Note di Matematica Vol. 15 - n. 2, 203-213 (1995)

ISOPTICS OF ROSETTES AND ROSETTES OF CONSTANT WIDTH

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Abstract. *This paper presents many geometric properties of isoptics of rosettes which complete and deepen the results obtained in* **[CM]** *and* **[Weg]**.

1. INTRODUCTION

This article is concerned with some geometric properties of isoptics of rosettes which complete and deepen the results obtained in **[CM]** and **[Weg]**. We begin by recalling the basic notions and results necessary in this paper.

Definition 1.1. *A plane, closed positively oriented curve of positive curvature is said to be a rosette.*

We introduce now a special parametrization using an oriented support function which is a natural generalization of the ordinary one (cf. **[S]**). Let us consider a rosette $C : z = z(s)$ parametrized by arc length. Let $n(s) = \frac{z''(s)}{|z''(s)|}$ and suppose that the index of C is equal to j. Choose a point O as the origin of our coordinate system and suppose that the curve C is considered in this system.

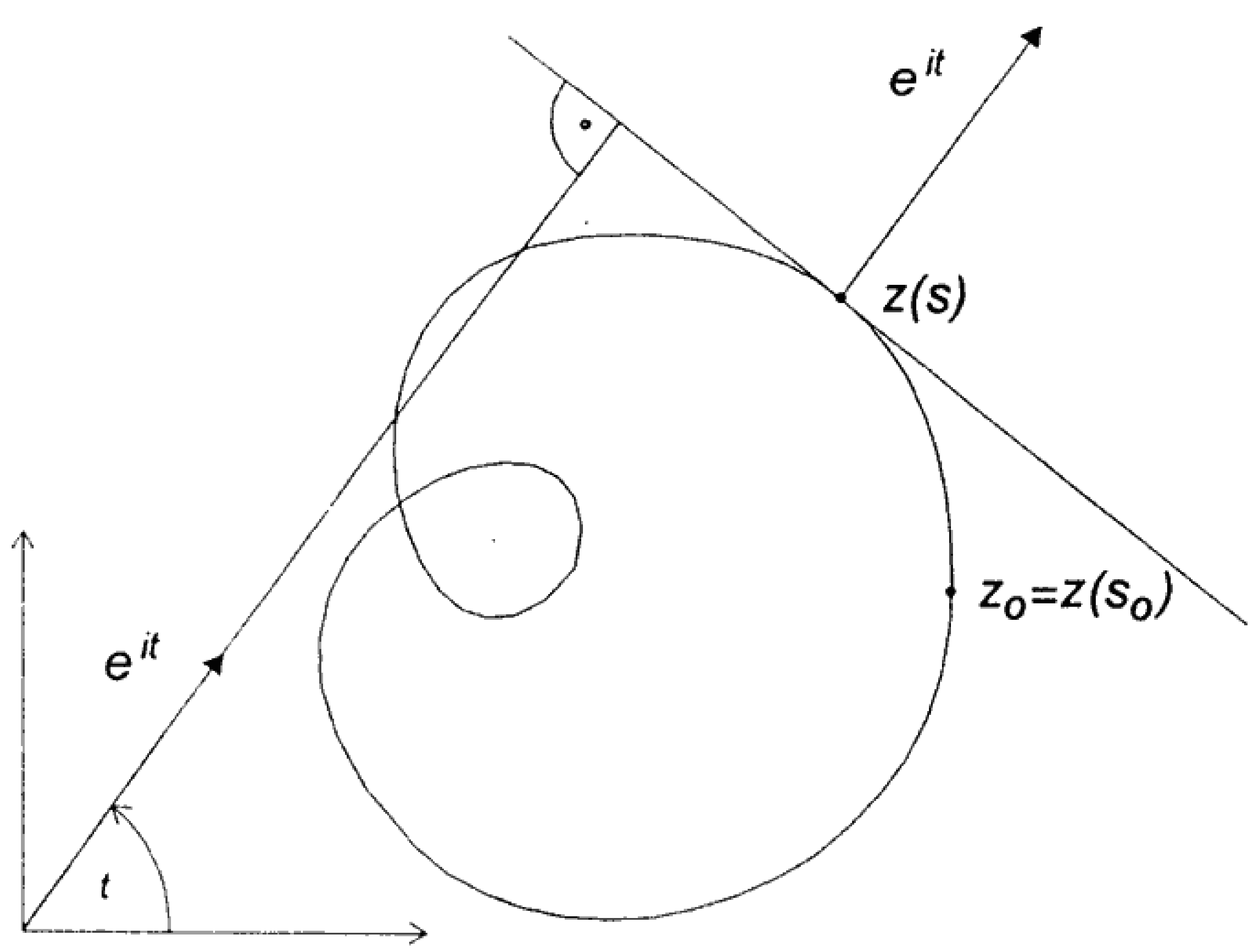

FIG. 1

We define an oriented support function of C in the following way. Let us fix a point $z_0 = z(s_0)$ and consider the tangent line at z_0. We can suppose that z_0 is chosen in such a way that the tangent line is perpendicular to the x-axis. For an arbitrary point $z(s)$ we define a vector $e^{it} = \cos t + i \sin t$ as on fig. 1 where t is an oriented angle between the positive direction of the x-axis and vector e^{it}. Since the curve C has the index j, we have $t \in [0, 2\pi j]$. Now we define an oriented distance $p(t)$ from the origin O of the coordinate system to the tangent line to C. Fix a point $z(s)$. Then we take e^{it} as a normal vector to C at this point (cf. fig. 1). If the vector e^{it} points to this half-plane which contains O then we put $p(t)$ equal the negative of the ordinary distance between O and the tangent line at $z(s)$. If not we define $p(t)$ as the ordinary distance between O and the tangent line at $z(s)$. Since the rosette is locally convex then the $p(t)$, $t \in [0, 2\pi j]$, is at least at the class C^1 (cf. [**San**]). Using $p(t)$ we obtain a new parametrization of C given by

$$z(t) = p(t)e^{it} + p'(t)ie^{it}.$$

However, in the later part of this paper we always assume that $p(t)$ is at least of the class C^2.

Definition 1.2. *A function p constructed above is called an oriented support function.*

Remark 1.1. *The length L of C is given by the formula*

$$L = \int_0^{2\pi j} p dt.$$

Remark 1.2. *Since the turning tangent line makes j turns, p(t) can have at most 2j zeros.*

Let us fix a point $z(t) \in C$ and consider the tangent line to C at $z(t)$. Let $z(t')$, $t' < t$, be the closest point (in the sense of parametrization) such that the angle between the tangent lines at $z(t)$ and $z(t')$ is equal to $\pi - \alpha$.

Definition 1.3. *The cut locus C_α of the intersection points of the above defined pairs of tangent lines is said to be an α-isoptic of C.*

Note that from the above considerations it follows that $t' = t + \alpha$

In the same way as in [**CMM1**] and [**CMM2**] we obtain the following parametrization of C_α

$$z_\alpha(t) = p(t)e^{it} + \left(-p(t)\cot\alpha + \frac{1}{\sin\alpha}p(t+\alpha)\right) ie^{it}.$$

2. INDEX OF AN ISOPTIC

Since the computation of the curvature of a curve is a local matter then reasoning as in [**CMM1**] we get the following formula for the curvature k_α of isoptic C_α

$$k_\alpha(t) = \frac{1}{|z'(t)|^3}\left\{z'(t), z''(t)\right\} = \frac{\sin\alpha}{|q(t)|^3}\left(2|q(t)|^2 - \{q(t), q'(t)\}\right),$$

where $q(t) = z(t) - z(t+\alpha)$ and $\{,\}$ denote the determinant of coordinates of vectors.

Now we prove the following theorem

Theorem 2.1. *If C_α is an α-isoptic of a rosette C then*

$$\text{Index } C_\alpha = \text{ Index } C.$$

Proof. Suppose that Index $C = j$. Geometrically this means that the tangent indicatrice makes j turns. Let k_α denote the curvature of C_α parametrized by arc length and L the length of C_α. If $\widehat{j}$ is the index of C_α then

$$2\pi\widehat{j} = \int_0^L k_\alpha(s)ds.$$

Using the theorem of change of variables we have

$$\begin{aligned}\int_0^L k_\alpha(s)ds &= \int_0^{2\pi j} k_\alpha(t)|z'(t)|dt = \int_0^{2\pi j} k_\alpha(t)\frac{|q(t)|}{\sin\alpha}dt = \\ &= \int_0^{2\pi j} \frac{\sin\alpha}{|q(t)|^3}\left(2|q(t)|^2 - \{q(t), q'(t)\}\right)\frac{|q(t)|}{\sin\alpha}dt = \int_0^{2\pi j}\left(2 - \left|\frac{d}{dt}\left(\frac{q(t)}{|q(t)|}\right)\right|\right)dt = \\ &= 4\pi j - \int_0^{2\pi j}\left|\frac{d}{dt}\left(\frac{q(t)}{|q(t)|}\right)\right|dt.\end{aligned}$$

But $\frac{q(t)}{|q(t)|}$ is a parametrization of a circle. Since the angle between vectors $\frac{z'}{|z'|}$ and $\frac{q}{|q|}$ is less then $\frac{\pi}{2}$ then the vector $\frac{q(t)}{|q(t)|}$ runs around the circle the same number of times that $\frac{z'}{|z'|}$, namely j times. Therefore

$$\int_0^{2\pi j}\left|\frac{d}{dt}\left(\frac{q(t)}{|q(t)|}\right)\right|dt = 2\pi j$$

and

$$2\pi \;\text{ Index } C_\alpha = 4\pi j - 2\pi j = 2\pi j. \quad \square$$

Remark 2.3. *Number of self-intersections of the starting rosette and its isoptic need not to be equal, however their Whitney numbers are always equal (cf.* **[Whi]***).*

3. SINE THEOREM FOR ROSETTES

In the paper [**CMM1**] we have proved the sine theorem for ovals. Let's note that the proof of this formula is of purely local nature, so the proof of the following theorem is similar.

Theorem 3.1. *Under the notations from fig. 2 we have*

$$\frac{|q|}{\sin\alpha} = \frac{t_1}{\sin\alpha_1} = \frac{t_2}{\sin\alpha_2}.$$

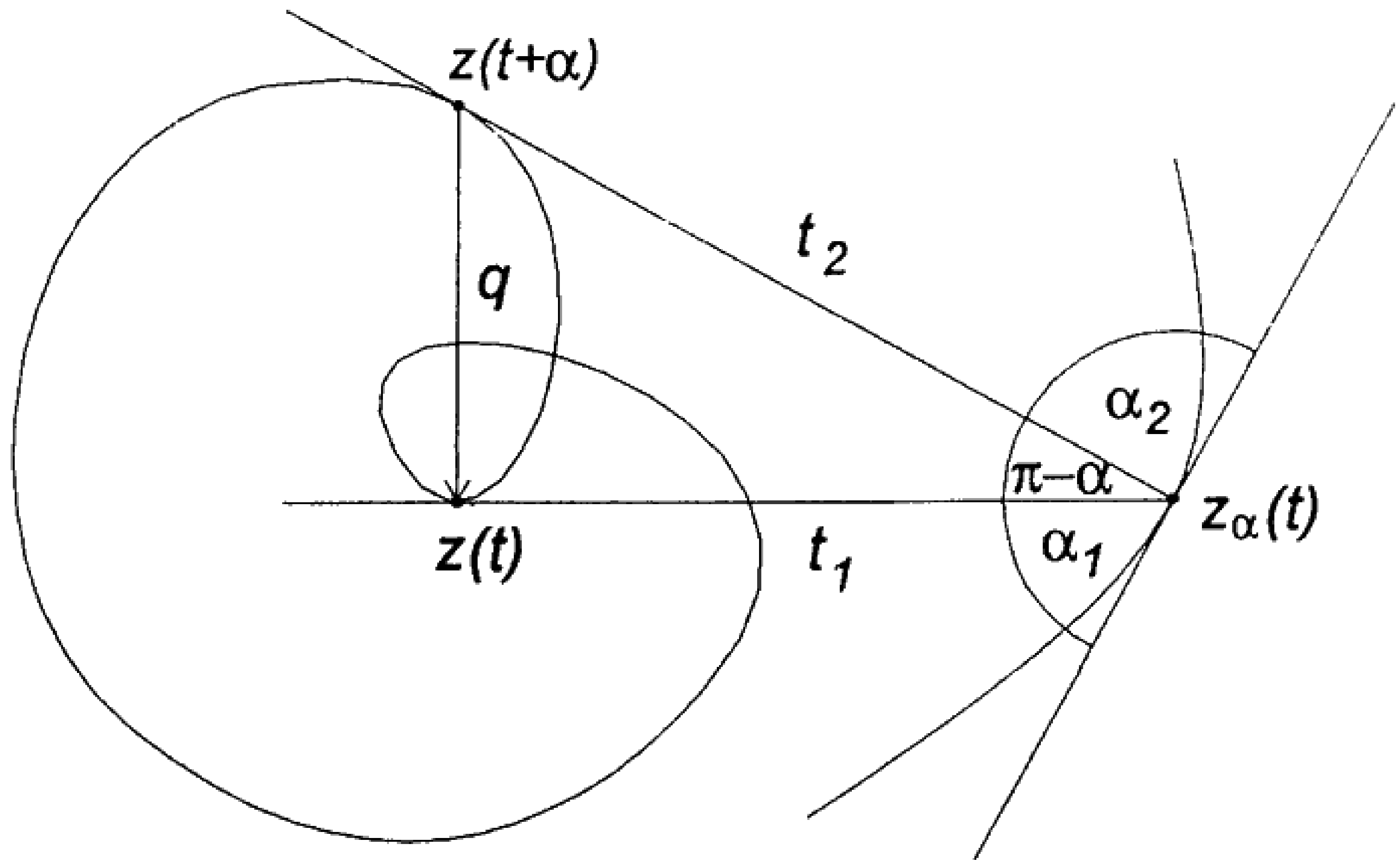

FIG. 2

4. THEOREM ON TANGENTS TO ISOPTICS OF ROSETTES

Now we prove

Theorem 4.1. *Let C_α be an α-isoptic of rosette C of index j. Consider a sequence of numbers $\tau_1, \tau_2, \ldots, \tau_n$, $\tau_i > 0$ such that $\tau_1 + \tau_2 + \ldots + \tau_n < 2\pi j$ and put $t_1 = t, t_2 = t + \tau_1, \ldots, t_{n+1} = t + \tau_1 + \tau_2 + \ldots + \tau_n$. Then*

$$\sum \measuredangle\left(z'_{\alpha,i}, z'_{\alpha,i+1}\right) + \sum \measuredangle\left(q_i, q_{i+1}\right) = 2(\tau_1 + \tau_2 + \ldots + \tau_n),$$

where $z'_{\alpha,i} = z'_\alpha(t_i)$, $q_i = q(t_i)$.

Proof. Let us fix an index i Then reasoning analogously as in proof of theorem 4.1 from [**CMM**] we obtain

$$\measuredangle\left(z'_{\alpha,i}, z'_{\alpha,i+1}\right) + \measuredangle\left(q_i, q_{i+1}\right) = 2\tau_i.$$

Summing up the obtained formulas, we get our theorem. □

Corollary 4.1. *If we put $\tau_1 = \tau_2 = \ldots = \tau_{2j-1} = \pi$ then for any rosette of index j we have*

$$\sum_{i=1}^{2j-1} \measuredangle\left(z'_{\alpha,i}, z'_{\alpha,i+1}\right) + \measuredangle\left(q_i, q_{i+1}\right) = 2(2j-1)\pi.$$

Corollary 4.2. *Let $\tau_1 = \tau_2 = \ldots = \tau_{2j-1} = \pi$. Then vectors $z'_{\alpha,1}, z'_{\alpha,2}, \ldots, z'_{\alpha,2j-1}$ are parallel if and only if vectors $q_1, q_2, \ldots, q_{2j-1}$ are parallel.*

5. ROSETTES OF CONSTANT WIDTH

Let $C : z(t) = p(t)e^{it} + p'(t)ie^{it}$ be a rosette of index j, where p is an oriented support function of period $2\pi j$.

Definition 5.1. *A rosette C is a rosette of constant width with respect to the origin O if $p(t+\pi j)+p(t)=const$ holds for each $t \in \mathbb{R}$.*

We are going to find a necessary and sufficient condition for a support function to give a rosette of constant width. We consider the Fourier expansion of p. Thus

$$p(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\Big(a_n\cos\frac{nt}{j}+b_n\sin\frac{nt}{j}\Big),$$

where

$$a_n=\frac{1}{\pi j}\int_0^{2\pi j}p(t)\cos\frac{nt}{j}dt,\qquad n=0,1,2,\dots$$
$$b_n=\frac{1}{\pi j}\int_0^{2\pi j}p(t)\sin\frac{nt}{j}dt,\qquad n=1,2,\dots.$$

Then

$$p(t+\pi j)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\Big(a_n\cos\frac{2kt}{j}+b_n\sin\frac{2kt}{j}\Big)+\sum_{n=1}^{\infty}\Big(-a_n\cos\frac{(2k+1)t}{j}-b_n\sin\frac{(2k+1)t}{j}\Big).$$

Since $p(t+\pi j)+p(t)=const$ then $a_{2k}=b_{2k}=0$ for each positive integer k. On the contrary, a_{2k+1} and b_{2k+1} are arbitrary - provided that $p+p''>0$.

Corollary 5.1. *Each rosette of constant width of index j is given by the following support function*

$$p(t)=\frac{a_0}{2}+\sum_{k=0}^{\infty}\Big(a_{2k+1}\cos\frac{(2k+1)t}{j}+b_{2k+1}\sin\frac{(2k+1t}{j}\Big),$$

where $a_0, a_{2k+1}, b_{2k+1}, k=0,1,2,\dots$ are such $p+p''>0$ for each $t\in\mathbb{R}$.

This is a generalization of the Tennison theorem for ovals (cf. [**Ten**]).

Note that the condition $p(t+\pi j)+p(t)=const$ depends on a choice of the origin O.

Theorem 5.1. *The following conditions are equivalent:*

(1) *The rosette is of constant width with respect to any point of the plane.*

(2) *The rosette is of constant width in sense of definition 7 from* [**CM**].

(3) *The support function p satisfies the following conditions:*

(a) $$p(t)=\frac{a_0}{2}+\sum_{k=0}^{\infty}\Big(a_{2k+1}\cos\frac{(2k+1)t}{j}+b_{2k+1}\sin\frac{(2k+1)t}{j}\Big),$$

(b) $p+p''>0$,

(c) *j is odd.*

Proof.

(1) $\Rightarrow$ (3)

Let us suppose that C is a rosette of constant width with respect to any point. Then (a) and (b) of (3) are obviously satisfied. A support function $\tilde{p}$ of C with respect to an arbitrary point (a, b) is of the form

$$\tilde{p}(t) = p(t) - a\cos t - b\sin t$$

Since $\tilde{p}(t+\pi j) + \tilde{p}(t) = p(t+\pi j) + p(t) = const$, thus j must be odd.
(3) $\Rightarrow$ (1)
This step is obvious.
(1) $\Rightarrow$ (2)

In both cases we know that j is odd. This means that the normal vectors at t and $t+\pi j$ are oppositely directed and $p(t+\pi j) + p(t)$ is an ordinary distance between the tangent lines. □

Corollary 5.2. *The following conditions are equivalent:*

(1) *Rosette is of constant width with respect to the origin O.*

(2) *Conditions* (a) *and* (b) *from the above theorem hold.*

We shall study now the case of rosettes of even index. In this direction we have the following

Theorem 5.2. *Let C be a rosette of even index j and of constant width with respect to the origin O. Then O is the Steiner centroid of C (this means that this point is geometrically determined).*

Proof. We have to prove that $A = \int_0^{2\pi j}(p(t)e^{it} + p'(t)ie^{it})dt = 0$. Since $\left(p(t)ie^{it}\right)' = $ $= p'(t)ie^{it} - p(t)e^{it}$ then

$$\begin{aligned}
A &= 2\int_0^{2\pi j} p(t)e^{it}dt = 2\int_0^{\pi j} p(t)e^{it}dt + 2\int_{\pi j}^{2\pi j} p(t)e^{it}dt = \\
&= 2\int_0^{\pi j} p(t)e^{it}dt + 2\int_0^{\pi j} p(t+\pi j)e^{i(t+\pi j)}dt = \\
&= 2\int_0^{\pi j} p(t)e^{it}dt + 2\int_0^{\pi j}\Big(c - p(t)\Big)e^{i(t+\pi j)}dt = \\
&= 2\int_0^{\pi j} ce^{it}dt = 0. \quad \square
\end{aligned}$$

We will show now that some results from **[CM]** extend to the class of rosettes of constant width of even index. Let us consider a rosette $C : z(t) = p(t)e^{it} + p'(t)ie^{it}$ of constant width $c = p(t+\pi j) + p(t)$ of even index j where a support function p is taken with respect to the Steiner centriod of C. Let $\tilde{C}$ denote the rosette symmetric to C with respect to O.

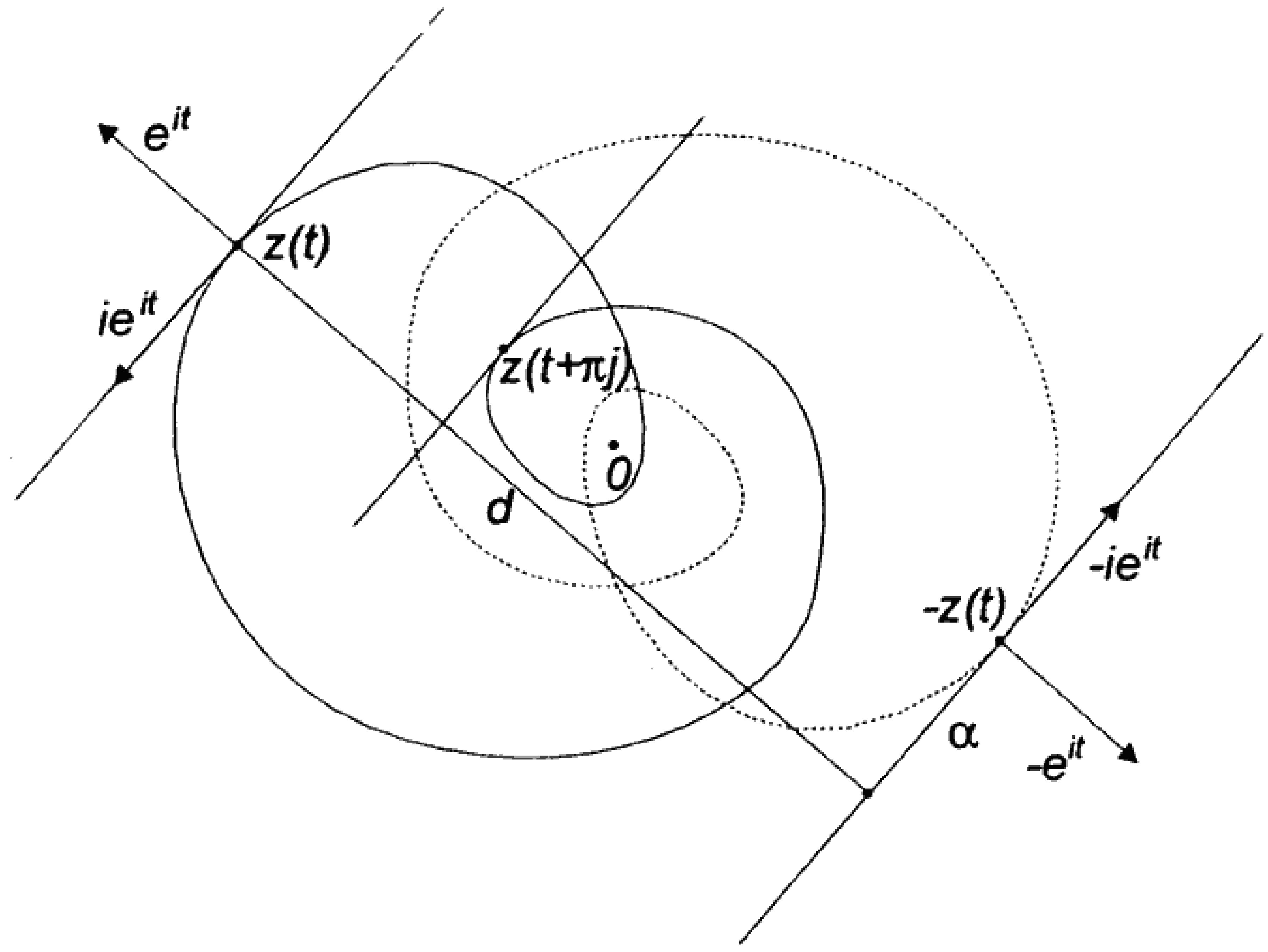

FIG. 3

We have (see fig. 3):

$$z(t+\pi j) - (-z(t)) = \alpha(t) i e^{it} + c e^{it}.$$

Hence

$$z'(t+\pi j) - (-z'(t)) = \alpha'(t) i e^{it} - \alpha(t) e^{it} + c i e^{it}.$$

On the other hand,

$$z'(t+\pi j) + z'(t) = \Big(p(t) + p''(t)\Big) i e^{it} + \Big(p(t+\pi j)) + p''(t+\pi j)\Big) i e^{it}.$$

Therefore

$$\alpha(t) \equiv 0$$

and

$$\alpha' + c = p(t) + p''(t) + p(t+\pi j)) + p''(t+\pi j).$$

Integrating the last formula over the integral $[0, 2\pi j]$ we get

$$L = c\pi j.$$

Taking into considerations the results from [**CM**] concerning only the rosettes of odd index we can state the following

Theorem 5.3.
(1) *The length of a rosette of constant width c is equal to $c\pi j$.*
(2) *The normal lines at $z(t)$ and $z(t+\pi j)$ of any rosette of constant width coincide.*

Let us note that from the above considerations and [**CM**] it follows

$$c = \frac{1}{k(t)} + \frac{1}{k(t+\pi j)}.$$

Consider now the reciprocal situation for even index.

Theorem 5.4. *Assume that $z(t) = p(t)e^{it} + p'(t)ie^{it}$ is a rosette of even index j such that*
(1) $c = \frac{1}{k(t)} + \frac{1}{k(t+\pi j)} = const.$
(2) $\int_0^{2\pi j} z(t)dt = 0.$

Proof. We have

$$\frac{1}{k(t)} = p(t) + p''(t) \qquad \frac{1}{k(t+\pi j)} = p(t+\pi j) + p''(t+\pi j).$$

Consider an equation (cf. fig. 2.)

$$z(t+\pi j) + z(t) = \alpha(t)ie^{it} + d(t)e^{it}.$$

Then differentiating the above formula we get

$$\Big[p(t) + p''(t) + p(t+\pi j) + p''(t+\pi j)\Big] ie^{it} = \alpha'(t)ie^{it} - \alpha(t)ie^{it} + d'(t)e^{it} + d(t)ie^{it}.$$

Hence

$$\begin{aligned} c &= \alpha'(t) + d(t) \\ 0 &= -\alpha(t) + d'(t) \end{aligned}$$

Therefore

$$\alpha(t) + \alpha''(t) = 0.$$

Consequently

$$\alpha(t) = a\cos t + b\sin t, \quad d(t) = a\sin t - b\cos t + c.$$

Thus we have

$$\begin{aligned} z(t+\pi j) + z(t) &= (a\cos t + b\sin t)ie^{it} + (a\sin t - b\cos t + c)e^{it} = \\ &= (-b + c\cos t, a + c\sin t). \end{aligned}$$

Since

$$\int_0^{2\pi j} z(t)dt = \int_0^{2\pi j} z(t+\pi j)dt = 0$$

then $a = b = 0$. Finally we get $\alpha(t) \equiv 0$ and $d(t) \equiv c$. □

6. AXES OF SYMMETRY OF ROSETTES OF CONSTANT WIDTH

S. Góźdź in [**Góź**] studied axes of symmetry of plane curves using the Fourier expansion of their radius of curvature. In this section we examine the same problem for rosettes of constant width using our oriented support function.

We are looking for the conditions for rosette to have one axis of symmetry. We can assume that the x-axis is the axis of symmetry. This means that $p(2\pi j - t) = p(t)$. Then from the equality

$$\frac{a_0}{2} + \sum_{k=0}^{\infty}\left(a_{2k+1}\cos\frac{(2k+1)t}{j} + b_{2k+1}\sin\frac{(2k+1)t}{j}\right)$$
$$= \frac{a_0}{2} + \sum_{k=0}^{\infty}\left(a_{2k+1}\cos\frac{(2k+1)(2\pi j - t)}{j} + b_{2k+1}\sin\frac{(2k+1)(2\pi j - t)}{j}\right)$$

we get $b_{2k+1} = 0$ for $k = 0, 1, 2, \dots$.

We shall consider now a rosette with n axes of symmetry. These axes must have a common point, since otherwise the boundedness of C leads to a contradiction. In this case $p(\frac{2\pi j}{n} - t) = p(t)$ for $t \in \mathbb{R}$. We suppose that x-axis is one of the axes of symmetry. We have then

$$p(t) = \frac{a_0}{2} + \sum_{k=0}^{\infty} a_{2k+1}\cos\frac{(2k+1)t}{j}.$$

Then from $p(\frac{2\pi j}{n} - t) = p(t)$ we have

$$a_{2k+1}\sin\frac{(2k+1)\pi}{n}\cos\frac{2k+1}{n} = 0$$
$$a_{2k+1}\sin^2\frac{(2k+1)\pi}{n} = 0.$$

Hence either $a_{2k+1} = 0$ or $\sin\frac{(2k+1)\pi}{n} = 0$. If $\sin\frac{(2k+1)\pi}{n} = 0$ then $2k+1 = np$ where p is an arbitrary odd positive integer.

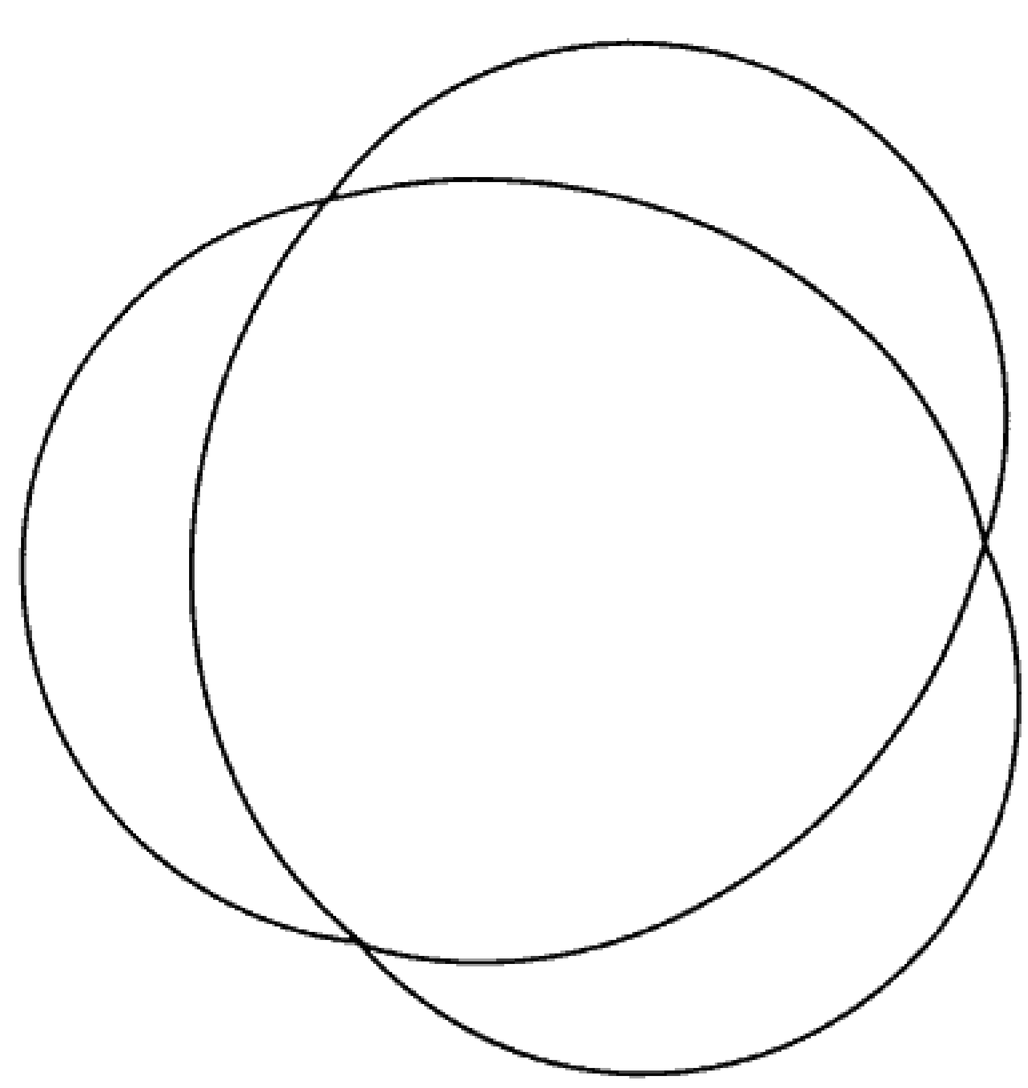

FIG. 4 A rosette of index two with three axes of symmetry

Theorem 6.1. *Any rosette of constant width with n symmetry axes where n is odd is given by an oriented support function p of the form*

$$\frac{a_0}{2} + \sum_{k=0}^{\infty} a_{2k+1}\cos\frac{(2k+1)t}{j},$$

where $a_{2k+1} = 0$ *for* $k \neq \frac{nq-1}{2}$, *where* q *is an arbitrary odd positive integer, provided that* $p + p'' > 0$.

Corollary 6.1. *There exists no rosette of constant width with an even number of symmetry axes.*

Corollary 6.2. *A rosette of constant width with a center of symmetry reduces to a circle.*

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Received January 22, 1997

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