0\), there is \(\delta = \delta(\epsilon, \Omega')\) so that \(|w_h(x) - w(x)| < \epsilon\) \((\|\gamma\|_{L^p(\Omega')}^{-1})^{-1}\), for any \(|h| < \delta, x \in \Omega'\). Then \(\|w - w_h\|_{L^p(\Omega', \gamma)} < \epsilon, \ |h| < \delta\). The proof of Lemma 5 is complete.

Let us go back to the proof of (54). By (55) and Lemma 5 it follows that \(\int_{\Omega} |u_h \phi_k - u \phi| \gamma dx \to 0\). The integral:
\[
\int_{\Omega} \sum_{i=1}^{N} |D_i(u_h \phi_k - u \phi)| \gamma dx
\]
may be dealt with in a similar way (by using \(D_i u \in L^2(\Omega, \gamma)\) and \(D_i u_h = (D_i u)_h\), as in particular \(u \in L^1_{loc}(\Omega)\)).

Let \(\phi \in W^{1,2}(\Omega, \gamma)\) so that \(\phi \geq 0\) and \(u \phi \geq 0\) in \(\Omega\). Then \(Lu \geq 0\) and (53) yield:
\[
\int_{\Omega} \{a^j D_j u D_i (\gamma \phi) - (b^i + c^i) \phi \gamma D_i u\} dx \leq
\]
\[
\leq \int_{\Omega} \{du \gamma \phi \ b^i D_i(u \phi \gamma)\} dx \leq 0
\]

Next we use (52) in the form:
\[
\sum_{i=1}^{N} (|b^i|^2 + |c^i|^2) \leq C_0^2 v^2
\]
so that to perform the estimates:
\[
\int_{\Omega} a^j D_j u D_i (\gamma \phi) dx \leq \int_{\Omega} (b^i + c^i) \phi \cdot D_i u \cdot \gamma dx \leq
\]
\[
\leq \int_{\Omega} |b^i + c^i| |\phi||D_i u| \gamma dx \leq 2C_0 v \int_{\Omega} |\nabla u| \phi \gamma dx
\]

Let \(\ell = \sup_{\partial \Omega} u^+\). The rest of the proof of (16) is by contradiction. Assume that \(\sup_{\Omega} u > \ell\). Then there is \(k \in \mathbb{R}\) so that \(\ell < k < \sup_{\Omega} u\). As previously established:
\[
\int_{\Omega} a^j D_j u \cdot D_i (\phi \gamma) dx \leq 2C_0 v \int_{\Omega} |\nabla u| \phi \gamma dx
\]
for any \( \phi \in W^{1,2}_0(\Omega, \gamma) \) with \( \phi \geq 0, \phi u \geq 0 \). Then:

\[
\int_\Omega a^{ij} D_j u \cdot D_i \phi \cdot \gamma dx \leq 2 C_0 v \int_\Omega |\nabla u| \phi \gamma dx - \int_\Omega a^{ij} D_j u \cdot D_i \gamma dx \leq 2 C_0 v \int_\Omega |\nabla u| \phi \gamma dx + \int_\Omega |a^{ij}||D_j u||\phi||D_i \gamma| dx \leq (2 C_0 v + NMC^*) \int_\Omega |\nabla u| \phi \gamma dx
\]

(where \( M = \max_{1 \leq i,j \leq N} ||a^{ij}||_\infty \)). Note that we used (7). At this point we may use:

\[
\int_\Omega a^{ij} D_j u \cdot D_i \phi \cdot \gamma dx \leq (2 C_0 v + N^2 MC^*) \int_\Omega |\nabla u| \phi \gamma dx \tag{61}
\]

for \( \phi = (u - k)^+ \). As \( k > 0 \), \( \phi \) vanishes at the points where \( u < 0 \) (thus \( \phi u \geq 0 \)). Set \( C_1 = 2 C_0 v + N^2 MC^* > 0 \). Then (61) furnishes:

\[
\int_\Omega a^{ij} D_j \phi \cdot D_i \phi \cdot \gamma dx \leq C_1 \int_\Gamma |\nabla \phi| \gamma dx
\]

where \( \Gamma = \text{supp}(\nabla \phi) \subset \text{supp}(\phi) \). Hence (by (51)):

\[
C_0 \int_\Omega |\nabla \phi|^2 \gamma dx \leq C_1 \int_\Gamma |\nabla \phi| \gamma dx \leq C_1 ||\phi||_{L^2(\Gamma, \gamma)} ||\nabla \phi||_{L^2(\Gamma, \gamma)} ||\nabla \phi||_{L^2(\Omega, \gamma)}
\]

so that:

\[
||\nabla \phi||_{L^2(\Omega, \gamma)} \leq C ||\phi||_{L^2(\Gamma, \gamma)} \tag{62}
\]

where \( C = C_1 / C_0 > 0 \). Let us now apply the weighted inequality (49). Indeed:

\[
||\phi||_{L^q(\Omega, \gamma q^{q/2})} \leq B \max_{|\alpha| = 1} ||a^{\alpha} \phi||_{L^2(\Omega, \gamma)} \leq B ||\nabla \phi||_{L^2(\Omega, \gamma)}
\]

where \( q = 2N / (N - 2) \) and \( B = B(N, A_0, A_1, A_2, C) \) is some positive constant. Set \( \lambda = BC \). Then:

\[
||\phi||_{L^q(\Omega, \gamma q^{q/2})} \leq \lambda ||\phi||_{L^2(\Gamma, \gamma)} \leq \lambda |\Gamma|^{1/n} \left( \int_\Gamma |\phi|^q \gamma q^{q/2} dx \right)^{1/q} \leq \lambda |\Gamma|^{1/n} ||\phi||_{L^q(\Omega, \gamma q^{q/2})}
\]
so that:

$$|\text{supp}(\nabla \phi)| \geq \lambda^{-n}$$  \hspace{1cm} (63)

This inequality is independent of $k$ so it must hold as $k$ tends to $\sup_{\Omega} u$. The contradiction is attained by Lemma 7.7 in [8], p. 152.

7. ON A TRACE THEOREM BY J. NECAS

We recall the notion of domain of class $C^0$ (respectively $C^{0,1}$). This is best understood in the context of domains in $C^\infty$ manifolds. Let $X$ be a real $N$-dimensional $C^\infty$ differentiable manifolds. Set $\Delta = \{y = (y_1, \ldots, y_{N-1}) \in \mathbb{R}^{N-1} : |y_1| < \delta, 1 \leq i \leq N - 1\}$. Consider a domain $\Omega \subseteq X$. We say that $\Omega \in C^0$ iff there exist $m$ local charts $(U_\alpha, \varphi_\alpha)$ of $X$, $\varphi_\alpha(U_\alpha) = \mathbb{R}^N$, and there exist $m$ functions $a_\alpha : \Delta \to \mathbb{R}, 1 \leq \alpha \leq m$, so that:

i) For any $x \in \partial \Omega$, there is $\alpha \in \{1, \ldots, m\}$ so that:

$$x'(x) = (x^1(x), \ldots, x^{N-1}(x)) \in \Delta$$

and:

$$x^N(x) = a_\alpha(x'(x))$$

where $\varphi_\alpha = (x^1, \ldots, x^N)$.

ii) Each $a_\alpha : \Delta \to \mathbb{R}$ is continuous, $1 \leq \alpha \leq m$.

iii) There is $0 < \beta < 1$ so that the sets $\Omega_\alpha = B_\alpha \cap \Omega$, $\Gamma_\alpha = B_\alpha \cap \partial \Omega$ satisfy:

$$\Omega_\alpha = \{x \in U_\alpha : x'(x) \in \Delta, a_\alpha(x'(x)) - \beta < x^N(x) < a_\alpha(x'(x))\}$$

and:

$$\Gamma_\alpha = \{x \in U_\alpha : x'(x) \in \Delta, x^N(x) = a_\alpha(x'(x))\}$$

for any $1 \leq \alpha \leq m$, where $B_\alpha$ are given by:

$$B_\alpha = \{x \in U_\alpha : x'(x) \in \Delta, a_\alpha(x'(x)) - \beta < x^N(x) < a_\alpha(x'(x)) + \beta\}$$

Next:

$$\Omega \in C^{0,1}$$

iff $\Omega \in C^0$ and the functions $a_\alpha$ in ii) are Lipschitz on $\Delta$:

$$|a_\alpha(y) - a_\alpha(z)| \leq A|y - z|$$

for some $A > 0$ and any $y, z \in \Delta$.

Let $\Omega \subseteq \mathbb{R}^N$ be a domain and let $\gamma(x)$ be given by (14) for $s(t) = t^\epsilon, t > 0, \epsilon \in \mathbb{R}$. If $\Omega$ is bounded and $\epsilon \geq 0$ then $C^\infty(\overline{\Omega}) \subset W^{1,p}(\Omega, \gamma)$, by Theor. 6.3 in [10], p. 48. If additionally $\Omega \in C^{0,1}$, $0 \leq \epsilon < p - 1$ and $M = \partial \Omega$ then (by Theor. 1.2 in [14], p. 309) $W^{1,p}(\Omega, \gamma) \to L^p(\partial \Omega)$, i.e. there is a continuous linear map $Z : W^{1,p}(\Omega, \gamma) \to L^p(\partial \Omega)$ so that $Z(\phi) = \phi|_{\partial \Omega}$ for any $\phi \in C^\infty(\overline{\Omega})$. 


Let us recall that $L^p(\partial \Omega, \gamma)$ consists of all $u(x)$ so that:

$$\|u\|_{L^p(\partial \Omega, \gamma)} = \left( \sum_{\alpha=1}^{m} \int_{\Delta} |u_{\alpha}(y, a_{\alpha}(y))|^{\gamma_{\alpha}(y, a_{\alpha}(y))} dy \right)^{1/p} < \infty$$

where $\Omega \subset C^{0,1}$ and $u_{\alpha} = u \circ \varphi^{-1}_{\alpha}$. The space $L^p(\partial \Omega, \gamma)$ is denoted by $L^p(\partial \Omega, d_M, \omega)$ (cf. [10], p. 100) when $\gamma(x) = d_M(x)^{\omega}$. If $M = \partial \Omega$ then $d_M(y, a_{\alpha}(y)) \equiv 0$ and the space $L^p(\partial \Omega, d_M, \omega)$ makes sense only for $\omega = 0$ (and then $L^p(\partial \Omega, d_{\partial \Omega}, 0) = L^p(\partial \Omega)$). We obtain:

**Theorem 4.** Let $1 < p < \infty$. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\gamma \in C^1(\Omega) \cap C^0(\overline{\Omega})$ a weight possessing the property $(P_1)$. If $\Omega \subset C^{0,1}$ and $C^\infty(\overline{\Omega})$ is dense in $W^{1,p}(\Omega, \gamma)$ then there is a continuous linear map $Z : W^{1,p}(\Omega, \gamma) \to L^p(\partial \Omega, \gamma)$ so that $Z(\phi) = \phi|_{\partial \Omega}$ for any $\phi \in C^\infty(\overline{\Omega})$.

**Proof.** Let $y \in \Delta$ and $0 < \eta < \beta$. Let $\phi \in C^\infty(\overline{\Omega})$. Then:

$$\phi_{\alpha}(y, a_{\alpha}(y))_{Y_{\alpha}(y, a_{\alpha}(y))}^{1/p} = \phi_{\alpha}(y, a_{\alpha}(y) - \eta)_{Y_{\alpha}(y, a_{\alpha}(y) - \eta)}^{1/p} + \int_{a_{\alpha}(y) - \eta}^{a_{\alpha}(y)} \frac{\partial}{\partial t} (\phi_{\alpha} Y_{\alpha}^{1/p})(y, t) dt$$

where $\phi_{\alpha} = \phi \cdot \varphi^{-1}_{\alpha}$, $Y_{\alpha} = \gamma \circ \varphi^{-1}_{\alpha}$. Consequently:

$$|\phi_{\alpha}(y, a_{\alpha}(y))|_{Y_{\alpha}(y, a_{\alpha}(y))}^{1/p} \leq |\phi_{\alpha}(y, a_{\alpha}(y) - \eta)|_{Y_{\alpha}(y, a_{\alpha}(y) - \eta)}^{1/p} + |J_1| + \frac{1}{p} |J_2|$$

where:

$$J_1 = \int_{a_{\alpha}(y) - \eta}^{a_{\alpha}(y)} \frac{\partial}{\partial t} \phi_{\alpha}(y, t)_{Y_{\alpha}(y, t)}^{1/p} dt$$

$$J_2 = \int_{a_{\alpha}(y) - \eta}^{a_{\alpha}(y)} \phi_{\alpha}(y, t) Y_{\alpha}(y, t)_{Y_{\alpha}(y, t)}^{(1-p)/p} \frac{\partial}{\partial t} Y_{\alpha}(y, t) dt$$

We estimate the integrals $J_i, i = 1, 2$, as follows (cf. Hölder’s inequality):

$$|J_1| \leq \beta^{1/q} \left( \int_{a_{\alpha}(y) - \beta}^{a_{\alpha}(y)} \left| \frac{\partial}{\partial t} \phi_{\alpha}(y, t) \right|_{Y_{\alpha}(y, t)}^{p} dt \right)^{1/p}$$

where $q = p/(p - 1)$. Next, using (7) we have:

$$|J_2| \leq \int_{a_{\alpha}(y) - \eta}^{a_{\alpha}(y)} |\phi_{\alpha}(y, t)|_{Y_{\alpha}(y, t)}^{(1-p)/p} C^*_Y \gamma_{\alpha}(y, t) dt \leq$$
\[
\leq \beta^{1/q} C^* \left( \int_{a_\alpha(y) - \beta}^{a_\alpha(y)} |\phi_\alpha(y, t)|^p \gamma_\alpha(y, t) dt \right)^{1/p}
\]

Using these estimates and the inequality \((a + b)^p \leq 2^{p-1}(a^p + b^p)\), for \(a > 0, b > 0\), we obtain:

\[
|\phi_\alpha(y, a_\alpha(y))|^p \gamma_\alpha(y, a_\alpha(y)) \leq \\
\leq 2^{p-1}(|\phi_\alpha(y, a_\alpha(y) - \eta)|^p \gamma_\alpha(y, a_\alpha(y) - \eta) + \\
+ 2^{p-1} \beta^{p/q} \int_{a_\alpha(y) - \beta}^{a_\alpha(y)} \left| \frac{\partial \phi_\alpha}{\partial t}(y, t) \right|^p \gamma_\alpha(y, t) dt + \\
+ \left( \frac{C^*}{p} \right)^p \int_{a_\alpha(y) - \beta}^{a_\alpha(y)} |\phi_\alpha(y, t)|^p \gamma_\alpha(y, t) dt)
\]

Furthermore, we integrate with respect to \(\eta\) on the interval \([\beta / 2, \beta]\) so that to yield:

\[
\frac{\beta}{2} |\phi_\alpha(y, a_\alpha(y))|^p \gamma_\alpha(y, a_\alpha(y)) \leq \\
\leq 2^{p-1} \left\{ \int_{a_\alpha(y) - \beta}^{a_\alpha(y)} |\phi_\alpha(y, t)|^p \gamma_\alpha(y, t) dt + \\
+ 2^{p-2} \beta^{1+p/q} \left[ \left( \frac{C^*}{p} \right)^p \int_{a_\alpha(y) - \beta}^{a_\alpha(y)} |\phi_\alpha(y, t)|^p \gamma_\alpha(y, t) dt \right] \right\} \leq \\
\leq C_1 \int_{a_\alpha(y) - \beta}^{a_\alpha(y)} |\phi_\alpha(y, t)|^p \gamma_\alpha(y, t) dt + \\
+ C_2 \int_{a_\alpha(y) - \beta}^{a_\alpha(y)} \left| \frac{\partial \phi_\alpha}{\partial t}(y, t) \right|^p \gamma_\alpha(y, t) dt
\]

where \(C_1 = 2^{p-1}(1 + 2^{p-2}p^{-p}(C^*p^{1+p/q})\text{ and } C_2 = 2^{p-3} \beta^{1+p/q}.\) Finally, we integrate with respect to \(y\) on \(\Delta\). We also make use of the identity:

\[
\int_{\Omega} f(x) dx = \int_{\Delta} dy \int_{a_\alpha(y) - \beta}^{a_\alpha(y)} f_\alpha(y, t) dt
\]

where \(f_\alpha = f \circ \phi_\alpha^{-1}.\) We obtain:

\[
\frac{\beta}{2} \int_{\Delta} |\phi_\alpha(y, a_\alpha(y))|^p \gamma_\alpha(y, a_\alpha(y)) dy \leq \\
\leq C_1 \int_{\Omega} |\phi(x)|^p \gamma(x) dx + C_2 \int_{\Omega} |D_\phi(x)|^p \gamma(x) dx \leq
\]
\[ Q \| \Phi \|_{W^{1,p}(\Omega, \gamma)} \leq \Phi \|_{W^{1,p}(\Omega, \gamma)} \]

where \( Q = \max \{ C_1, C_2 \} \). We now take the sum over \( 1 \leq \alpha \leq m \) and obtain:

\[
\| \Phi \|_{L^p(\partial \Omega, \gamma)} \leq \left( \frac{2mQ}{\beta} \right)^{1/p} \| \Phi \|_{W^{1,p}(\Omega, \gamma)}
\]

(64)

Thus \( C^\infty(\overline{\Omega}) \hookrightarrow L^p(\partial \Omega, \gamma) \) and due to the hypothesis:

\[
\overline{C^\infty(\Omega)} = W^{1,p}(\Omega, \gamma)
\]

(65)

we obtain \( Z \) by continuous extension.

Recall that, given \( x \in \partial \Omega \), we say that \( \Omega \in \mathcal{C}(x) \) iff \( \Omega \in \mathcal{C}^0 \) and there is an open bounded cone \( \mathcal{C}_x \) with vertex at \( x \) and such that \( \mathcal{C}_x \cap \overline{\Omega} = \emptyset \) (cf. [10], p. 29).

**Remark 1.** Assume that \( \Omega \in \mathcal{C}(\overline{M}) \), i.e. for any \( x \in \overline{M} \), \( \Omega \in \mathcal{C}(x) \) and the cones \( \mathcal{C}_x \) are mutually congruent. Let \( \gamma \) be given by (14) with \( s(t) = t^\epsilon, t > 0, \epsilon \geq 0 \). Then (65) holds (cf. Prop. 7.6 in [10], p. 62) and our Theorem 4 applies.

**Theorem 5.** Let \( 1 < p < \infty \). Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain of class \( \mathcal{C}^{0,1} \) and \( \gamma(x) \) a weight on \( \Omega \). Assume that i) (65) holds and ii) there is a constant \( 0 < C < \infty \) so that:

\[
\int_{a(y) - \beta}^{a(y)} \gamma_\alpha(y, t)^{1/(1-p)} dt \leq C
\]

(66)

for any \( y \in \Delta \) and any \( 1 \leq \alpha \leq m \). Then \( W^{1,p}(\Omega, \gamma) \rightarrow L^p(\partial \Omega) \). Moreover, if additionally \( \Omega \in \mathcal{C}(\overline{M}) \) and \( \gamma(x) \) is given by (14) with \( s(t) = t^\epsilon, t > 0, 0 \leq \epsilon < p - 1 \), then the hypothesis i) - ii) are satisfied (with a constant \( C > 0 \) depending only on \( \epsilon, p, \beta \) and \( A \)).

**Proof.** Let \( y \in \Delta, 0 < \eta < \beta \) and \( \Phi \in C^\infty(\overline{\Omega}) \). We wish to derive an estimate similar to (64). The proof is analogous to that of Theorem 4, so that we allow ourselves to be somewhat sketchy. Using (66) and Hölder’s inequality we obtain:

\[
\| \Phi_\alpha(y, a(y)) \| \leq \| \Phi_\alpha(y, a(y) - \eta) \| +
\]

\[+ C^{(p-1)/p} \left[ \int_{a(y) - \beta}^{a(y)} \left| \frac{\partial \Phi_\alpha}{\partial t}(y, t) \right|^{p} \gamma_\alpha(y, t) dt \right]^{1/p} \]

for any \( y \in \Delta, 1 \leq \alpha \leq m \). Moreover, we integrate with respect to \( \eta \) from \( \beta / 2 \) to \( \beta \) and use the estimate:

\[
\int_{a(y) - \beta}^{a(y) - \beta / 2} \| \Phi_\alpha(y, t) \| dt \leq C^{(p-1)/p} \left[ \int_{a(y) - \beta}^{a(y)} \| \Phi_\alpha(y, t) \|^{p} \gamma_\alpha(y, t) dt \right]^{1/p}
\]

so that to yield (as \( \beta / 2 < 1 \)):

\[
\left( \frac{\beta}{2} \right)^{p} \| \Phi_\alpha(y, a(y)) \|^{p} \leq
\]
\[ \leq (2C)^{p-1} \int_{a(x)-y}^{a(x)+y} \left\{ |\Phi_\alpha(y,t)|^p + \left| \frac{\partial \Phi_\alpha(y,t)}{\partial t} \right|^p \right\} \gamma_\alpha(y,t) dt \]

Finally, let us integrate with respect to \( y \) on \( \Delta \) and sum over \( 1 \leq \alpha \leq m \). This procedure furnishes:

\[ \frac{\beta}{2} \| \Phi \|_{\partial \Omega} \| \nu \|_{U(\partial \Omega)} \leq m^{1/p} (2C)^{(p-1)/p} \| \Phi \|_{W^{1,p}(\Omega, \gamma)} \]

and we are done. To check the second statement in Theorem 5 it suffices to show that:

\[ \int_{a_\alpha(y)-\beta}^{a_\alpha(y)} d_M(\varphi_\alpha^{-1}(y,t))^{\epsilon/(1-p)} dt < \infty \]

for any \( y \in \Delta \), provided that \( 0 \leq \epsilon < p - 1 \). Set \( x = \varphi_\alpha^{-1}(y,t), y \in \Delta, a_\alpha(y) - \beta < t < a_\alpha(y) \).

By Lemma 4.6 in [10], p. 25.

\[ |a_\alpha(y) - t| \leq (A + 1)d_{\partial \Omega}(x) \leq (A + 1)d_M(x) \]

Set \( q = \epsilon/(1-p) \). As \( 0 \leq q < 1 \), if \( a \geq b > 0 \) then \( a^{-q} \leq b^{-q} \). Thus:

\[ |a_\alpha(y) - t|^{-q} \geq (A + 1)^{-q} \gamma_\alpha(y,t)^{1/(1-p)} \]

and we obtain:

\[ \int_{a_\alpha(y)-\beta}^{a_\alpha(y)} \gamma_\alpha(y,t)^{1/(1-p)} dt \leq (A + 1)^q \frac{\beta^{1-q}}{1-q} \]

and Theorem 5 is completely proved.
REFERENCES


