A CHARACTERIZATION OF CURVES OF MINIMAL ORDER AS REGARDS SINGULAR POINTS AND THEIR MULTIPLICITIES

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1. INTRODUCTION

In [6] the following theorem was proved: "Let $\mathcal{K}$ be a set of order-characteristics in the plane with fundamental number $k$. Then a normal arc or curve of $\mathcal{K}$-order $k+1$ contains at most $k+1$ $\mathcal{K}$-singular points".

Moreover, $\mathcal{K}$-multiplicities were assigned to $\mathcal{K}$-singular points and a stronger result was obtained:

"The sum of the $\mathcal{K}$-multiplicities of the $\mathcal{K}$-singular points is at most $k+1$".

Here it is shown:

**Theorem.** Let $\mathcal{K}$ be a set of order-characteristics in the plane with fundamental number $k$. Then the sum of the $\mathcal{K}$-multiplicities of the $\mathcal{K}$-singular points of a curve $C_{k+1}$ of $\mathcal{K}$-order $k+1$ is at least $k+1$.

2. ORDER CHARACTERISTICS

2.1. Let $\mathcal{K}$ be a family of order characteristic arcs or curves in $G = \overline{G}$, a closed disk in the euclidean plane with fundamental number $k$ satisfying the following axioms ([1]; 1.1 and 2.4).

(I) If $K \in \mathcal{K}$ is an arc then $K$ has exactly its two end-points $e', e''$ in common with the boundary $G_b$ of $G$.

If $K \in \mathcal{K}$ is a curve, then $K$ has at most one point in common with $G_b$. Hence for each $K \in \mathcal{K}$, $G - G \cap K$ ($G$ is the interior of $G$) is the union of two disjoint open connected sets $K(\alpha, G)$ in which $\alpha = +$ or $\alpha = -$; these two global sides of $K$ in $G$ are denoted by $K(\pm, G)$.

(II) There exists a natural number $k \geq 1$, the so-called fundamental number with the following properties:

1. Let $x_\lambda, \lambda = 1, 2, \ldots, k$ be $k$ distinct points of $G$. Then there is a unique $K \in \mathcal{K}$ with $x_\lambda \in K$.

2. Let $x'_\lambda$ be close to $x_\lambda, \lambda = 1, 2, \ldots, k$. Then there exists $K' \in \mathcal{K}$, $K' = K(x'_1, x'_2, \ldots, x'_k)$ and $K(x'_1, x'_2, \ldots, x'_k)$ varies continuously with the $x'_\lambda$. Note: The class of all compact sets in $G$ is a metric space where $d(A_1, A_2) = \inf(\epsilon > 0; A_1 \subseteq U_2, A_2 \subseteq U_1)$ where $U_i = U(A_i, \epsilon)$ is the $\epsilon - G$-neighbourhood of $A_i$ (i.e., the union of $\epsilon - G$-neighbourhoods of all points of $A_i$).

We use this metric for $\mathcal{K}$.

(III) Let $K_n \in \mathcal{K}, n = 1, 2, \ldots, \lambda = \lim x_{n\lambda}$, $u = 1, 2, \ldots, k, x_u$ distinct. Then there exists $K \in \mathcal{K}$ with $x_u \in K, u = 1, 2, \ldots, k$ (by Axiom II 2), $K = \lim K_n$.

(IV) For any $K \in \mathcal{K}$ and any point $a \in K$, let $y_1, y_2, \ldots, y_i$ be $i$ arbitrary, distinct points of $G$, $1 \leq i \leq k - 1, y_i \neq a$. Further, let $x_1, x_2, \ldots, x_{k-i}$ be distinct points of $K$ converging to $a$ on
$K$. Then $\lim K(y_1, y_2, \ldots, y_i, x_1, x_2, \ldots, x_{k-i}) = K(y_1, y_2, \ldots, y_i; a^{k-i})$ exists uniquely in $\mathcal{K}$. This condition saying that the order characteristics themselves are strongly differentiable arcs or curves was previously denoted $EP_k$ by Haupt and Künneth ([1], 4.1.4).

Remark. These Axioms I - IV are more restrictive than those in 1.1 and 2.4 of [1]. Also one can see 1.1 of [5] for the conformal definition of strong differentiability.

2.2. Let $A$ be an arc in $G$ (we use the same letter for a point on the parameter interval and its image).

Definition. An order characteristic $K$ has $j$-point contact (is $j$-osculating) with an arc $A$ at $a \in A$ if for any two-sided neighbourhood (subarc) $N$ of $a$ there exists a $K$ close to $K$ that intersects $N$ at $j$ distinct points.

In particular one can consider the subsystem $\mathcal{K}(a^j)$ of $\mathcal{K}$ having $j$-point contact with each other studied in ([6]; 2.3). It was shown that:

1) $j$-point contact is a "transitive" relation on $\mathcal{K}$.
2) The subsystem $\mathcal{K}(a^j)$ also satisfies Axioms (I) - (IV) with fundamental number $k - j$.

3. MULTIPLEXITIES OF $\mathcal{K}$-SINGULAR POINTS

3.1. Let $a$ be an end-point of a normal arc $A_{k+1}$ of $\mathcal{K}$-order $k + 1$. Assume that $a < r$ for all $r \in A_{k+1}$. It is also assumed that $\mathcal{K}$ is a family of order characteristics with fundamental number $k$ satisfying Ax. I - IV.

3.2. Next let $i = k, k - 1, \ldots, 2, 1$. There are subfamilies $\mathcal{K}(a^{k-i})$ of $\mathcal{K}$ for each $i$ with fundamental number $i$ having $k - i$ point contact at $a$. Moreover, at each interior point $t \in A_{k+1}$ there are unique one-sided osculating characteristics $K_t^- = K(a^{k-i}, t^i)$ and $K_t^+ = K(a^{k-i}, t^i)$ of $A_{k+1}$ ([1]; 4.2.6.3 and 4.2.6.4).

3.3. An interior point $t \in A_{k+1}$ is said to be $\mathcal{K}(a^{k-i})$-singular if for any two-sided neighbourhood $N = L \cup \{ t \} \cup M$ ($L$ below $t$ on the parameter interval, $M$ above) of $t$ on $A_{k+1}$ there is a member of $\mathcal{K}(a^{k-i})$ that intersects $N$ at $i + 1$ distinct points. Note that this member of $\mathcal{K}(a^{k-i})$ cannot then meet $A_{k+1}$ again; otherwise the order of $A_{k+1}$ is $> k + 1$.

One can now classify all the $\mathcal{K}(a^{k-i})$-singular points $t$ of $A_{k+1}$ as follows by specifying the kinds of pairs of one-sided osculating characteristic curves of $A_{k+1}$ at $t$:

(a) $(i, 1)(1, i)$ point
(b) $(i, 1)(0, i)$ point
(c) $(i, 0)(i, 1)$ point
(d) $(i, 0)(0, i)$ point

where $t$ is assigned the symbol $(i, r)$ $(s, i)$ if for any neighbourhood $N = L \cup \{ t \} \cup M$ of $t$ on $A_{k+1}$ there are members $K_{Ni}^-, K_{Ni}^+$ of $\mathcal{K}(a^{k-i})$ one, $K_{Ni}^-$, that intersects $L$ at $i$ points, $M$ at $r$ points and one, $K_{Ni}^+$, that intersects $M$ at $i$ points and $L$ at $s$ points; $r, s = 0, 1$.

Remarks. 1) $\lim K_{Ni}^- = K_t^-$, $\lim K_{Ni}^+ = K_t^+$ ([1]; 4.2.6.3 and 4.2.6.4).

2) In words
(a) $t$ is a $\mathcal{K}(a^{k-i})$-singular point with respect to both $K_t^- (a^{k-i}, t^i)$ and $K_t^+ (a^{k-i}, t^i)$.
(b) \( t \) is \( K \) \((a^{k-i})\)-singular with respect to \( K_i^- (a^{k-i}, t^i) \) only,
(c) \( t \) is \( K \) \((a^{k-i})\)-singular with respect to \( K_i^+ (a^{k-i}, t^i) \) only,
(d) \( t \) is \( K \) \((a^{k-i})\)-singular with respect to neither \( K_i^- (a^{k-i}, t^i) \) nor \( K_i^+ (a^{k-i}, t^i) \).

3.4. Again let \( z \) be an interior \( K \)-singular point and let \( i = k, k - 1, \ldots, 2 \). At each stage each of the types (a), (b), (c), (d) may be subclassified as

\( \alpha^i \): \( z \) is of type \( \alpha^i \) if \( z \) is \( K, K (a), \ldots, K (a^{k-i}) \)-singular but not \( K (a^{k-i+1}) \)-singular,
\( \beta^i_1 \): \( z \) is \( \beta^i_1 \) if \( z \) is \( K, K (a), \ldots, K (a^{k-i}) \), \( K (a^{k-i+1}) \)-singular and both \( K_i^-, K_i^+ \) do not meet the arc \((a, z)\) again.
\( \beta^i_2 \): \( z \) is \( \beta^i_2 \) if \( z \) is \( K, K (a), \ldots, K (a^{k-i}) \), \( K (a^{k-i+1}) \)-singular and both \( K_i^-, K_i^+ \) meet \((a, z)\) again.
\( \gamma^i_1 \): \( z \) is \( \gamma^i_1 \) if \( z \) is \( K, K (a), \ldots, K (a^{k-i}) \), \( K (a^{k-i+1}) \)-singular, \( K_i^- \), meets \((a, z)\) again but \( K_i^+ \) does not.
\( \gamma^i_2 \): \( z \) is \( \gamma^i_2 \) if \( z \) is \( K, K (a), \ldots, K (a^{k-i}) \), \( K (a^{k-i+1}) \)-singular, \( K_i^- \), does not meet \((a, z)\) again but \( K_i^+ \) does.

**Definition.** Each \( K \)-singular point \( z \) is given an initial multiplicity equal to 1. At each stage (i) changes to the multiplicities may be assigned as indicated by the following chart. The total is then called the \( K \)-multiplicity of \( z \).

<table>
<thead>
<tr>
<th>( (i, i) ), ((1, i))</th>
<th>((i, 1), (0, i))</th>
<th>((i, 0), (1, i))</th>
<th>((i, 0), (0, i))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha^i )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( \beta^i_1 )</td>
<td>( +1 )</td>
<td>( 0 )</td>
<td>( +1 )</td>
</tr>
<tr>
<td>( \beta^i_2 )</td>
<td>( - )</td>
<td>( - )</td>
<td>( - )</td>
</tr>
<tr>
<td>( \gamma^i_1 )</td>
<td>( - )</td>
<td>( - )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( \gamma^i_2 )</td>
<td>( - )</td>
<td>( +1 )</td>
<td>( - )</td>
</tr>
</tbody>
</table>

The entry - indicates that this situation cannot occur. The motivation for this definition comes from [5] for the conformal case with \( k = 3 \). Note that the \( K \)-multiplicity is always non-negative even though there is a possibility to decrease it in the case where \( z \) is \( \gamma^i_1 \).

**Remarks.**

(A) In the case \( k = 2 \) ([1], 3.2.1) the points of inflection and the cusps of the second kind each have multiplicity 1 (using the definition of multiplicity in 3.4) while a cusp of the first kind has multiplicity 2.

(B) In the case \( k = 3 \) in [2] the differentiable singular points with the characteristic \((1, 1, 2), (1, 1, 2)_0, (1, 2, 1)_0, (2, 1, 1)_0\) have multiplicities 1, 1, 2, 3, respectively.
4. \( \mathcal{K}(a) \)-SINGULAR POINTS

Since induction will be the method of proof for the main result, it is necessary to see how \( \mathcal{K}(a) \)-singular points with certain \( \mathcal{K}(a) \)-multiplicities give rise to \( \mathcal{K} \)-singular points with their \( \mathcal{K} \)-multiplicities using the monotony, contraction and expansion theorems of Haupt and Künneth ([1]; 2.4).

4.1. Let \( z \) be a \( \mathcal{K}(a) \)-singular point of \( A_{k+1} \) with \( \mathcal{K}(a) \)-multiplicity \( s \). If \( z \) is \( \mathcal{K} \)-singular, then

<table>
<thead>
<tr>
<th>Possible type</th>
<th>( \mathcal{K} )-multiplicity</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta_1^k (k, 1) (1, k) )</td>
<td>( s + 1 )</td>
</tr>
<tr>
<td>( \beta_1^k (k, 0) (1, k) )</td>
<td>( s + 1 )</td>
</tr>
<tr>
<td>( \gamma_2^k (k, 1) (0, k) )</td>
<td>( s + 1 )</td>
</tr>
<tr>
<td>( \gamma_1^k (k, 0) (0, k) )</td>
<td>( s - 1 )</td>
</tr>
<tr>
<td>( \beta_1^k (k, 1) (0, k) )</td>
<td>( s )</td>
</tr>
<tr>
<td>( \gamma_1^k (k, 0) (1, k) )</td>
<td>( s )</td>
</tr>
<tr>
<td>( \beta_1^k (k, 0) (0, k) )</td>
<td>( s )</td>
</tr>
<tr>
<td>( \beta_2^k (k, 0) (0, k) )</td>
<td>( s )</td>
</tr>
</tbody>
</table>

Notice that an \( \alpha^k \) \( \mathcal{K} \)-singular point is not possible since \( z \) is already \( \mathcal{K} (a) \)-singular.

4.2. Let \( z \) be a \( \mathcal{K} (a) \)-singular point of \( A_{k+1} \) which is not \( \mathcal{K} \)-singular. Then \( z \) has \( \mathcal{K} (a) \)-multiplicity 1.

**Proof.** Since \( z \) is of \( \mathcal{K} \)-order \( k \), \( A_{k+1} \) satisfies \( EP_{k-1} \) at \( z \) (i.e. the \((k - 2)\) strong differentiability condition). Hence there is only one \( \mathcal{K} (a) \)-osculating characteristic at \( a \) and \( z \) is not \( \mathcal{K} (a^2) \)-singular. Thus \( z \) has \( \mathcal{K} (a) \)-multiplicity 1.

4.3. Let \( z_1 < z_2 \) be two \( \mathcal{K} (a) \)-singular points on \( A_{k+1} \).

If \( z_1 \) is one of \( \left\{ \begin{array}{c}
\beta_1^k (k, 1)(0, k) \\
\gamma_1^k (k, 0)(0, k)
\end{array} \right. \) and if \( z_2 \) is one of \( \left\{ \begin{array}{c}
\gamma_1^k (k, 0)(0, k) \\
\gamma_2^k (k, 0)(0, k)
\end{array} \right. \), then there is at least one \( \mathcal{K} \)-singular point in \((z_1, z_2)\).

**Proof.** Consider the case where \( z_1 \) is \( \beta_1^k (k, 1)(0, k) \) and \( z_2 \) is \( \gamma_1^k (k, 0)(1, k) \). The proof for the other cases is similar.

Since \( z_1 \) is \( \beta_1^k (k, 1)(0, k) \) there is an order-characteristic close to \( K_k^+ (z_1) \) that meets \( A_{k+1} \) at \( k \) points \( z_1 < y_1 < y_2 < \ldots < y_k \) near \( z_1 \) and does not meet \((a, z_1)\). Since \( z_2 \) is \( \gamma_1^k (k, 0)(1, k) \) there is a member of \( \mathcal{K} \) close to \( K_k^- (z_2) \) that meets \( A_{k+1} \) at \( k \) points \( x_2 < x_3 < \ldots < x_{k+1} < z_2 \) near \( z_2 \) and meets \((a, z_1)\) at one point \( x_1 \).

Now let a point \( t \) move from \( z_2 \) toward \( y_1 \), keeping \( x_3, \ldots, x_{k+1} \) fixed. Then there is a point \( u \) which moves from \( x_1 \) toward \( z_1 \). If \( u \) reaches \( z_1 \) first, then there is a \( k + 1 \)-tuple of points
in \([z_1, z_2]\) and an order-characteristic containing these points. By contraction, one obtains a 
\(K\)-singular point in \((z_1, z_2)\). If \(t_2\) reaches \(y_1\) first, then let \(t_3\) move from \(x_3\) toward \(y_2\), \(t_4\) move from \(x_4\) toward \(y_3\), \ldots, \(t_{k+1}\) move from \(x_{k+1}\) toward \(y_k\), if necessary. The point \(u\) must reach \(z_1\) first. Otherwise there is a member \(K\) \((t_2, t_3, \ldots, t_{k+1}) = K(y_1, y_2, \ldots, y_k)\) meeting \((a, z_1)\); contradiction. Then one obtains a \(K\)-singular point in \((z_1, z_2)\) as above.

Similarly one obtains

4.4. Let \(z_1\) be \(K(a)\)-singular but not \(K\)-singular.

\[
\begin{align*}
\gamma_1^k & \quad (k, 0)(1, k) \\
\gamma_2^k & \quad (k, 0)(0, k) \\
\beta_2^k & \quad (k, 0)(0, k)
\end{align*}
\]

where \(z_1 < z_2\), then there is a \(K\)-singular point in \((z_1, z_2)\).

4.5. Let \(z_2\) be \(K(a)\)-singular but not \(K\)-singular.

\[
\begin{align*}
\beta_1^k & \quad (k, 1)(0, k) \\
\gamma_1^k & \quad (k, 0)(0, k) \\
\beta_2^k & \quad (k, 0)(0, k)
\end{align*}
\]

where \(z_1 < z_2\), then there is a \(K\)-singular point in \((z_1, z_2)\).

4.6. Let \(a < z_1 < z_2 < \ldots < z_r < a\) where \(z_j\) is a \(K(a)\)-singular point of \(K(a)\)-multiplicity \(m_j\), \(j = 1, 2, \ldots, r\) on a curve \(C_{k+1}\) of \(K\)-order \(k + 1\).

Then the sum of the \(K\)-multiplicities of the \(K\)-singular points is at least \(\left(\sum_{j=1}^{r} m_j\right) + 1\).

**Proof.** (A) Each \(z_j\) is \(K\)-singular and not \(\gamma_1^k (k, 0)(0, k)\).

By 4.1, the sum of the \(K\)-multiplicities is at least \(\sum_{j=1}^{r} m_j\) and will be at least \(\left(\sum_{j=1}^{r} m_j\right) + 1\) if any of the \(z_j\) are \(\beta_1^k (k, 1)(1, k)\) or \(\beta_2^k (k, 0)(1, k)\).

If \(z_1\) is \(\gamma_1^k (k, 0)(1, k)\) then \(K_k^{-}(z_1)\) meets \((a, z_1)\) and a characteristic close to \(K_k^{-}(z_1)\) meets \((a, z_1)\) and meets \(C_{k+1}\) at \(k\) points \(x_1 < x_2 < \ldots < x_k < z_1\). By contraction there is a \(K\)-singular point in \((a, z_1)\). Similarly, if \(z_r\) is \(\beta_1^k (k, 1)(0, k)\) there is a \(K\)-singular point in \((z_r, a)\).

Hence assume that \(z_1\) is \(\beta_1^k (k, 1)(0, k)\) and \(z_r\) is \(\gamma_1^k (k, 0)(1, k)\), and \(z_2, \ldots, z_{r-1}\) are

\[
\begin{align*}
\beta_1^k & \quad (k, 1)(0, k) \\
\gamma_1^k & \quad (k, 0)(1, k) \\
\beta_1^k & \quad (k, 0)(0, k) \\
\beta_2^k & \quad (k, 0)(0, k)
\end{align*}
\]

If \(z_2\) is \(\gamma_1^k (k, 0)(1, k)\) then there is a \(K\)-singular point in \((z_1, z_2)\), by 4.3 and the desired result is obtained. If \(z_2\) is not \(\beta_1^k (k, 0)(0, k)\) then \(z_2\) is \(\beta_1^k (k, 1)(0, k)\) the same as \(z_1\).
Continue this process with \( z_3, z_4, \ldots \). Either one obtains a \( \mathcal{K} \)-singular point in one of the \((z_j, z_{j+1})\) or all of \( z_1, z_2, \ldots, z_{r-1} \) are \( \{\beta_1^k (k, 1)(0, k) \} \). But then \( z_r \) is \( \{\gamma_1^k (k, 0)(1, k) \} \) and there is a \( \mathcal{K} \)-singular point in \((z_{r-1}, z_r)\) by 4.3.

Hence the sum of the \( \mathcal{K} \)-multiplicities of the \( \mathcal{K} \)-singular points is at least \( \left( \sum_{j=1}^{r} m_j \right) + 1 \).

(B) Each \( z_j \) is \( \mathcal{K} \)-singular and at least one of the \( z_j \) is \( \gamma_1^k (k, 0)(0, k) \).

If \( z_1 \) is \( \gamma_1^k (k, 0)(0, k) \) then \( z_1 \) is of \( \mathcal{K} \)-multiplicity \( m_1 - 1 \). Also \( P_k^- (z_1) \) meets \((a, z_1)\) and one obtains a \( \mathcal{K} \)-singular point in \((a, z_1)\).

If \( z_r \) is \( \gamma_1^k (k, 0)(0, k) \) then again \( z_r \) is of \( \mathcal{K} \)-multiplicity \( m_r - 1 \) and there is a \( \mathcal{K} \)-singular point in \((z_r, a)\).

Let \( J \) be the first index other than \( J = 1 \) or \( J = r \) for which \( z_J \) is \( \gamma_1^k (k, 0)(0, k) \). Hence the \( \mathcal{K} \)-multiplicity of \( z_J \) is \( m_J - 1 \). If \( z_{J-1} \) is \( \{\beta_1^k (k, 1)(0, k) \} \) then there is a \( \mathcal{K} \)-singular point in \((z_{J-1}, z_J)\) by 4.3.

If \( z_{J-1} \) is \( \{\gamma_1^k (k, 0)(1, k) \} \) then \( z_{J-1} \) has \( \mathcal{K} \)-multiplicity \( m_{J-1} + 1 \) by 4.1. Hence one

\[
\gamma_1^k (k, 0)(1, k) \quad \beta_1^k (k, 0)(0, k)
\]

is left with \( z_{J-1} \) being \( \{\gamma_1^k (k, 0)(1, k) \} \). Considering \( z_{J-2} \), one either obtains one more multiplicity in \((z_{J-2}, z_{J-1})\) or one more multiplicity for \( z_{J-2} \), or \( z_{J-2} \) is \( \{\gamma_1^k (k, 0)(1, k) \} \). Proceeding one obtains one more multiplicity or all of \( z_{J-1}, z_{J-2}, \ldots, z_1 \) are \( \{\gamma_1^k (k, 0)(1, k) \} \) in which case there is a \( \mathcal{K} \) (a)-singular point in \((a, z_1)\) since \( \mathcal{K}_k^- (z_1) \) meets \((a, z_1)\). Hence the total sum of the \( \mathcal{K} \)-multiplicities of the \( \mathcal{K} \)-singular points on \((a, z_1)\) is at least \( \sum_{j=1}^{J} m_j \). But now if \( z_{J+1} \) is \( \{\gamma_1^k (k, 0)(1, k) \} \), there is a \( \mathcal{K} \)-singular point in \((z_J, z_{J+1})\), by 4.3 and the total sum of the \( \mathcal{K} \)-multiplicities of the \( \mathcal{K} \)-singular points on \((a, z_{J+1})\) is at least \( \left( \sum_{j=1}^{J+1} m_j \right) + 1 \).

Also if \( z_{J+1} \) is \( \{\beta_1^k (k, 1)(1, k) \} \) then \( z_{J+1} \) is of \( \mathcal{K} \)-multiplicity \( m_{J+1} + 1 \) and again the total sum of the \( \mathcal{K} \)-multiplicities of the \( \mathcal{K} \)-singular points on \((a, z_{J+1})\) is at least \( \left( \sum_{j=1}^{J+1} m_j \right) + 1 \). If \( z_{J+1} \) is \( \{\beta_1^k (k, 0)(0, k) \} \) we move to the next index where a \( \gamma_1^k (k, 0)(0, k) \) occurs and treat as we did for \( z_{J-1} \) being \( \{\beta_1^k (k, 1)(0, k) \} \) as above. If \( z_{J+1} \) is \( \gamma_1^k (k, 0)(0, k) \) then treat \( z_{J+1} \) as we did above.

The only possibility for which we do not get the total sum of the \( \mathcal{K} \)-multiplicities of the \( \mathcal{K} \)-singular points as being at least \( \left( \sum_{j=1}^{r} m_j \right) + 1 \) occurs if \( z_J, z_{J+1}, \ldots, z_r \) are all \( \gamma_1^k (k, 0)(0, k) \). But then there are \( \mathcal{K} \)-singular points in all of \((z_J, z_{J+1}), (z_{J+1}, z_{J+2}), \ldots, (z_{r-1}, z_r)\) and one in \((z_r, a)\); altogether \( \left( \sum_{j=1}^{r} m_j \right) + 1 \).

(C) At least one of the \( z_j \) is not \( \mathcal{K} \)-singular.
If $z_1$ is not $\mathcal{K}$-singular then there is a $\mathcal{K}$-singular point in $(a, z_1)$ since $z_1$ is $\mathcal{K}$-singular.

If $z_r$ is not $\mathcal{K}$-singular then there is a $\mathcal{K}$-singular point in $(z_r, a)$.

Let $J$ be the first index other than $J = 1$ or $J = r$ for which $z_J$ is not $\mathcal{K}$-singular. If $z_{J-1}$ is
\[
\begin{cases}
\beta_1^J (k, 1)(0, k) \\
\beta_2^J (k, 0)(0, k)
\end{cases}
\]
then there is a $\mathcal{K}$-singular point in $(z_{J-1}, z_J)$, by 4.5, and if $z_{J-1}$ is
\[
\begin{cases}
\gamma_1^J (k, 0)(0, k) \\
\gamma_2^J (k, 1)(0, k)
\end{cases}
\]
then the $\mathcal{K}$-multiplicity of $z_{J-1}$ is $m_{J-1} + 1$. Hence one is left with $z_{J-1}$ being
\[
\begin{cases}
\gamma_1^J (k, 0)(1, k) \\
\beta_2^J (k, 0)(0, k)
\end{cases}
\]
As in (B) the total sum of the $\mathcal{K}$-multiplicities of the $\mathcal{K}$-singular points on $(a, z_J)$ is at least $\sum_{j=1}^{J} m_j$. But if $z_{J+1}$ is
\[
\begin{cases}
\beta_1^J (k, 1)(0, k) \\
\beta_2^J (k, 0)(0, k)
\end{cases}
\]
point in $(z_J, z_{J+1})$, by 4.4. If $z_{J+1}$ is
\[
\begin{cases}
\beta_1^J (k, 0)(0, k) \\
\gamma_2^J (k, 1)(0, k)
\end{cases}
\]
then $z_{J+1}$ has $\mathcal{K}$-multiplicity $m_{J+1} + 1$. Hence the total sum of the $\mathcal{K}$-multiplicities of the $\mathcal{K}$-singular points on $(a, z_{J+1})$ is at least $\left(\sum_{j=1}^{J+1} m_j\right) + 1$. If $z_{J+1}$ is
\[
\begin{cases}
\beta_1^J (k, 1)(0, k) \\
\beta_2^J (k, 0)(0, k)
\end{cases}
\]
we move to the next index where a non $\mathcal{K}$-singular, $\mathcal{K}$-singular point occurs and treat as we did above for $z_{J-1}$ being
\[
\begin{cases}
\beta_1^J (k, 1)(0, k) \\
\beta_2^J (k, 0)(0, k)
\end{cases}
\]
Finally, if $z_{J+1}$ is not $\mathcal{K}$-singular, then treat $z_{J+1}$ as we did $z_J$ above.

The only possibility for which we do not get the total sum of the $\mathcal{K}$-multiplicities of the $\mathcal{K}$-singular points as being at least $\left(\sum_{j=1}^{r} m_j\right) + 1$ occurs if $z_J, z_{J+1}, \ldots, z_r$ are all not $\mathcal{K}$-singular. But then there are $\mathcal{K}$-singular points in all of $(z_J, z_{J+1}), (z_{J+1}, z_{J+2}), \ldots, (z_r, z_r)$ and one in $(z_r, a)$; altogether $\left(\sum_{j=1}^{r} m_j\right) + 1$.

5. THE MAIN RESULT

**Theorem 1.** Let $C = C_{k+1}$ be a curve of $\mathcal{K}$-order $k + 1$ with respect to a system $\mathcal{K}$ of order-characteristics with fundamental number $k$. Then the sum of the $\mathcal{K}$-multiplicities of the $\mathcal{K}$-singular points of $C$ is at least $k + 1$.

**Proof.** The proof is by induction. The result is valid for $k = 2 ([1]; 3.2.6)$.

Now assume that the result is true for $n = k - 1$ and show that it is true for $n = k$. Take any strongly differentiable ($C$ satisfies $EP_k$ at $a$) non-singular point $a$ on $C$. Now $C$ is of $\mathcal{K}$-order $k$ with respect to the system $\mathcal{K}$ $(a)$ whose fundamental number is $k - 1$.

By induction the sum of the $\mathcal{K}$ $(a)$-multiplicities of the $\mathcal{K}$ $(a)$-singular points is at least $k$.

Denote the $\mathcal{K}$ $(a)$-singular points $z_j$ with $\mathcal{K}$ $(a)$-multiplicity $m_j$ and $a < z_1 < z_2 < \ldots < z_r < a$.

Then $\sum_{j=1}^{r} m_j \geq k$. By 4.6, the sum of the $\mathcal{K}$-multiplicities of the $\mathcal{K}$-singular points is at least $\left(\sum_{j=1}^{r} m_j\right) + 1$; i.e. $\geq k + 1$.

Hence the theorem is true by induction.

**Theorem 2.** Let $C = C_{k+1}$ be a curve of $\mathcal{K}$-order $k + 1$ with respect to a system $\mathcal{K}$ of
order-characteristics with fundamental number k. Then the sum of the \( K \)-multiplicities of the \( K \)-singular points is exactly \( k + 1 \).

**Proof.** Combine Theorem 1 and section 5 of [6].

**Corollary.** A curve \( C_{k+1} \) of \( K \)-order \( k + 1 \) satisfying \( EP_k \) at each point contains exactly \( k + 1 \) singular points.

**Proof.** Use Theorem 2 and the fact that \( K \)-singular points satisfying \( EP_k \) are of \( K \)-multiplicity 1.
REFERENCES


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