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EVALUATION OF MEAN-VALUES OF PRODUCTS OF SHIFTED ARITHMETICAL FUNCTIONS[1]

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Dedicated to the Memory of Gottfried Köthe

1. INTRODUCTION

1.1. Lucht's Results

In [16], L. Lucht showed the existence of mean-values

$$M(F) = \lim_{x\to\infty} \frac{1}{x}.\sum_{n\leq x} F(n) \tag{1}$$

for arithmetical functions $F : \mathbb{N} \to \mathbb{C}$, defined by

$$F(n) = \prod_{\kappa=1}^{k} f_\kappa(L_\kappa(n)). \tag{2}$$

The functions $f_\kappa (\kappa = 1, \ldots, k)$ are *multiplicative* functions, satisfying

$$|f_\kappa| \leq 1, \text{ and } \sum_p \left|\frac{f_\kappa(p)-1}{p}\right| < \infty, \tag{3}$$

$L_\kappa : n \mapsto \frac{1}{\gamma_\kappa}.(\beta_\kappa n + \alpha_\kappa)$ are linear forms with $\alpha_\kappa \in \mathbb{Z}$, $\beta_\kappa, \gamma_\kappa \in \mathbb{N}$[2]

Representing f_κ as a *convolution product* $f_\kappa = 1 * g_\kappa$, the functions $g_\kappa = \mu * f_\kappa$: $n \mapsto \sum_{d|n} \mu(d) \cdot f_\kappa\left(\frac{n}{d}\right)$ are "small"; according to (3), the series $\sum_p \left|\frac{g_\kappa(p)}{p}\right|$ is absolutely convergent. In the terminology of [12], the functions f_κ are *related* to the constant function 1.

In addition, in [16], L. Lucht gave a *product representation* for the mean-value (1); apart from some factor γ, depending on $\gamma_1, \ldots, \gamma_k$, it is

$$M(F) = \gamma \cdot \prod_p \left(\sum_{\nu=0}^{\infty} \frac{1}{p^\nu} \cdot G(p^\nu)\right), \tag{4}$$

[1]In February 1996, at the university of Hong Kong, resp. in May 1996, at the Kyoto Conference on Analytic Number Theory, the author gave lectures on the topics dealt with in this paper. The author would like to express his gratitude to Prof. Dr. M.C. Liu, Hong Kong, and to the organizers of the Kyoto Conference, Profs. Dr. Y. Motohashi, N. Hirata-Kohno and Leo Murata.

[2]$f(m) = 0$, if $m \notin \mathbb{N}$. Therefore, the denominator γ in the definition of the linear form L amounts to some congruence condition; in the future, we restrict ourself to the special case $\gamma = 1$. The letter p is reserved for primes.

where the function G is, apart from some factor $\psi^*(\overrightarrow{q})$, a convolution of the functions $g_1, \ldots, g_k$, $G(n) = \sum_{q_1 \ldots q_k = n} g_1(q_1) \ldots g_k(q_k) \cdot \psi^*(q_1, \ldots, q_k)$.

1.2. Spaces of Arithmetical Functions

As explained in [19] and [21], for arithmetical functions in rather large classes of almost-periodic functions, the question of the *existence* of mean-values is easy to answer. Denote by c_r resp. e_α Ramanujan sums resp. exponential functions, defined by

$$c_r(n) = \sum_{\substack{d|r \\ d|n}} d\mu\left(\frac{r}{d}\right) = \sum_{\substack{a=1 \\ (a,r)=1}}^{r} \exp\left\{2\pi i \frac{a}{r} \cdot n\right\}, \text{ resp. } e_\alpha(n) = \exp\{2\pi i \alpha n\}. \tag{5}$$

Denote further by $\mathcal{B}$, $\mathcal{D}$, resp. $\mathcal{A}$ the vector spaces

$\mathcal{B}$	of linear combinations of Ramanujan sums	$c_r,\ r \in \mathbb{N}$,
$\mathcal{D}$	of linear combinations of exponential functions	$e_\alpha,\ \alpha \in \mathbb{Q}/\mathbb{Z}$,
$\mathcal{A}$	of linear combinations of exponential functions	$e_\alpha,\ \alpha \in \mathbb{R}/\mathbb{Z}$.

For $q \geq 1$, the Banach spaces

$$\mathcal{B}^q = \|\cdot\|_q - \text{closure of } \mathcal{B}, \mathcal{D}^q = \|\cdot\|_q - \text{closure of } \mathcal{D}, \mathcal{A}^q = \|\cdot\|_q - \text{closure of } \mathcal{A} \tag{6}$$

of $(q-)$ *almost-even*, (q) *limit-periodic, and* $(q-)$ *almost-periodic* arithmetical functions are the closures of $\mathcal{B}$, $\mathcal{D}$, resp. $\mathcal{A}$ with respect to the *(semi-) norms*

$$\|f\|_q = \sqrt[q]{\limsup_{x\to\infty} \frac{1}{x} \cdot \sum_{n\leq x} |f(n)|^q}. \tag{7}$$

Then (see, for example, [21], VI.1-VI.3) functions f in $\mathcal{A}^1$ do have *mean-values* $M(f)$, and *Fourier coefficients* and *Ramanujan coefficients* $a_r(f)$, defined by

$$\hat{f}(\alpha) \overset{\text{def}}{=} M(f \cdot e_{-\alpha}), \qquad a_r(f) \overset{\text{def}}{=} \frac{1}{\varphi(r)} \cdot M(f \cdot c_r). \tag{8}$$

For functions in $\mathcal{B}^2$ resp. $\mathcal{D}^2$ the Parseval equation

$$\|f\|^2 = \sum_{r=1}^{\infty} \varphi(r) \cdot |a_r(f)|^2, \quad \text{resp. } \|f\|^2 = \sum_{r=1}^{\infty} \sum_{\substack{1\leq a\leq r \\ (a,r)=1}} \left|\hat{f}\left(\frac{a}{r}\right)\right|^2 \tag{9}$$

holds. If the functions $f_1, \ldots, f_k$ are in $\mathcal{A}^k$, then the function F, defined by

$$F(n) = \prod_{\kappa=1}^{k} f_\kappa(\beta_\kappa n + \alpha_\kappa), \tag{10}$$

is in $\mathcal{A}^1$, and so it has a mean-value. Thus the *existence* of $M(F)$ is settled.

For multiplicative functions $f : \mathbb{N} \to \mathbb{C}$, with mean-value $M(f) \neq 0$, there are convenient necessary and sufficient conditions for membership in $\mathcal{A}^q$, $q > 1$ (Theorems of P.D.T.A. Elliott and H. Daboussi; see [21], VII, in particular Theorem 5.1, p. 257, and [5], [7], [6], [1], [2], and [3]. K.-H. Indelekofer [13] proved similar theorems for larger classes of multiplicative functions. A short report is given in section 2).

The functions considered by L. Lucht (see condition (3)) are rather special multiplicative functions in $\bigcap_{q \geq 1} \mathcal{A}^q$. Our aim is to give some formulae for mean-values $M(F)$ for functions F defined in (10), for $k = 2$, under suitable assumptions, more general than those used by L. Lucht.

1.3. Further Motivation

In recent papers, attention was given to the question of obtaining results on the *correlation* of multiplicative functions (see, for example, [8], [9], [14], [22], [23]), which have *mean value zero*.

The motivating problem is the (up to now unproven) conjecture

$$?? \quad \limsup_{x \to \infty} \frac{1}{x} \sum_{n \leq x} \lambda(n) \cdot \lambda(n+1) < 1 \quad ??$$

for the (completely multiplicative) Liouville function $\lambda(n) = (-1)^{\Omega(n)}$.

P. Elliott [8], as a special case of a more general theorem on

$$\limsup_{x \to \infty} \frac{1}{x} \cdot \sum_{n \leq x} g(n)^{d_0} \ldots g(n+k)^{d_k},$$

where g is completely multiplicative[3], was able to prove that

$$\limsup_{x \to \infty} \frac{1}{x} \cdot \sum_{n \leq x} \lambda(n) \cdot \lambda(n+1) \cdot \lambda(n+2) < \frac{20}{21}.$$

Khripunova generalized Halász' theorem to sums $\sum_{n \leq x} f(n) \cdot r(n-1)$, where f is multiplicative, and Timofeev gave sufficient (rather restrictive) conditions for the estimate $\left| \sum_{|a| < n \leq x} f(n) \cdot g(n+a) \right| \leq (1 - \delta) \cdot x$.

Stepanauskas ([22]) proved remainder term estimates for sums

$$\sum_{n \leq x} g_1(n+1) \cdot g_2(n) - \textit{main term}$$

for multiplicative functions g_i, satisfying further conditions, and also, more generally, for

$$\sum_{n \leq x} g_1(a_1 n + b_1) \ldots g_s(a_s n + b_s) - \textit{main term}.$$

[3]Further it is assumed that there is a prime q with the property $g(p)^q = 1$ for all primes, $\sum_p \frac{1}{p} (1 - \Re g(p))$ is divergent, and $d_0 + \ldots + d_k \not\equiv 0 \bmod q$.

Results on functions with mean value zero are definitely **not** contained in our line of approach, which is based on simple ideas from functional analysis - so our results do not contribute to these [difficult and deep] investigations.

2. THE RESULTS OF ELLIOTT-DABOUSSI

Define the set $\mathcal{E}_q$ as follows: $f \in \mathcal{E}_q$ if and only if f is multiplicative and the four series

$$S_1(f) = \sum_p \frac{1}{p} \cdot (f(p) - 1), \quad S_2'(f) = \sum_{\substack{p, \\ |f(p)| \leq \frac{5}{4}}} \frac{1}{p} \cdot |f(p) - 1|^2, \tag{11}$$

$$S''_{2,q}(f) = \sum_{\substack{p, \\ |f(p)| > \frac{5}{4}}} \frac{1}{p} \cdot |f(p)|^q, \quad S_{3,q}(f) = \sum_p \sum_{k \geq 2} \frac{1}{p^k} \cdot |f(p^k)|^q$$

are convergent. P.D.T.A. Elliott [5], [7] showed the following result.

Assume that $q > 1$, and that f is a multiplicative function with bounded semi-norm $\|f\|_q$. Then the mean value $M(f)$ of f exists and is non-zero if and only if

$$\left\{ (i) \quad f \in \mathcal{E}_q, \quad \text{and if} \quad (ii) \quad \sum_{k=0}^{\infty} \frac{1}{p^k} \cdot f(p^k) \neq 0 \text{ for every prime } p \right\}.$$

H. Daboussi [3] gave another proof for this result, and he extended it ([2]) to multiplicative functions f having at least one non-zero Fourier coefficient $\hat{f}(\alpha) = M\ (n \mapsto f(n) \cdot \exp\{2\pi i \cdot \alpha n\})$; the necessary and sufficient conditions for this to happen are the convergence of the series $S_1(\chi f)$, $S_2'(\chi f)$, $S''_{2,q}(f)$, and $S_{3,q}(f)$ for some Dirichlet character χ. In fact, the conditions of the Elliott-Daboussi theorem ensure that f is in the space $\mathcal{B}^q$. For details see [2], and [21], Chaper VI, VII.

3. GENERATING FUNCTIONS, LUCHT'S METHOD

If the mean-value of $f : \mathbb{N} \to \mathbb{C}$ *exists*, a standard procedure for its *calculation* is to study generating Dirichlet series $\sum_{n=1}^{\infty} f(n) \cdot n^{-s}$. Abels partial summation immediately gives[4]:

If $f : \mathbb{N} \to \mathbb{C}$ *has a mean-value $M(f)$, then*

$$M(f) = \lim_{\sigma \to 1+} \sum_{n=1}^{\infty} \frac{f(n)}{n^\sigma} \cdot \frac{1}{\zeta(\sigma)}. \tag{12}$$

The application of this result rests on the possibility to cope sufficiently well with the generating function $\sum_{n=1}^{\infty} f(n) \cdot n^{-s}$. For the study of functions F, defined by (10), even with multiplicative f_κ, this method is not very convenient. However, if $f \in \mathcal{A}^q$, $q > 1$ is multiplicative, in the way described, one obtains (see [21], VII.2) [complicated] product representations for the mean-values $M(f)$, $\lim_{x\to\infty} \frac{1}{x} \cdot \sum_{n \leq x} \chi(n) \cdot f(n)$, where χ is a Dirichlet

[4]See, for example, [21], VI. Theorem 7.1, p. 217.

character, for $\lim_{x\to\infty} \frac{1}{x} \cdot \sum_{\substack{n\le x\\ gcd(n,d)=1}} f(n)$, for $M_d(f) = \lim_{x\to\infty} \frac{1}{x} \cdot \sum_{\substack{n\le x\\ n\equiv 0 mod\, d}} f(n)$, and for the Ramanujan coefficients $a_r(f) = \frac{1}{\varphi(r)} \cdot \sum_{d|r} d\mu\left(\frac{r}{d}\right) \cdot M_d(f)$.

If we try to apply this method in a straightforward manner to our problem we don't get better results than Lucht, who used the following method.

3.1. Lucht's Method

In [16] L. Lucht represented the functions f_κ as a convolution $f_\kappa = 1 * g_\kappa$. Due to the assumption (3) the values $g_\kappa\ (p)$ are small in the mean. Then

$$\frac{1}{x} \cdot \sum_{n\le x} F(n) = \frac{1}{x} \cdot \sum_{n\le x} \prod_{\kappa=1}^{k} f_\kappa(L_\kappa(n))$$

$$= \sum_{q_1\le L_1(x)} \cdots \sum_{q_k\le L_k(x)} g_1(q_1)\ldots g_k(q_k) \cdot \left(\frac{1}{x} \sum_{\substack{n\le x\\ \beta_1 n+\alpha_1 \equiv 0 \ mod q_1\gamma_1\\ \cdots\\ \beta_k n+\alpha_k \equiv 0 \ mod q_k\gamma_k}} 1 \right).$$

A careful investigation of the solubility of the system of congruences involved allows to pass to the limit $x \to \infty$, and the existence of the limit and an evaluation of it is obtainable with some care.

4. PROPERTIES OF RAMANUJAN SUMS

Remark. The main result of this section is for use in another paper.

Ramanujan sums are multiplicative functions of the index, $c_r(n) \cdot c_s(n) = c_{rs}(n)$, if gcd $(r, s) = 1$. The mean-value of c_r is $M(c_r) = 0$, if $r \neq 1$, and $M(c_1) = M(1) = 1$. In order to generalize the *orthogonality relations*,

$$M(c_{r_1} \cdot c_{r_2}) = \begin{cases} 0 & \text{if } r_1 \neq r_2, \\ \varphi(r), & \text{if } r_1 = r_2, \end{cases}$$

some results on difference equations are useful. Assume that the quadratic equation $x^2 - Ax - B = 0$ has two different zeros $\eta_{1/2} = \frac{1}{2} A \pm \frac{1}{2}\sqrt{A^2 + 4B}$; then the set of solutions of $K(d) = A \cdot K(d-1) + B \cdot K(d-2)$ is a two-dimensional vector space, generated by the functions $d \mapsto \eta_1^d$, $d \mapsto \eta_2^d$. Therefore:

Proposition 4.1. *Assume that p is a prime, and ℓ a positive integer. Then the solution $K(d)$ of the difference equation*

$$K(d) = (p^\ell - 2p^{\ell-1}) \cdot K(d-1) + p^{\ell-1}\varphi(p^\ell) \cdot K(d-2), \quad d \geq 3,$$

with initial values $K(2) = \varphi(p^\ell), K(3) = \varphi(p^\ell) \cdot (p^\ell - 2p^{\ell-1})$, is given by

$$K(d) = \frac{1}{p^\ell} \cdot (\varphi(p^\ell))^d - \frac{\varphi(p^\ell)}{p^\ell} \cdot (-p^{\ell-1})^{d-1}, \quad d \in \mathbb{N}, d \geq 2.$$

Proof. The zeros of $x^2 = Ax + B$ with $A = p^\ell - 2p^{\ell-1}$, $B = p^{\ell-1} \cdot (p^\ell - p^{\ell-1})$ are $\eta_1 = p^\ell - p^{\ell-1}$ and $\eta_2 = -p^{\ell-1}$. The initial values easily give the constants $\gamma_1 = \frac{1}{p^\ell}$, $\gamma_2 = \frac{\varphi(p^\ell)}{p^\ell}$ in the relation $K(d) = \gamma_1 \cdot \eta_1^d + \gamma_2 \cdot \eta_2^d$. □

Proposition 4.2. *If p is a prime, and ℓ, $m \in \mathbb{N}$, then*

$$\begin{aligned} c_{p^m} \cdot c_{p^\ell} &= \varphi(p^\ell) \cdot c_{p^m}, \text{ if } m > \ell, \\ c_{p^\ell} \cdot c_{p^\ell} &= \varphi(p^\ell) \cdot (c_1 + c_p + \ldots + c_{p^{\ell-1}}) + (p^\ell - 2p^{\ell-1}) \cdot c_{p^\ell}. \end{aligned}$$

For a proof, by straightforward calculation, see [20].

Proposition 4.3. *Let $m_1 \geq m_2 \geq \ldots \geq m_k > 0$. Write*

$$\mathcal{C} = c_{p^{m_1}} \ldots c_{p^{m_k}} = (c_{p^{\ell_1}})^{d_1} \ldots (c_{p^{\ell_s}})^{d_s},$$

where $d_1, \ldots, d_s > 0$, and $\ell_1 > \ell_2 > \ldots > \ell_s > 0$. Then the mean-value $M(\mathcal{C})$ equals 0, if $d_1 = 1$, and $M(\mathcal{C})$ equals

$$\{\varphi(p^{\ell_s})\}^{d_s} \ldots \{\varphi(p^{\ell_2})\}^{d_2} \cdot \left[\frac{1}{p^{\ell_1}} \cdot \{\varphi(p^{\ell_1})\}^{d_1} - \frac{\varphi(p^{\ell_1})}{p^{\ell_1}} \cdot (-p^{\ell_1 - 1})^{d_1 - 1}\right],$$

if $d_1 \geq 2$.

Proof. The first formula of Proposition 4.2 gives

$$\mathcal{C} = \{\varphi(p^{\ell_s})\}^{d_s} \ldots \{\varphi(p^{\ell_2})\}^{d_2} \cdot (c_{p^{\ell_1}})^{d_1}.$$

By the second formula of Proposition 4.2 it is possible, in principle, to replace the last factor $(C_{p^\ell})^d$ by a linear combination of Ramanujan sums $c_1, c_p, \ldots, c_{p^\ell}$. For $d = 2$ this is Proposition 4.2; if $d \geq 3$, the reduction argument is as follows:

$$\begin{aligned} (c_{p^\ell})^d &= (c_{p^\ell})^{d-2} \cdot \{(p^\ell - 2p^{\ell-1}) \cdot c_{p^\ell} + \varphi(p^\ell) \cdot (c_1 + c_p + \ldots + c_{p^{\ell-1}})\} \\ &= (c_{p^\ell})^{d-1} \cdot (p^\ell - 2p^{\ell-1}) + (c_{p^\ell})^{d-2} \cdot (\varphi(p^\ell) \cdot (1 + \varphi(p) + \ldots + \varphi(p^{\ell-1}))) \\ &= (c_{p^\ell})^{d-1} \cdot (p^\ell - 2p^{\ell-1}) + (c_{p^\ell})^{d-2} \cdot \varphi(p^\ell) \cdot p^{\ell-1}. \end{aligned} \tag{13}$$

It is not necessary to know the full Ramanujan expansion of the function $(c_{p^\ell})^d = \sum_{\lambda=0}^{\ell} a_{p^\lambda} \cdot c_{p^\lambda}$, which is implicit in the last formula; we are only interested in the mean-value of this function, which is equal to the coefficient $a_1((c_{p^\ell})^d)$ in the Ramanujan expansion. Using Prop. 4.2. resp. (14), we find

$$\begin{aligned} d = 2 : (c_{p^\ell})^2 &= \varphi(p^\ell) \cdot c_1 + \ldots \\ d = 3 : (c_{p^\ell})^3 &= \varphi(p^\ell) \cdot (p^\ell - 2p^{\ell-1}) \cdot c_1 + \ldots \\ d \geq 3 : (c_{p^\ell})^d &= \left\{(p^\ell - 2p^{\ell-1}) a_1(c_{p^\ell}^{d-1}) + p^{\ell-1} \varphi(p^\ell) a_1(c_{p^\ell}^{d-2})\right\} \cdot c_1 + \ldots \end{aligned}$$

Thus, the coefficients $a_1(c_{p^\ell}^d)$, considered as functions of d, satisfy the difference equation of Proposition 4.1, and, for $d = 2, 3, \ldots$, we obtain

$$M((c_{p^\ell})^d) = a_1((c_{p^\ell})^d) = \frac{1}{p^\ell} \cdot \{\varphi(p^\ell)\}^d - \frac{\varphi(p^\ell)}{p^\ell} \cdot (-p^{\ell-1})^{d-1}. \qquad \square$$

The general case is dealt with in

Theorem 4.1. *Given integers $r_1, \ldots, r_k$ with prime factor decomposition*

$$r_\kappa = \prod_p p^{\rho_\kappa(p)}, \quad \kappa = 1, \ldots, k, \; accordingly \; c_{r_\kappa} = \prod_p c_{p^{\rho_\kappa(p)}},$$

put

$$\mathcal{C}_{(\vec{r})} \overset{def}{=} c_{r_1} \ldots c_{r_k} = \prod_p c_{p^{\rho_1(p)}} \ldots c_{p^{\rho_k(p)}}.$$

Denote the maximum of the exponents $\rho_\kappa(p)$ by $\rho(p)$. Then

$$M(\mathcal{C}_{(\vec{r})}) \neq 0,$$

if *for every prime p (for which $\rho(p) > 0$) there exist <u>at least two indices</u> $i \neq j$ such that*

$$\rho(p) = \rho_i(p), \quad and \; \rho(p) = \rho_j(p). \tag{14}$$

Otherwise the mean-value is zero.

If (14) is fulfilled for every prime p, then order the set $\{\rho_1(p), \ldots, \rho_k(p)\}$ according to the size of the elements as

$$\underbrace{\ell_1(p), \ldots, \ell_1(p)}_{d_1(p) \text{ times}}, \ldots, \underbrace{\ell_s(p), \ldots, \ell_s(p)}_{d_s(p) \text{ times}},$$

where $\ell_1(p)[= \rho(p)] > \ell_2(p) > \ldots > \ell_s(p)$, $d_1(p) \geq 2$, and $d_1(p) + \ldots + d_s(p) = k$. Then

$$M(\mathcal{C}_{(\vec{r})}) = \prod_p \varphi(p^{\ell_s})^{d_s} \ldots \varphi(p^{\ell_2})^{d_2} \left[\frac{1}{p^{\ell_1}}\{\varphi(p^{\ell_1})\}^{d_1} - \frac{\varphi(p^{\ell_1})}{p^{\ell_1}}(-p^{\ell_1-1})^{d_1-1}\right],$$

where $\ell_\sigma = \ell_\sigma(p), d_\sigma = d_\sigma(p)$, for short.

Proof. Consider the coefficient a_1 of c_1 in the Ramanujan expansion of $\mathcal{C}_{(\vec{r})}$. Using the multiplicativity of c_r as a function of the index r, this coefficient can be calculated from the results of Proposition 4.3. Put $\mathrm{C}_p := c_{p^{\rho_1}} \ldots c_{p^{\rho_k}} = a_{0,p} + a_{1,p} \cdot c_p + a_{2,p} \cdot c_{p^2} + \ldots$, then $\mathcal{C}_{\vec{r}} = \prod_p a_{0,p} + \sum_{q \neq 1} b_q \cdot c_q$. Therefore, using Proposition 4.3, $M(\mathcal{C}_{\vec{r}}) = \prod_p a_{0,p} = \prod_p M(\mathcal{C}_p)$.
$\square$

Corollary 4.1. *Given positive integers $r_1, \ldots, r_k$, denote by $L_{(\vec{r})}$ the number of integer solutions*

$$A_1, \ldots, A_k, \quad where \; 1 \leq A_\kappa \leq r_\kappa, \quad and \; \gcd(A_\kappa, r_\kappa) = 1 \quad for \; \kappa = 1, \ldots, k,$$

of the congruence

$$\frac{A_1}{r_1} + \ldots + \frac{A_k}{r_k} \equiv 0 \bmod 1.$$

Then $L_{(\vec{r})} = M(C_{(\vec{r})})$.

In particular, $L_{(\vec{r})} \neq 0$ *only if for every prime p the maximal prime power* $p^{\rho(p)}$ *diving any of the* r_κ, *is a divisor of at least two of the* r_κ.

Proof. Using (5), we obtain

$$\frac{1}{x}\sum_{n\le x} c_{r_1}(n)\ldots c_{r_k}(n) = \sum_{\substack{1\le A_1 r_1 \\ \gcd(A_1,r_1)=1}} \cdots \sum_{\substack{1\le A_k r_k \\ \gcd(A_k,r_k)=1}} \frac{1}{x}\sum_{n\le x} e^{2\pi i n\left(\frac{A_1}{r_1}+\ldots+\frac{A_k}{r_k}\right)}.$$

The left hand side tends to $M(C_{(\vec{r})})$, as $x \to \infty$, and the right hand side tends to the number $L_{(\vec{r})}$ described in the corollary. □

5. USING RAMANUJAN COEFFICIENTS

5.1. Special Case: Two Additively Shifted Functions

Given f_1, f_2 in some space $\mathcal{B}^q$,[5] we are interested in the mean-value

$$\mathcal{M} = \lim_{x\to\infty} \frac{1}{x} \cdot \sum_{n\le x} f_1(n+\alpha_1)\cdot f_2(n+\alpha_2), \tag{16}$$

where α_1, α_2 are fixed integers[6]. For shifted functions, we use the abbreviations

$$F_\kappa : n \mapsto f_\kappa(n+\alpha_\kappa), \quad C_r^\kappa(n) = c_r(n+\alpha_\kappa), \quad G_\kappa(n) = g_\kappa(n+\alpha_\kappa).$$

By definition of $\mathcal{B}^q$ the functions f_κ are $\|\cdot\|_q$-approximable by finite linear combinations of Ramanujan sums. Assume that $q \ge 2$; then, by Parseval's equation, the functions f_κ are $\|\cdot\|_2$-approximable by partial sums of the Ramanujan expansion: given $\varepsilon > 0$, there exist integers R_κ with the property

$$\|f_\kappa - g_\kappa\|_2 < \varepsilon, \quad \text{where } g_\kappa(n) = \sum_{r=1}^{R_\kappa} a_r(f_\kappa)\cdot c_r(n), \tag{18}$$

with the Ramanujan coefficients (see (8)) $a_r(f) = \frac{1}{\varphi(r)} \cdot M(f\cdot c_r)$. Using (18) and $\|F_1 \cdot F_2 - G_1 \cdot G_2\|_1 \le \|F_1 - G_1\|_2 \cdot \|F_2\|_2 + \|F_2 - G_2\|_2 \cdot \|G_1\|_2$, we obtain

$$\|F_1\cdot F_2 - G_1\cdot G_2\|_1 \le (\|f_2\|_2 + \|f_1\|_2 + \varepsilon)\cdot\varepsilon.$$

[5] Here it is important to restrict ourselves to functions in $\mathcal{B}^q$-spaces, because we are going to use Ramanujan-expansions. - Recall that $f(m) = 0$, if m is an integer ≤ 0.

[6] In principle it would be possible to use the Parseval equation $M(f\cdot g) = \sum_{r=1}^{\infty} \varphi(r)\cdot a_r(f)\cdot a_r(g)$. The calculations of special mean values of Ramanujan sums must be done in the same manner.

By partial summation this estimate implies

$$\limsup_{\sigma\to 1+} \frac{1}{\zeta(\sigma)} \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \cdot |F_1(n)F_2(n) - G_1(n)G_2(n)| \leq (\|f_2\|_2 + \|f_1\|_2 + \varepsilon) \cdot \varepsilon =: \rho\varepsilon.$$

Therefore, the mean-value $\mathcal{M}$ is $\rho\varepsilon$-near the mean-value

$$\lim_{\sigma\to 1+} \frac{1}{\zeta(\sigma)} \cdot \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \cdot G_1(n)G_2(n), \tag{19}$$

and so knowledge of the asymptotic behaviour of the Dirichlet series

$$\mathcal{G}(s) = \sum_{r_1\leq R_1} \sum_{r_2\leq R_2} a_{r_1}(f_1)a_{r_2}(f_2) \cdot \sum_{n=1}^{\infty} c_{r_1}(n+\alpha_1)c_{r_2}(n+\alpha_2) \cdot \frac{1}{n^\sigma}$$

is important, as $\sigma \to 1+$. Using (5) we easily see that

$$\sum_{n=1}^{\infty} c_{r_1}(n+\alpha_1) \cdot c_{r_2}(n+\alpha_2) \cdot \frac{1}{n^\sigma} = \tag{21}$$

$$\sum_{\substack{b_1=1\\(b_1,r_1)=1}}^{r_1} \sum_{\substack{b_2=1\\(b_2,r_2)=1}}^{r_2} \exp\left\{2\pi i \left(\frac{b_1\alpha_1}{r_1} + \frac{b_2\alpha_2}{r_2}\right)\right\} \cdot \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \exp\left\{2\pi i \left(\frac{b_1r_2+b_2r_1}{r_1r_2} n\right)\right\}.$$

The mean-value of the function $n \mapsto \exp\left\{2\pi i \left(\frac{b_1r_2+b_2r_1}{r_1r_2} \cdot n\right)\right\}$ is zero unless $\frac{b_1r_2+b_2r_1}{r_1r_2}$ is an integer, and this occurs if and only if $r_1 = r_2 = r$ and $b_1 + b_2 \equiv 0 \bmod r$. Therefore, as $\sigma \to 1+$, the right hand side in (21) is

$$\begin{aligned} &= (1+o(1)) \cdot \zeta(\sigma) \cdot \sum_{\substack{1\leq b\leq r\\(b,r)=1}} \exp\left\{2\pi i \frac{1}{r}(\alpha_1 b + \alpha_2(r-b))\right\} \\ &= (1+o(1)) \cdot \zeta(\sigma) \cdot c_r(|\alpha_1 - \alpha_2|), \end{aligned}$$

if $r_1 = r_2 = r$, and it is $o(\zeta(\sigma))$, if $r_1 \neq r_2$. Parseval's equation and Cauchy's inequality show that $\left|\sum_{r>R} a_r(f_1) \cdot a_r(f_2) \cdot \varphi(r)\right|^2$ is small; therefore

$$\mathcal{M} = \sum_{r=1}^{\infty} a_r(f_1) \cdot a_r(f_2) \cdot c_r(|\alpha_1 - \alpha_2|), \text{ if } \alpha_1 \neq \alpha_2, \tag{22a}$$

$$\mathcal{M} = \sum_{r=1}^{\infty} a_r(f_1) \cdot a_r(f_2) \cdot \varphi(r), \text{ if } \alpha_1 = \alpha_2. \tag{22b}$$

(22b) is nothing else than Parseval's equation. Both series are absolutely convergent. Collating, the following result is obtained.

Theorem 5.1. *Assume that the integers α_1, α_2 are distinct and that f_1, f_2 are in $\mathcal{B}^q$, where $q \geq 2$. Then for the mean-value $\mathcal{M} = \lim_{x\to\infty} \frac{1}{x} \cdot \sum_{n\leq x} f_1\ (n+\alpha_1) \cdot f_2\ (n+\alpha_2)$ formula (22a) holds.*

If, *in addition*, f is supposed to be multiplicative and to have non-zero mean-value $M(f) \neq 0$, then the map

$$r \mapsto a_r^*(f) := \frac{1}{M(f)} \cdot a_r(f) \tag{23}$$

is multiplicative (see [21], VIII, Theorem 4.4), and therefore we obtain the

Corollary 5.1.1. *Let α_1, α_2 be integers, and put $\alpha = |\alpha_1 - \alpha_2|$. If f_1,f_2 are multiplicative functions in $\mathcal{B}^2$ with non-zero mean-values, then*

$$\lim_{x\to\infty} \frac{1}{x} \cdot \sum_{n\leq x} f_1(n+\alpha_1) \cdot f_2(n+\alpha_2) = \tag{24}$$

$$= M(f_1) \cdot M(f_2) \cdot \prod_p (1 + a_p^*(f_1)a_p^*(f_2)c_p(\alpha) + a_{p^2}^*(f_1)a_{p^2}^*(f_2)c_{p^2}(\alpha) + \ldots)$$

$$= M(f_1) \cdot M(f_2) \cdot \prod_{p\nmid\alpha} (1 - a_p^*(f_1)a_p^*(f_2)) \times$$

$$\times \prod_{p|\alpha} (1 + a_p^*(f_1)a_p^*(f_2)c_p(\alpha) + a_{p^2}^*(f_1)a_{p^2}^*(f_2)c_{p^2}(\alpha) + \ldots).$$

a_r^* is defined by (23), and $c_r(0)$ means $\varphi(r)$.

Remark 1. Using

$$c_{p^k}(\alpha) = \begin{cases} \varphi(p^k), & \text{if } p^k|\alpha, \\ -p^{k-1}, & \text{if } p^{k-1}|\alpha, \text{but } p^k \nmid\alpha, \\ 0, & \text{if } p^{k-1} \nmid\alpha, \end{cases}$$

it is possible, to give formula (24) a slightly different shape.

Remark 2. In principle, the same method works for the evaluation of

$$\lim_{x\to\infty} \frac{1}{x} \cdot \sum_{n\leq x} f_1(n+\alpha_1)\ldots f_k(n+\alpha_k).$$

Example. If $f_1 = f_2 = \mu^2, \alpha_1 = 0, \alpha_2 = 1$, then $a_p^* = -\frac{1}{p^2-1}$, $a_{p^2}^* = -\frac{p}{(p-1)(p^2-1)}$, $a_{p^\ell}^* = 0$, if $\ell \geq 3$. Therefore

$$\lim_{x\to\infty} \frac{1}{x} \cdot \sum_{n\leq x} \mu^2(n) \cdot \mu^2(n+1) = \frac{6}{\pi^2} \cdot \prod_p \left(1 - \frac{1}{(p^2-1)^2}\right).$$

5.2. Generalization to More General Linear Forms

In order to obtain similar results for linear forms $n \mapsto \beta \cdot n + \alpha$ with an additive *and* a multiplicative shift, we prove

Lemma 5.2.1. *Denote by r, β positive integers, and put $B = gcd\,(r, \beta)$. Then*

$$c_r(\beta \cdot n) = \frac{\varphi(r)}{\varphi\left(\frac{r}{B}\right)} \cdot c_{r\,/\,B}(n). \tag{25}$$

Proof. The function $c_{r,\beta} : n \mapsto c_r(\beta n) = \sum_{d|gcd(r,\beta n)} d \cdot \mu\left(\frac{r}{d}\right)$ is r-even, in fact $\frac{r}{B}$ - even. The Ramanujan-coefficients of its [finite] Ramanujan-expansion are calculated from

$$\begin{aligned} a_k(c_{r,\beta}) &= \frac{1}{\varphi(k)} \cdot M(c_{r,\beta} \cdot c_k) = \frac{1}{\varphi(k)} \cdot M\left(\sum_{\substack{\rho=1 \\ (\rho,k)=1}}^{r} e^{e\pi i \frac{\rho}{r} \cdot \beta n} \cdot \sum_{\substack{\kappa=1 \\ (\kappa,k)=1}}^{k} e^{2\pi i \frac{\kappa}{k} \cdot n} \right) \\ &= \frac{1}{\varphi(k)} \cdot \sum_{\substack{1\leq\kappa\leq k \\ (\kappa,k)=1}} \sum_{\substack{1\leq\rho\leq r \\ (\rho,r)=1}} \lim_{x\to\infty} \frac{1}{x} \cdot \sum_{n\leq x} \exp\left\{2\pi i n \cdot \frac{\kappa r + \beta\rho k}{kr}\right\}. \end{aligned} \tag{26}$$

The mean-value of the exponential function in (26), second line, is zero unless $\kappa r + \beta\rho k \equiv 0$ mod kr. Paying attention to gcd $(\rho, r) = 1$ and gcd $(\kappa, k) = 1$, this congruence implies $k|r$ and $\left(\frac{r}{B}\right)|k$, therefore

$$k = \frac{r}{B} \cdot \ell, \quad \text{where } \ell | B.$$

Given ρ, κ is uniquely determined from $\kappa + \rho\frac{\beta}{B}\ell \equiv 0 \bmod \left(\frac{r}{B}\right)\ell$, and

$$1 = \ \gcd(\kappa, k) = \ \gcd\left(\rho \cdot \frac{\beta}{B} \cdot \ell, \frac{r}{B} \cdot \ell\right)$$

implies $\ell = 1$. Thus, $a_k(c_{r,\beta}) = 0$ *except* if $k = \frac{r}{B}$, in which case

$$a_{\frac{r}{B}}(c_{r,\beta}) = \frac{1}{\varphi\left(\frac{r}{B}\right)} \cdot \sum_{\substack{1\leq\rho\leq r \\ (\rho,r)=1}} 1 = \frac{\varphi(r)}{\varphi\left(\frac{r}{B}\right)}.$$

□

Lemma 5.2.2. *Let positive integers t and β be given. Then the set of solutions r of the equation $t = \frac{r}{\gcd(r,\beta)}$ is given by*

$$\left\{ r, r = t \cdot \ell, \ell | \beta, \ and \gcd\left(t, \frac{\beta}{\ell}\right) = 1 \right\}.$$

Thus, if p divides t and β, say $p^b \| \beta$, then ℓ is restricted to those divisors of β, which are divisible by p^b.

Proof. If t is given, then $r = t\cdot \gcd(r, \beta)$ is of the shape $\ell \cdot t$, where $\ell|\beta$. Then

$$\frac{r}{\gcd(r,\beta)} = \frac{t\cdot\ell}{\gcd(t\cdot\ell,\beta)} = \frac{t}{\gcd(t,\frac{\beta}{\ell})} = t$$

if and only if $\gcd(t, \frac{\beta}{\ell}) = 1$. □

Using Lemma 5.2.1 and Lemma 5.2.2, it is easy to obtain the Ramanujan-expansion of the function ${}_{\beta}\cdot f : n \mapsto f(\beta\cdot n)$, where $f \in \mathcal{B}^q$, $q \geq 2$, in the sense that partial sums of this expansion are near ${}_{\beta}\cdot f$.

Lemma 5.2.3. *Assume that* β *is a positive integer. If* $f \in \mathcal{B}^q$, $q \geq 2$, *and* $\|f - g\|_2 < \varepsilon$, *where* $g(n) = \sum_{r=1}^{R} a_r(f)\cdot c_r(n)$, *then*

$$\|{}_{\beta}\cdot f - G\|_2 < \varepsilon\cdot\beta, \quad \textit{where } G(n) = \sum_{r\leq R} A_r \cdot c_r(n),$$

and where the coefficients are

$$A_r = \sum_{\substack{\ell|\beta,\ \ell\leq R/r,\\ \gcd\left(R,\frac{\beta}{\ell}\right)=1}} a_{r\ell}\cdot\frac{\varphi(r\ell)}{\varphi(r)}. \tag{27}$$

By Bessel's inequality, the series $\sum_{r=1}^{\infty} |A_r|^2\cdot\varphi(r)$ *is absolutely convergent.*

Proof. Using (25) first, then Lemma 5.2.2, the function ${}_{\beta}\cdot f$ is well approximable[7] by the function

$$\begin{aligned}
G(n) &= \sum_{r\leq R} a_r(f)c_r(\beta n) = \sum_{r\leq R} a_r(f)\cdot\frac{\varphi(r)}{\varphi(r/\gcd(\beta,r))}\cdot c_{r/\gcd(\beta,r)}(n)\\
&= \sum_{t\leq R} c_t(n) \sum_{\substack{r\leq R\\ \frac{r}{\gcd(\beta,r)}=t}} a_r(f)\cdot\frac{\varphi(r)}{\varphi(r/\gcd(\beta,r))}\\
&= \sum_{r\leq R} c_t(n) \sum_{\substack{\ell|\beta,\ \ell\leq R/t,\\ \gcd\left(t,\frac{\beta}{\ell}\right)=1}} a_{t\ell}\cdot\frac{\varphi(t\ell)}{\varphi(t)}.
\end{aligned}$$
□

Using the Lemma and Theorem 5.1, we easily obtain

Theorem 5.2. *Assume that* f_1, f_2 *are in* $\mathcal{B}^q$, *where* $q \geq 2$, *and that* $\beta_1 > 0$, $\beta_2 > 0$, α_1, α_2 *are distinct integers. Then the mean-value*

$$\mathcal{M}_{\vec{\beta},\vec{\alpha}} = \lim_{x\to\infty}\frac{1}{x}\cdot\sum_{n\leq x} f_1(\beta_1 n+\alpha_1)\cdot f_2(\beta_2 n+\alpha_2) \tag{28}$$

[7] This means $\|{}_{\beta}\cdot f - G\|_2^2 \leq \beta\cdot\|f-g\|_2^2$.

equals

$$\mathcal{M}_{\vec{\beta},\vec{\alpha}} = \sum_{r=1}^{\infty} c_r(|\alpha_1 - \alpha_2|) \sum_{\substack{\ell_1 | \beta_1 \\ \gcd\left(r, \frac{\beta_1}{\ell_1}\right)=1}} a_{r\ell_1}(f_1) \cdot \frac{\varphi(r\ell_1)}{\varphi(r)} \sum_{\substack{\ell_2 | \beta_2 \\ \gcd\left(r, \frac{\beta_2}{\ell_2}\right)=1}} a_{r\ell_2}(f_2) \cdot \frac{\varphi(r\ell_2)}{\varphi(r)}.$$

Remark. In the case $\alpha_1 = \alpha_2$, replace $c_r(|\alpha_1 - \alpha_2|)$ by $\varphi(r)$.

In the case of *multiplicative* functions one would like to have a product representation for the mean-value $\mathcal{M}_{\vec{\beta},\vec{\alpha}}$. The following result is needed.

Proposition 5.2.1. *Assume that g is a multiplicative function, and that*

$$1 + g(p) + \ldots + g(p^b) \neq 0$$

for every prime power p^b, dividing β exactly ($p^b \| \beta$). Then the map $r \mapsto G_\beta(r)$, where

$$G_\beta(r) = \prod_{p|\beta} (1 + g(p) + \ldots + g(p^b))^{-1}. \sum_{\substack{\ell | \beta \\ \gcd\left(r, \frac{\beta}{\ell}\right)=1}} g(r\ell),$$

is multiplicative.

Proof. Write (with exponents ρ, λ, b depending on p),

$$r = \prod_p p^\rho, \qquad \beta = \prod_p p^b, \qquad \ell = \prod_p p^\lambda.$$

Then the condition $\gcd\left(r, \frac{\beta}{\ell}\right) = 1$ implies $\lambda = b$, if $\rho > 0$; if $\rho = 0$, all values $\lambda = 0, 1, \ldots, b$ are possible. Thus

$$\begin{aligned}
\sum_{\substack{\ell | \beta \\ \gcd(r, \frac{\beta}{\ell})=1}} g(r\ell) &= \prod_{p|r} g(p^{\rho+b}) \sum_{\substack{\ell | \beta \\ \gcd(r, \beta/\ell)=1}} \prod_{\substack{p|\ell \\ p\nmid r}} g(p^\lambda) \\
&= \prod_{p|r} g(p^{\rho+b}) \sum_{\lambda=0}^{b} \prod_{\substack{p|\ell \\ p\nmid r}} g(p^\lambda) \\
&= \prod_{p|r} g(p^{\rho+b}) \cdot \prod_{\substack{p|\beta \\ \nmid r}} (1 + g(p) + \ldots + g(p^b)) \\
&= \prod_{p|\beta} (1 + g(p) + \ldots + g(p^b)) \prod_{\substack{p|r \\ p|\beta}} \left(\frac{g(p^{\rho+b})}{1 + g(p) + \ldots + g(p^b)} \right) \prod_{\substack{p|r \\ p\nmid\beta}} g(p^\rho).
\end{aligned}$$

This formula shows the multiplicativity of $G_\beta(r)$. □

Using this result and the multiplicativity of the map $r \mapsto a_r^*(f) = \frac{1}{M(f)}\, a_r(f)$, Theorem 5.2 implies the following

Theorem 5.3. *Assume that f_1, f_2 are multiplicative functions in $\mathcal{B}^q$, $q \geq 2$, with non-zero mean-values $M(f_\kappa)$. Write for short*

$$g_\kappa(t) = a_t^*(f_\kappa) \cdot \varphi(t).$$

Given integers $\beta_1 = \prod_p p^{b_1(p)} > 0$, $\beta_2 = \prod_p p^{b_2(p)} > 0$, α_1, α_2, assume that

$$\prod_{p|\beta_\kappa} \left(1 + g_\kappa(p) + \ldots + g_\kappa(p^{b_\kappa(p)})\right) \neq 0, \text{ for } \kappa = 1, 2.$$

Then

$$\mathcal{M}_{\vec{\beta}, \vec{\alpha}} = M(f_1) \cdot M(f_2) \cdot \prod_{p|\beta_1} (1 + g_1(p) + \ldots + g_1(p^{b_1})) \tag{29}$$

$$\cdot \prod_{p|\beta_2} (1 + g_2(p) + \ldots + g_2(p^{b_2})) \cdot \sum_{r=1}^{\infty} \frac{c_r(|\alpha_1 - \alpha_2|) \cdot G_1(r) \cdot G_2(r)}{\varphi^2(r)},$$

where

$$G_\kappa(r) = \sum_{\substack{\ell | \beta_\kappa \\ \gcd\left(r, \frac{\beta_\kappa}{\ell}\right) = 1}} a_{r\ell}^*(f_\kappa) \cdot \varphi(r\ell) \cdot \prod_{p|\beta_\kappa} \left(1 + g_\kappa(p) + \ldots g_\kappa(p^{b_\kappa(p)})\right)^{-1}$$

is multiplicative, and therefore the function following the $\sum$-sign in (29) is a multiplicative function, and thus $\mathcal{M}_{\vec{\beta}, \vec{\alpha}}$ may be expressed as an infinite product over the primes.

H. DELANGE (*On products of multiplicative functions of absolute value at most 1 which are composed with linear functions*, Analytic Number Theory Vol. I (Allerton park Ill. 1995), 245-263, Progr. Math. 138, Birkhäuser 1996) proved similar results for multiplicative functions f_κ, $\kappa = 1, \ldots, k$, satisfying $|f_\kappa| \leq 1$ and $\left\{ \sum_p 1\ overp \cdot \left(\Re(f_1(p) + \ldots + f_\kappa(p))\right) \text{ is convergent}\right\}$.

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