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A CLASS OF TOPOLOGICAL SPACE GEOMETRIES

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Abstract. *The purpose of this paper is to introduce R^2-divisible R^3-spaces. This is a branch of topological space geometries in the sense of Betten. We define a few more topologies on the line set in R^2-divisible R^3-spaces and investigate under the consequences of these topologies basic properties of these space geometries.*

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1. INTRODUCTION

In this paper we investigate a new class of topological space geometries, so-called *R^2-divisible R^3-spaces*. A topological projective plane $\mathcal{P}$ is a projective plane with point set P and line set $\mathcal{L}$, where both P and $\mathcal{L}$ carry topologies such that the operations of joining and intersecting are continuous in their domains of definition. A topological projective plane is called *n-dimensional* if P and $\mathcal{L}$ are n-dimensional, locally compact, connected topological spaces. As in the case of projective planes, we will call a locally compact, connected affine plane n-dimensional if its point set and line set are n-dimensional, locally compact, connected topological spaces. The lines in 2-(4-)dimensional affine planes are homeomorphic to R (R^2). For general information about topological planes the reader is referred to [**Sal95**]. Since the fundamental papers of Salzmann [**Sal70**, **Sal 71**], Betten has tried to classify all 4-dimensional compact flexible projective planes. A toplogical projective plane is called *flexible* if the collineation group has an open orbit in the space of flags(flag=incident point-line pair). In a series of papers of Betten and Knarr many different types of 4-dimensional projective planes were found. These planes can be represented by 4-dimensional affine planes, and R^2-divisible R^3-spaces are derived from 4-dimensional affine planes. We can regard an R^2-divisible R^3-space as an intersection of a 4-dimensional affine plane and R^3. In order to explain of this geometrical structure, we consider the classical 4-dimensional affine plane $\mathcal{A}_2C$ over the complex field C and the induced R^2-divisible R^3-spaces. The affine plane $\mathcal{A}_2C$ consists of point set $C \times C$ and the following subsets of $C \times C$ are called lines: $L(s,t) = \{(x, sx + t) : x \in C\}$ for $s, t \in C$, $\{c\} \times C$ for $c \in C$. If we identify C^2 with $R^4 = \{(x, y, u, v) : x, y, u, v \in R\}$, then we can identify the lines with the following forms: $L(a, b, \xi, \eta) = \{(x, y, ax-by+\xi, ay+bx+\eta) : x, y \in R\}$ for $(a, b, \xi, \eta) \in R^4$, $\{(x, y)\} \times R^2$ for $(x, y) \in R^2$. Let $R^3_{x=0} := \{(0, y, u, v) : y, u, v \in R\}$ and let $l(a, b, \xi, \eta) := L(a, b, \xi, \eta) \cap R^3_{x=0} = \{(0, y, -by+\xi, ay+\eta) : y \in R\}$. $\mathcal{L}$ denote the set of all lines $l(a, b, \xi, \eta)$ with $(a, b, \xi, \eta) \in R^4$. If we identify $R^3_{x=0}$ with $R^3 = \{(y, u, v) : y, u, v \in R\}$, then we get a geometrical structure $(R^3, \mathcal{L}, \Lambda)$ on R^3, that is, for two points $(y_1, u_1, v_1), (y_2, u_2, v_2) \in R^3$ with $y_1 \neq y_2$ there exists a unique joining line $l(a, b, \xi, \eta)$, and $\Lambda = \{\{y\} \times R^2 : y \in R\}$ is a partition of R^3. In the same way we get also a geometrical structure on $R^3_{y=0} := \{(x, 0, u, v) : x, u, v \in R\}$. Hence we have an abstraction, so-called *R^2-divisible R^3-spaces*. The two geometrical structures are equal to the classical R^3-space without vertical lines. We call the classical R^3-space without vertical lines as *the classical R^2-divisible R^3-space*. In the classical R^2-divisible R^3-space on $R^3 = \{(x,y,z) : x,y,z \in R\}$ we can consider two projections on $<x,y>$-coordinate plane and $<x,z>$-coordinate plane, respectively. We get also two affine planes on $<x,y>$-coordinate plane and $<x,z>$-coordinate plane, respectively, where the line set is the set of all projections of lines in R^3 on $<x,y>$-coordinate plane and $<x,z>$-coordinate plane, respectively. In a series of papers of Betten and Knarr we have lots of examples of R^2-divisible R^3-spaces which are induced from 4-dimensional affine planes.

After inspection of all flexible 4-dimensional translation planes we see: the induced R^2-divisible R^3-spaces by translation planes are the classical R^2-divisible R^3-spaces. The affine planes in **[Bet84]** give rise to R^2-divisible R^3-spaces which are non-classical, that is, if we consider two projections on $<x,y>$-coordinate plane and $<x,z>$-coordinate plane, respectively, one of the projection is the classical affine plane and the other is a Moulton plane. Knarr studied 4-dimensional shift planes. The shift planes give also rise to non-classical R^2-divisible R^3-spaces. In this case one of the projection is the classical affine plane and the other is a 2-dimensional shift plane. Conversely, we can reconstruct R^2-divisible R^3-spaces in the following manner. An R^2-plane $(R^2, \mathcal{L})$ is called *standard* if all vertical lines $\{x\} \times R$ are in $\mathcal{L}$ and the other lines $l \in \mathcal{L}$ can be written as the graph(f) of a continuous mapping $f : R \longrightarrow R$. Let $E_1 = (R^2, \mathcal{L})$ and $E_2 = (R^2, \Im)$ be two standard R^2-planes. If we identify E_1 with the horizontal plane $z = 0$ and E_2 with the vertical plane $y = 0$ in $R^3 = \{(x,y,z) : x,y,z \in R\}$, respectively. We define on R^3 the following curves as lines: $f \times g := \{(x, f(x), g(x)) : x \in R\}$, where f and g are lines in E_1 and E_2, respectively. Let $\mathcal{L} \times \Im = \{f \times g : f \in \mathcal{L}, g \in \Im\}$ and $\Lambda = \{\{x\} \times R^2 : x \in R\}$. The incidence structure $(R^3, \mathcal{L} \times \Im, \Lambda)$ is called the product space of two planes E_1 and E_2. Denote by in particular $(R^3, \mathcal{L} \times \Im, \Lambda)_{E_1 \times E_2}$, and it is an R^2-divisible R^3-space. In this viewpoint one of the induced R^2-divisible R^3-spaces in **[Bet84]** are the product spaces of the classical R^2-plane and a Moulton plane. The R^2-divisible R^3-spaces which are induced from 4-dimensional shift planes are the classical R^2-plane and a 2-dimensional shift plane. These obsertions give the motive of the study of R^2-divisible R^3-spaces.

For the purpose of a systematic study of R^2-divisible R^3-spaces we are at first interested in topological structure of these geometries. In **[Bet81]** Betten studied R^3-spaces. An R^3-space in the sense of Betten **[Bet81]** is an incidence structure $(R^3, \mathcal{L})$ which satisfies the following three axioms; (1) each pair p, q of distinct points is contained in a unique line $p \vee q \in \mathcal{L}$, (2) each line $l \in \mathcal{L}$ is closed in R^3 and homeomorphic to the real line R and (3) the mapping $\vee : R^3 \times R^3 - \triangle \longrightarrow \mathcal{L}$ is continuous, where $\triangle := \{(p,p) : p \in R^3\}$ is the diagonal and $\mathcal{L}$ carries the topology of Hausdorff-convergence. A plane in a R^3-space $(R^3, \mathcal{L})$ is a closed subset $E \subseteq R^3$ which is homeomorphic to R^2 such that $p \vee q \subseteq E$ for each pair of distinct points $p, q \in E$. Obviously, $(E, \mathcal{L}_E)$ is an R^2-plane, where $\mathcal{L}_E := \{l \in \mathcal{L} : l \subseteq E\}$. It is a generalization of the classical R^3-space. In the viewpoint of Betten, the geometrical structure of R^2-divisible R^3-spaces is closely related to the case of R^3-spaces, that is, it can be regarded as a variation of R^3-spaces.

In topological space geometries there can be besides the joining operation of distinct pair of points the following join and intersection operations:

(G1) $\vee : D \subseteq R^3 \times R^3 \times R^3 \longrightarrow \mathcal{E}$,

(G2) $\vee : D \subseteq R^3 \times \mathcal{L} \longrightarrow \mathcal{E}$,
(G3) $\wedge : D \subseteq \mathcal{L} \times \mathcal{L} \setminus \{(g,h) : |g \wedge h| \neq 1\} \longrightarrow R^3$,
(G4) $\wedge : D \subseteq \mathcal{E} \times \mathcal{L} \setminus \{(E,g) : |E \wedge g| \neq 1\} \longrightarrow R^3$,
(G5) $\wedge : D \subseteq \mathcal{E} \times \mathcal{E} \setminus \{(E,F) : E = F \text{ or } E \wedge F = \emptyset\} \longrightarrow \mathcal{L}$,
where $\mathcal{E}$ is the set of all planes and D is a suitable domain.

We investigate a suitable topology on the line set such that the above geometrical operations are continuous. We claim some additional condition, so-called bounded condition, in order to guarantee of continuity of these geometrical operations. In this paper we define a few topologies on the line set $\mathcal{L}$(see 1.7 and 1.9). In a topological R^2-divisible R^3-space if the two topologies H and COT are equal, then the above operations are also continuous, and two topologies H and COT on the line set are coincide if and only if the space satisfies the bounded condition in most cases. In this paper we will study basic properties of these space geometries which need to develope of topological R^2-divisible R^3-spaces.

Let X be a topological space and $(A_n)_{n\in N}$ be a sequence of subsets of X. Denote by $\liminf A_n$ the set of all limit points of sequences $(a_n)_{n\in N}$ with $a_n \in A_n$, and denote by $\limsup A_n$ the set of all accumulation points of such sequences. The sequence $(A_n)_{n\in N}$ is Hausdorff-convergent to $A \subseteq X$ if and only if $\liminf A_n = \limsup A_n = A$ (written by $\lim A_n = A$ or $A_n \longrightarrow A$).

$\mathcal{P}^n$ denote a topological space which is homeomorphic to R^n. A partition $\Lambda := \{S_i : i \in \mathcal{A}\}$ in $\mathcal{P}^n$ ($n \geq 2$) is divisible if each S_i is closed in $\mathcal{P}^n$ and homeomorphic to $\mathcal{P}^{n-1}$.

Definition 1.1. Let $\mathcal{L}$ be a system of subsets of $\mathcal{P}^3$, and let $\Lambda = \{S_i : i \in \mathcal{A}\}$ be a divisible partition in $\mathcal{P}^3$. The elements of $\mathcal{P}^3$ are called points, and the elements of $\mathcal{L}$ are called lines. We say that $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ is an R^2-divisible R^3-space if the following axioms hold:

(1) Each line is closed in the topological space $\mathcal{P}^3$ and is homeomorphic to R.

(2) For all $x \in S_i, y \in S_j$ with $i \neq j$ there is a unique line $l \in \mathcal{L}$ with $x, y \in l$. For $i = j$ there are no lines $l \in \mathcal{L}$ with $x, y \in l$.

The joining line in (2) is denoted by $l = x \vee y$. If $\Lambda = \{S_i : i \in \mathcal{A}\}$ is a divisible partition in $\mathcal{P}^2$, then we can similarly define an R-divisible R^2-plane $(\mathcal{P}^2, \mathcal{L}, \Lambda)$. If we think the partition as the added line set, we can regard an R-divisible R^2-plane as an R^2-plane (Salzmann-plane).

Definition 1.2. Let $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ be an R^2-divisible R^3-space. A subset $E \subseteq \mathcal{P}^3$ is called a plane of $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ if the following conditions hold:

(1) E is closed in $\mathcal{P}^3$ and homeomorphic to R^2,

(2) $(E, \mathcal{L}_E, \Lambda_E)$ is an R-divisible R^2-plane, where $\mathcal{L}_E := \{l \in \mathcal{L} : l \subseteq E\}$ and $\Lambda_E = \{E \cap S_i : i \in \mathcal{A}\}$ is a divisible partition in E.

In an R^2-divisible R^3-space $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ let $\mathcal{E}$ denote the set of all planes of $(\mathcal{P}^3, \mathcal{L}, \Lambda)$. Furthermore, let $\mathcal{L}_s := \{E \cap S_i : E \in \mathcal{E} \text{ and } S_i \in \Lambda\}$ and $\overline{\mathcal{L}} := \mathcal{L} \cup \mathcal{L}_s$. If the join operation from two points to a unique line is written by $\vee$, then $\vee$ is the following mapping:

$$\vee : \mathcal{P}^3 \times \mathcal{P}^3 \setminus \cup_{i\in\mathcal{A}}(S_i \times S_i) \longrightarrow \mathcal{L}.$$

Let $\mathcal{P}^3_{S_i} \times \mathcal{P}^3_{S_i}$ denote the set $\mathcal{P}^3 \times \mathcal{P}^3 \setminus \cup_{i\in\mathcal{A}}(S_i \times S_i)$.

Lemma 1.3. *Let $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ be an R^2-divisible R^3-space, and let (l_n) be a sequence in $\overline{\mathcal{L}}$ with $\limsup l_n \neq \emptyset$. Then $|\limsup l_n| = \infty$.*

Proof. Let $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ be an R^2-divisible R^3-space, and let $\gamma : \mathcal{P}^3 \longrightarrow R^3$ be a homeomorphism. Then $(R^3, \mathcal{L}^\gamma, \Lambda^\gamma)$ is also an R^2-divisible R^3-space, where $\mathcal{L}^\gamma := \{\gamma(l) : l \in \mathcal{L}\}$ and $\Lambda^\gamma := \{\gamma(S_i) : S_i \in \Lambda\}$. Let $x \in \limsup l_n$, then $\gamma(x) \in \limsup \gamma(l_n)$. For $m \in N$ let $S(\gamma(x), 1/m) = \{p \in R^3 : ||\gamma(x) - p|| = 1/m\}$. Since each line $\gamma(l)$ is connected, unbounded and homeomorphic to R, hence $S(\gamma(x), 1/m) \cap \gamma(l_n) \neq \emptyset$ for infinitely many $n \in N$. Since $S(\gamma(x), 1/m)$ is compact, it follows that $S(\gamma(x), 1/m) \cap \limsup \gamma(l_n) \neq \emptyset$. Therefore, $\gamma^{-1}(S(\gamma(x), 1/m)) \cap \limsup l_n \neq \emptyset$. $\square$

Hausdorff metric: Let $\mathcal{U}$ be the set of the non-empty closed subsets of $\mathcal{P}^3$. We define on $\mathcal{U}$ the following metric:

$$\delta : \mathcal{U} \times \mathcal{U} \longrightarrow R : (A, B) \longrightarrow \sup\{|d(x, A) - d(x, B)|e^{-d(p,x)} : x \in \mathcal{P}^3\},$$

where d is the metric on $\mathcal{P}^3$ and $p \in \mathcal{P}^3$.

Theorem 1.4. *δ is a metric on $\mathcal{U}$. Let $(A_n)_{n\in N}$ be a sequence in $\mathcal{U}$ and $A \in \mathcal{U}$. Then $(A_n)_{n\in N}$ converges to A in $(\mathcal{U}, \delta)$ if and only if* $\lim A_n \longrightarrow A$.

Proof. [**Bu65**, Section 3]. $\square$

Since the line set $\mathcal{L}$ is a subset of $\mathcal{U}$, we can take on $\mathcal{L}$ the induced topology of $\mathcal{U}$, and it is called the *natural topology* and written by H. *From now on we always assume that $\mathcal{L}$, $\mathcal{L}_s$, $\overline{\mathcal{L}}$ and $\mathcal{E}$ have the induced topology of $\mathcal{U}$.*

Definition 1.5. An R^2-divisible R^3-space is called *topological* if the join map $\vee$ is continuous with the natural topology H on the line set $\mathcal{L}$.

This definition has quite intuitive interpretation: roughly speaking, it means that the points on the lines l_n approximate precisely the points on l, and that no parts of l_n can stay away from l. In the case R^2-planes(also R-divisible R^2-planes) with this notion of convergence, the join map of every R^2-plane (also R-divisible R^2-plane) is sequentially continuous [**Sal95**, chapter 3], but not in the case R^2-divisible R^3-spaces (also R^3-plane). We have a counterexample which is a modification of the counterexample in [**Bet87**].

Example 1.6. The *classical R^3-space* is the real affine space: the point set is $R^3 = \{(y, u, v) : y, u, v \in R\}$, and the lines are all 1-dimensional affine subspaces of R^3. We replace $< y, v >$-coordinate plane by an affine Moulton plane $\mathcal{M}_k$, $k > 1$ (see [**Sal95**, chapter 3]). Then we get an R^3-space which is not sequentially continuous. We choose $\Lambda = \{\{y\} \times R^2 : y \in R\}$ and take $\mathcal{L}$ as the set of all lines in the above defined R^3-space which are not contained in $\{y\} \times R^2$. Then $(R^3, \mathcal{L}, \Lambda)$ is an R^2-divisible R^3-space, which is obviously not sequentially continuous.

Definition 1.7. (1) The *final topology* F on $\mathcal{L}$ is the largest topology on $\mathcal{L}$ for which the mapping $\vee : \mathcal{P}^3 \times \mathcal{P}^3 \setminus \cup_{i\in\mathcal{A}}(S_i \times S_i) \longrightarrow \mathcal{L}$ is continuous.
(2) The *open join topology OJ* is generated by the subbasic elements $O_1 \vee O_2 = \{p \vee q \in \mathcal{L} : p \in O_1, q \in O_2\}$, where O_1, O_2 are disjoint open sets in $\mathcal{P}^3$.
(3) The *open meet topology OM* is defined by the subbasic sets $M_O = \{l \in \mathcal{L} : l \cap O \neq \emptyset\}$, where O is an open set in $\mathcal{P}^3$.

Theorem 1.8. *Let $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ be a topological R^2-divisible R^3-space. Then:*

(1) The topologies H, F, OJ, OM for $\mathcal{L}$ coincide.

(2) The join map $\vee : \mathcal{P}^3 \times \mathcal{P}^3 \setminus \cup_{i \in \mathcal{A}}(S_i \times S_i) \longrightarrow \mathcal{L}$ is open.

Proof. (1) We show that $OJ \subseteq OM \subseteq H \subseteq F \subset OJ$. Since $O_1 \vee O_2 = M_{O_1} \cap M_{O_2}$, the first inclusion holds. If O is open in $\mathcal{P}^3$, then $\mathcal{L} \setminus M_O = \{l \in \mathcal{L} : l \cap O = \emptyset\}$ is H-closed. Assume that $\mathcal{L} \setminus M_O$ is not H-closed. Then there exists a sequence (l_n) in $\mathcal{L} \setminus M_O$ which converges to $l \in M_O$. Choose a point $a \in l \cap O$. Then there exists a sequence (a_n) with $a_n \in l_n$ which converges to a. For sufficiently large $n \in N$ a_n lie in O, a contradiction to $l_n \in \mathcal{L} \setminus M_O$. Consequently, $OM \subseteq H$. The definition 1.5 implies that $H \subseteq F$. For the inclusion $F \subseteq OJ$, let $U \subseteq \mathcal{L}$ be F-open, and let $l = p_1 \vee p_2 \in U$. Then $\vee^{-1}(U) = \{(x, y) \in \mathcal{P}^3 \times \mathcal{P}^3 \setminus \cup_{i \in \mathcal{A}}(S_i \times S_i) : x \vee y \in U\}$ is open in $\mathcal{P}^3 \times \mathcal{P}^3 \setminus \cup_{i \in \mathcal{A}}(S_i \times S_i)$. Using the normality of $\mathcal{P}^3$, we can show that $\cup_{i \in \mathcal{A}}(S_i \times S_i)$ is closed in $\mathcal{P}^3 \times \mathcal{P}^3$. Consequently, $\vee^{-1}(U)$ is open in $\mathcal{P}^3 \times \mathcal{P}^3$, hence we can find two disjoint open sets O_i containing p_i such that $l \in O_1 \vee O_2 \subseteq U$.
(2) is immediate from (1). □

If $S_i \in \Lambda$, then $\mathcal{P}^3 \setminus S_i$ has precisely two components (denoted by S_i^+, S_i^-), of which S_i is the common (topological) boundary (see [**Mas80**, III. §6]). If we choose more $S_j \in \Lambda$ with $i \neq j$, then one of the components of $\mathcal{P}^3 \setminus S_i$ (for example S_i^+) is also separated by S_j. We can choose one of the components of $S_i^+ \setminus S_j$ which contains S_i and S_j as the topological boundaries. Let $\overline{S_{ij}^{+-}}$ denote the union of the components which has S_i and S_j as topological boundaries, S_i and S_j. We identify $\overline{S_{ii}^{+-}}$ with simply S_i

Definition 1.9. The *open partition meet topology OPM* on $\mathcal{L}$ is generated by the subbasis elements $S_i^O = \{l \in \mathcal{L} : l \cap O \neq \emptyset\}$, where O is an open set in S_i and $S_i \in \Lambda$.
The *compact open topology COT* on $\mathcal{L}$ is defined by the subbasis elements $S_{ij}^O = \{l \in \mathcal{L} : l \cap S_i = \{x\}, l \cap S_j = \{y\}, [x, y] \subseteq \overline{S_{ij}^{+-}} \cap O\}$, where S_i, $S_j \in \Lambda$, $\overline{S_{ij}^{+-}}$ is the union of the component which has S_i and S_j as topological boundaries, S_i and S_j, and O is open in $\mathcal{P}^3$.

Lemma 1.10. *Let $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ be a topological R^2-divisible R^3-space. Then $H \subseteq OPM \subseteq COT$.*

Proof. By theorem 1.8, we show that $OJ \subseteq OPM$. Let $p_1 \vee p_2 \in O_1 \vee O_2 \in OJ$. Then there exist $S_1, S_2 \in \Lambda$ with $p_1 \in S_1, p_2 \in S_2$, and $p_1 \vee p_2 \in S_1^{O_1} \cap S_2^{O_2} \subseteq O_1 \vee O_2$.

This implies that $OJ = H \subseteq OPM$.
By definition of COT, it is clear that $OPM \subseteq COT$. □

2. BASIC PROPERTIES

Let $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ be a topological R^2-divisible R^3-space. Since lines are homeomorphic to R, there is a natural notion of intervals in lines. If $l \in \mathcal{L}$ is a line and $p, q \in l$ are two (not necessarily distinct) points on l, then we denote the *interval* which consists of all points on l between p and q by the symbol $[p, q]$. The *open interval* between p and q is defined as $(p, q) := [p, q] \setminus \{p, q\}$. $\overrightarrow{p \vee q}$ is an emanating ray from p to q. A subset $K \subseteq \mathcal{P}^3$ is called *convex* if it contains with each pair $p, q \in K$ also the interval $[p, q]$. $\mathcal{E}$ is the set of planes of $(\mathcal{P}^3, \mathcal{L}, \Lambda)$, and $\mathcal{L}_s = \{E \cap S_i : E \in \mathcal{E}$ and $S_i \in \Lambda\}$, $\overline{\mathcal{L}} = \mathcal{L} \cup \mathcal{L}_s$. If $a, b, c \in \mathcal{P}^3$ are three non-collinear but coplanar points, then we denote the plane which contains all three points by

the symbol $a \vee b \vee c$. If $E \in \mathcal{E}$ is a plane and $l \in \mathcal{L}$ is a line with $|E \cap l| = 1$, then we denote the point of intersection of E and l by $E \wedge l$. If $E_1, E_2 \in \mathcal{E}$ are two planes with $E \cap F \neq \emptyset$, then their intersection is a line (see lemma 2.2), which will be denoted by $E \wedge F$.
If E is a plane of $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ or an element of partition Λ, then $\mathcal{P}^3 \setminus E$ has precisely two components, of which E is the common (topological) boundary (see [**Mas80**, III. §6]). The two components are called *halfspaces*.

Definition 2.1. Let $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ be an R^2-divisible R^3-space. Given two subsets $A, B \subseteq \mathcal{P}^3$, we define

$$[A, B] := \bigcup_{a \in A, b \in B} [a, b],$$

i.e., $[A, B]$ is the set of all points between A and B.

Let $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ be an R^2-divisible R^3-space. Then we will consider the following additional axioms:

(B) (Bounded-axiom) If $A, B \subseteq \mathcal{P}^3$ are compact, then $\overline{[A, B]}$ is also compact.

(Exc) (Continuously existence condition for planes) Given three points $a, b, c \in \mathcal{P}^3$ with $(a, b) \in \mathcal{P}^3_{S_i} \times \mathcal{P}^3_{S_j}$, $c \in S_c$ and $c \notin a \vee b$, then there exists a continuous mapping $\varphi : J \longrightarrow \mathcal{E}$ such that $c \vee z \subseteq \varphi(z)$ for all $z \in J$, where $J = [a, b]$ if $S_c \cap [a, b] = \emptyset$, $J = [a, b] \setminus \{w\}$ if $S_c \cap [a, b] = \{w\}$.

Lemma 2.2. *Let $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ be a topological R^2-divisible R^3-space. Then:*

(1) If $E \subseteq \mathcal{P}^3$ is a plane and $l \in \mathcal{L}$ is a line with $l \wedge E = \{y\}$, then the two components of $l \setminus \{y\}$ are contained in different halpspaces of E.

(2) If $E \subseteq \mathcal{P}^3$ is a plane and $l_s \in \mathcal{L}_s$ is a line with $l_s \wedge E = \{y\}$, then the two components of $l_s \setminus \{y\}$ are contained in different halpspaces of E.

(3) If E_1 and E_2 are two planes with $E_1 \neq E_2$ and $E_1 \cap E_2 \neq \emptyset$, then $E_1 \cap E_2 \in \overline{\mathcal{L}}$.

Proof. (1) Let l be a line and E be a plane with $l \wedge E = \{y\}$, and let E^+ and E^- be the two connected components of $\mathcal{P}^3 \setminus E$, $E^+ \cap l \neq \emptyset$. Let $x \in E^+$ with $l = x \vee y$ and a sequence $(y_n) \subseteq E^-$ with $y_n \to y$. Then it is clear that for sufficiently large $n \in N$ $y_n \notin S_i$, where $x \in S_i \in \Lambda$. Therefore, $\limsup(x \vee y_n) = l$. Let $a_n := (x \vee y_n) \wedge E$, then $\overrightarrow{a_n \vee y_n} \subseteq E^- \cup E$. Let $S := \{p \in \mathcal{P}^3 : ||y - p|| = 1\} \cap (E \cup E^-)$, then for infinitely many $n \in N$ $(\overrightarrow{a_n \vee y_n}) \cap S \neq \emptyset$. Since S is compact, the set of intersection points of $\overrightarrow{a_n \vee y_n}$ and S has an accumulation point, i.e., there is a $a \in S \cap \limsup(\overrightarrow{a_n \vee y_n}) \subseteq S \cap l$. Since $y \notin S$ and $|\ l \wedge E\ | = 1$, therefore, $a \in E^-$, so that $E^- \cap l \neq \emptyset$.
(2). Let $l_s \in \mathcal{L}_s$ be a line and E be a plane with $l_s \wedge E = \{y\}$. Since $l_s \in \mathcal{L}_s$, there exists a plane F with $l_s \subseteq F$. Let E^+ and E^- be the two connected components of $\mathcal{P}^3 \setminus E$, $E^+ \wedge l_s \neq \emptyset$. Choose $x \in E^+$ with $l_s = x \vee y$. Let $\mathcal{L}^F_y := \{l \in \mathcal{L} : y \in l \subseteq F\}$. Then by (1) each line $l \in \mathcal{L}^F_y$ and E are transversal, we can choose a sequence $(y_n) \subseteq E^- \cap F$ with $y_n \longrightarrow y$. Since F with lines $S_i \cap F, i \in \mathcal{A}$ is an R^2-plane, it is $\limsup(x \vee y_n) = l_s$. Then the rest part of proof is the same in proof (1).
(3). By definition 1.2, we can regard E_1 and E_3 as R^2-planes. The assertion can be proved in proofs [**Si85**, 1.2.4. Korollar 10 und 11]. □

Lemma 2.3. *Let $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ be a topological R^2-divisible R^3-space, and let E be a plane of $(\mathcal{P}^3, \mathcal{L}, \Lambda)$. If E^+ is one of the components of E, then E^+ and $\overline{E^+}$ are convex.*

Proof. Let $x, y \in E^+$ with $(x, y) \in \mathcal{P}^3_{S_i} \times \mathcal{P}^3_{S_j}$. Assume that $[x, y] \cap E \neq \emptyset$, then there exists $z \in [x, y]$ with $x, y \neq z \in E$. Now the points x and y lie on different sides of $(x \vee y) \setminus \{z\}$, and by lemma 2.2, the points x and y lie on different sides of E, a contradiction to $x, y \in E^+$. Therefore, $[x, y] \subseteq \mathcal{P}^3 \setminus E$, and since $[x, y]$ is connected, it follows that $[x, y] \subseteq E^+$. Therefore, E^+ is convex. Since $\partial E^+ = E$, hence $\overline{E^+} = E^+ \cup E$. Let $x, y \in \overline{E^+}$ with $(x, y) \in \mathcal{P}^3_{S_i} \times \mathcal{P}^3_{S_j}$. If $x, y \in E$ or $x, y \in E^+$, then $[x, y] \subseteq E^+ \cup E = \overline{E^+}$. Hence we may assume that $x \in E$ and $y \in E^+$. Then $(x, y] \subseteq \mathcal{P}^3 \setminus E$ is connected, therefore, $(x, y] \subseteq E^+$, and also $[x, y] = \{x\} \cup (x, y] \subseteq E \cup E^+ = \overline{E^+}$. □

Lemma 2.4. *Let $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ be a topological R^2-divisible R^3-space, and let $l \in \mathcal{L}$, $S_i \in \Lambda$. Then:*

(1) If $l \cap S_i = \{y\}$, then the two components of $l \setminus \{y\}$ are contained in different halpspaces of S_i.

(2) If S_i^+ is one of the components of S_i, then S_i^+ and $\overline{S_i^+}$ are convex.

Proof. Since $|l \cap S_i| \leq 1$ for $l \in \mathcal{L}, S_i \in \Lambda$, the assertation can be proved as in proof of lemmas 2.2 and 2.3.

Theorem 2.5. *(Order-condition) Let $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ be a topological R^2-divisible R^3-space. If the points sequences $(a_n)_{n\in N}, (b_n)_{n\in N}, (c_n)_{n\in N}$ have mutually distinct limits a, b, c. If $b_n \in [a_n, c_n]$ for all $n \in N$, then it is also that $b \in [a, c]$.*

Proof. Since $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ is topological, it is clear that a, b, c are collinear. Since the intervals $[a_n, c_n]$ and $[a, c]$ are defined, let $l_n = a_n \vee b_n$ and $l = a \vee b$. Suppose that $b \notin [a, c]$. Without loss of generality, we may assume that $c \in [a, b)$. Choose a point $p \in (c, b)$, and let $p \in S_i$. Let S_i^+ and S_i^- be two halfspaces of $\mathcal{P}^3 \setminus S_i$ and $b \in S_i^-$. If $a \in S_i^-$, then $p \in (c, b) \subseteq [a, b] \subseteq S_i^-$. Since S_i^- is convex and $p \in S_i$, a contradiction. Hence $a \in S_i^+$. Similiarly, $c \in S_i^+$. Since S_i^+ and S_i^- are open in $\mathcal{P}^3$ for sufficiently large $n \in N$, $a_n, c_n \in S_i^+$ and $b_n \in S_i^-$. Since S_i^+ is convex, it follows that $b_n \in [a_n, c_n] \subseteq S_i^+$, a contradiction. Therefore, $b \in [a, c]$. □

Lemma 2.6. *Let $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ be a topological R^2-divisible R^3-space satisfying the axiom (B). Then:*

(1) Let (a_n) and (b_n) be two sequences in $\mathcal{P}^3$ which converge both to a point $p \in \mathcal{P}^3$. Let (p_n) be a sequence in $\mathcal{P}^3$ with $p_n \in [a_n, b_n]$ for all $n \in N$. Then $p_n \longrightarrow p$.

(2) Let U be a neighborhood of a point $p \in \mathcal{P}^3$. Then there exists a neighborhood V of p with $[V, V] \subseteq U$.

Proof. The assertations can be proved in similar manner of the proofs of [10, lemma 2] □

Lemma 2.7. *Let $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ be a topological R^2-divisible R^3-space satisfying the axiom (B). Then $H = OPM = COT$.*

Proof. We show that $OPM \subseteq OJ = H$. Let $l \in S_i^U \in OPM$, $\{p\} = l \cap U$ and V be an open set in $\mathcal{P}^3$ such that $S_i \cap V = U$. Let $(V_n)_{n\in N}$ be decreasing sequence of neighborhoods of p such that $\{V_n : n \in N\}$ is a neighborhood basis at p. Let S_i^+ and S_i^- be the two connected components of $\mathcal{P}^3 \setminus S_i$, and let $V_n \cap (\mathcal{P}^3 \setminus S_i) = V_n^+ \cup V_n^-$ such that $V_n^+ \subseteq S_i^+$ and $V_n^- \subseteq S_i^-$.

We will show that there exists a number $n \in N$ such that $l \in V_n^+ \vee V_n^- \subseteq S_i^U$. Suppose that it is not true; for each $n \in N$ we can choose $p_n \in V_n^+, q_n \in V_n^-$ such that $(p_n \vee q_n) \cap U = \emptyset$. Since (p_n) and (q_n) converge to p, and by the axiom (B) and lemma 2.6, $(p_n \vee q_n) \cap S_i$ converges to p. Hence for sufficiently large $n \in N$ $(p_n \vee q_n) \cap S_i \subseteq V \cap S_i = U$, a contradiction.

We show that $COT \subseteq OPM$. Let $l \in S_{ij}^O \in COT$, and let $\{x\} = S_i \cap l, \{y\} = S_j \cap l$. If $i = j$, then it is clear that $S_{ij}^O \in OPM$. Hence let $i \neq j$. Let $(V_n(x))_{n \in N}$ and $(W_n(y))_{n \in N}$ be two decreasing sequences of neighborhoods of x and y such that $(V_n(x))_{n \in N}$ and $(W_n(y))_{n \in N}$ are neighborhood basis at x and y, respectively. Then we will show that there exists a number $n \in N$ such that $l \in S_i^{V_n(x)} \cap S_j^{W_n(x)} \subseteq S_{ij}^O$. If we assume that it is not true; for each $n \in N$ there exist $x_n \in V_n(x)$ and $y_n \in W_n(y)$ such that $[x_n, y_n] \nsubseteq S_{ij}^O$. For each $n \in N$ choose a point $p_n \in [x_n, y_n]$ such that $p_n \notin S_{ij}^O$. Then by the axiom (B) and theorem 2.4, the sequence (p_n) has an accmulation point on $[x, y]$, a contradiction. □

Theorem 2.8. *Let $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ be a topological R^2-divisible R^3-space. Let $H = COT$ and $l \cap S_i \neq \emptyset$ for each $l \in \mathcal{L}$ and $S_i \in \Lambda$. If $A, B \subseteq \mathcal{P}^3$ are compact with $A \times B \subseteq \mathcal{P}_{S_i}^3 \times \mathcal{P}_{S_j}^3$, then $\overline{[A, B]}$ is also compact.*

Proof. We assume that $\overline{[A, B]}$ is not compact. Then there exists a sequence $((a_n, b_n))$ in $A \times B$ and a sequence $(p_n), p_n \in [a_n, b_n] \setminus \{a_n, b_n\}$ such that (p_n) is unbounded. Since $A \times B$ is compact, there exists a convergent subsequence $((a_{n_k}, b_{n_k}))$ which converges to a point $(a, b) \in A \times B$. Since $A \times B \subseteq \mathcal{P}_{S_i}^3 \times \mathcal{P}_{S_j}^3$, $a \vee b$ is defined. Let $a \in S_a$ and $b \in S_b$. Since for each $l \in \mathcal{L}, S_i \in \Lambda$ it is $l \cap S_i \neq \emptyset$, and $H = OPM$. Without loss of generality, let $a_{n_k} \in S_a$ and $b_{n_k} \in S_b$. Choose a relative compact open set U which contains $[a, b]$. Since $H = COT$ and $a_{n_k} \vee b_{n_k} \longrightarrow a \vee b$, there exists N such that for all $k \geq N$ $\cup([a_{n_k}, b_{n_k}]) \cup [a, b] \subseteq U$. Therefore, $(\cup[a_{n_k}, b_{n_k}]) \cup [a, b]$ is bounded, a contradiction. □

Lemma 2.9. *Let $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ be a topological R^2-divisible R^3-space. Let $S_i \in \Lambda$ and (l_n) be a sequence in $\mathcal{L}$ such that $l_n \longrightarrow l \in \mathcal{L}$. Let $H = OPM$ and $|S_i \cap l| = 1$. Then $|S_i \cap l_n| = 1$ for sufficiently large $n \in N$, and $S_i \cap l_n \longrightarrow S_i \cap l$.*

Proof. Let $p \in S_i \cap l$ and U be an open neighborhood of p which is relative compact in S_i. By the assumption $H = OPM$, S_i^U is an open neighborhood of l. Since l_n converges to l, it follows that $l_n \in S_i^U$ for sufficiently large $n \in N$, so that $l_n \cap U \neq \emptyset$. Since U is relative compact, the sequence $(l_n \cap U)$ has an accumulation point. If x is a limes of convergent subsequence of this sequence, then $x \in S_i \cap \limsup l_n = S_i \cap l = p$. Therefore, $S_i \cap l_n \longrightarrow S_i \cap l$. □

Theorem 2.10. *Let $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ be a topological R^2-divisible R^3-space. Let $H = OPM$ and for each $l \in \mathcal{L}$ and $S_i \in \Lambda$ $l \cap S_i \neq \emptyset$. Then:*

(1) The line space $\mathcal{L}$ is homeomorphic to R^4.

(2) For $p \in \mathcal{P}^3$ the line pencil $\mathcal{L}_p = \{l \in \mathcal{L} : p \in l\}$ is homeomorphic to R^2.

Proof. (1) Let $S_i, S_j \in \Lambda$ such that $S_i \cap S_j = \emptyset$. Let $\mathcal{L}_{S_i} = \{l \in \mathcal{L} : |S_i \cap l| = 1\}$ and $\mathcal{L}_{S_j} = \{l \in \mathcal{L} : |S_j \cap l| = 1\}$. Set $\mathcal{L}_{i,j} = \mathcal{L}_{S_i} \cap \mathcal{L}_{S_j}$ By the assumption, it is clear that $\mathcal{L}_{i,j} = \mathcal{L}$. By lemma 2.9, the following maps are well defined and continuous:

$$\Phi : \mathcal{L}_{i,j} \longrightarrow S_i \times S_j : l \longrightarrow (S_i \cap l, S_j \cap l),$$

$$\Psi : S_i \times S_j \longrightarrow \mathcal{L}_{i,j} : (x, y) \longrightarrow x \vee y.$$

It is clear that $\Psi \circ \Phi = id$ and $\Phi \circ \Psi = id$, i.e., Φ is a homeomorphism.
(2) Choose $S_i \in \Lambda$ such that $p \notin S_i$. By the assumption and lemma 2.9, the following maps are well defined and continuous:

$$\Phi : \mathcal{L}_p \longrightarrow S_i : l \longrightarrow l \cap S_i,$$

$$\Psi : S_i \longrightarrow \mathcal{L}_p : x \longrightarrow p \vee x.$$

It is clear that $\Psi \circ \Phi = id$ and $\Phi \circ \Psi = id$, i.e., Φ is a homeomorphism. □

3. CONTINUITY OF THE GEOMETRIC OPERATIONS

In this section, let $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ be a topological R^2-divisible R^3-space. Let E denote the space of all planes of $(\mathcal{P}^3, \mathcal{L}, \Lambda)$. Furthermore, let $\mathcal{L}_s = \{E \cap S_i : E \in \mathcal{E} \text{ and } S_i \in \Lambda\}$ and $\overline{\mathcal{L}} = \mathcal{L} \cup \mathcal{L}_s$.

Lemma 3.1. *Let $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ hold the axiom (Exc). Then for each line $l \in \mathcal{L}$ there is a plane E with $l \subseteq E$.*

Proof. Let $l \in \mathcal{L}$ and $[c,p] \subseteq l$ with $c \neq p$, $c \in S_c \in \Lambda$. Choose a line $g \in \mathcal{L}$ with $p \in g$ and an interval $[a,b] \subseteq g$ with $[a,b] \cap S_c = \emptyset$. Then by the axiom (Exc), there is a continuous mapping $\varphi : [a,b] \longrightarrow \mathcal{E}$ with $c \vee z \subseteq \varphi(z)$ for all $z \in [a,b]$. It is clear that $l = c \vee p \subseteq \varphi(p)$. □

Lemma 3.2. *Let $H = COT$. Let $E \subseteq \mathcal{P}^3$ be a plane and (l_n) be a sequence in $\mathcal{L}$ with $l_n \longrightarrow l \in \mathcal{L}$. Let $|E \wedge l| = 1$. Then for sufficiently large $n \in N$ $|E \wedge l_n| = 1$, and $E \wedge l_n \longrightarrow E \wedge l$.*

Proof. Let E^+ and E^- be the components of $\mathcal{P}^3 \setminus E$. By lemma 2.2, the plane E and the line l are transversal, hence there exist $a \in l \wedge E^+$ and $b \in l \wedge E^-$, and let $a \in S_a$ and $b \in S_b$. Since $l_n \longrightarrow l$ and $H = OPM = COT$, we can choose sequences $(a_n), (b_n) \in l_n$ with $a_n \in l_n \cap S_a$ and $b_n \in l_n \cap S_b$ which converge to a and b, respectively. Since E^+ and E^- are open in $\mathcal{P}^3$, for sufficiently large $n \in N$ $a_n \in E^+$ and $b_n \in E^-$. Therefore, for sufficiently large $n \in N : |E \wedge l_n| = 1$ and $E \wedge l_n \in [a_n, b_n]$. By the assumption $H = COT$, the sequence $(E \wedge l_n)$ is bounded. If x is a limes of convergent subsequence of this sequence, then $x \in E \cap \limsup l_n = E \wedge l$. Therefore, $E \wedge l_n \longrightarrow E \wedge l$. □

Lemma 3.3. *Let $H = COT$. Let $E, F \subseteq \mathcal{P}^3$ be planes with $E \wedge F \in \mathcal{L}$. If $(E_n)_{n \in N}$ is a sequence of planes with $E_n \longrightarrow E \in \mathcal{E}$, then for sufficiently large $n \in N$ $E_n \wedge F \in \mathcal{L}$, and $E_n \wedge F \longrightarrow E \wedge F$.*

Proof. Let $l = x \vee y = E \wedge F$ with $x \in S_i, y \in S_j, i \neq j$. Choose a point $a \in E \setminus (E \wedge F)$ with $a \notin S_i \cup S_j$, and let $g := a \vee x$ and $h := a \vee y$. Since E_n converges to E, there are sequences $(x_n), (y_n), (a_n)$ with $x_n, y_n, a_n \in E_n$, which converge to x, y and a, respectively. It is clear that for sufficiently large $n \in N$ a_n is not contained $S_{i_n} \cup S_{j_n}$, where $x_n \in S_{i_n}, y_n \in S_{j_n}$. Let $g_n := a_n \vee x_n$, $h_n := a_n \vee y_n$, then $g_n \longrightarrow g, h_n \longrightarrow h$. Since $a \notin F$, $g \nsubseteq F$ and $F \wedge g = x$. by lemma 3.2, for sufficiently large $n \in N$ $F \wedge g_n$ is not emptyset, and $b_n := F \wedge g_n \longrightarrow F \wedge g = x$. Similiary, $c_n := F \wedge h_n \longrightarrow F \wedge h = y$. Since $x \in S_i$, $y \in S_j$ with $i \neq j$, b_n and c_n are not contained in the same partition set. Therefore, we set $l_n = b_n \vee c_n \subseteq F$, and $l_n \longrightarrow l$. Since $b_n, c_n \in E_n$, it follows that $E_n \wedge F = l_n$, and we have $l_n = E_n \wedge F \longrightarrow l = E \wedge F$.

Lemma 3.4. *Let $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ hold the axiom (Exc) and let $H = COT$. Let $l \in \mathcal{L}$, and E, $E_n (n \in N)$ be planes, and it be $x = E \wedge l$. Furthermore let $E_n \longrightarrow E$. Then there exist for sufficiently large $n \in N$ also $E_n \wedge l$, and it holds $E_n \wedge l \longrightarrow E \wedge l = x$.*

Proof. Choose a plane F which contains l. Then it is clear that $E \neq F, F \wedge E \in \overline{\mathcal{L}}$ and $x = (E \wedge F) \wedge l$. Case 1; let $F \wedge E \in \mathcal{L}$. By lemma 3.3, it holds that for sufficiently large $n \in N$ $E_n \wedge F \in \mathcal{L}$, and $l_n := E_n \wedge F \longrightarrow E \wedge F \in \mathcal{L}$. Since F is an R-divisible R^2-plane, and $x = (E \wedge F) \wedge l$, there exist for sufficiently large $n \in N$ also $(E_n \wedge F) \wedge l$, and $(E_n \wedge F) \wedge l \longrightarrow x$. Consequently, for sufficiently large $n \in N$ $E_n \wedge l = (E_n \wedge F) \wedge l$, and it holds that $E_n \wedge l \longrightarrow E \wedge l$.

Case 2; let $F \wedge E \in \mathcal{L}_s$. Then we will show that there exists a plane F' such that $l \subseteq F'$ and $F' \wedge E \in \mathcal{L}$. If it will be done, we return to the case 1. Let $l = x \vee p$, $x \in S_x$. Choose a line $g \in \mathcal{L}$ such that $F \wedge g = p$. By lemma 3.1, there exists a plane F' with $g \subseteq F'$. It is clear that $F \wedge F' = l$. Suppose that $F' \cap E = l_{s'} \in \mathcal{L}_s$. Since $x \in l_s, x \in l_{s'}$ and $l_s, l_{s'} \subseteq S_x$, therefore, $l_s = l_{s'}$, we have $F \wedge F' = \{l, l_s\}$, a contradiction. □

Lemma 3.5. *Let $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ hold the axiom (Exc) and let $H = COT$. Let a, b be two points on different sides of a plane E with $a \vee b \in \mathcal{L}$. If $(E_n)_{n \in N}$ is a sequence of planes with $E_n \longrightarrow E \in \mathcal{E}$, then there exist neighborhoods U and V of a and b, respectively, such that for sufficiently large $n \in N$ U and V are on different sides of E_n.*

Proof. Let $U(a)$ and $U(b)$ be two connected open neighborhoods of a and b, respectively with $\overline{U(a)} \cap E = \overline{U(b)} \cap E = \emptyset$. Then evidently $U(a) \cap E_n = U(b) \cap E_n = \emptyset$ for sufficiently large $n \in N$. By lemma 3.4, it follows that $E_n \wedge (a \vee b) \neq \emptyset$ for sufficiently large $n \in N$, and $E_n \wedge (a \vee b) \longrightarrow E \wedge (a \vee b) \in (a, b)$. Since U and V are connected, consequently $U(a)$ and $U(b)$ are on different sides of E_n for sufficiently large $n \in N$.

Theorem 3.6. *Let $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ hold the axiom (Exc) and let $H = COT$. Let $E \in \mathcal{E}$, and let $(E_n)_{n \in N}$ be a sequence of planes with $E_n \longrightarrow E$. Let $(l_n)_{n \in N}$ be a sequence in $\mathcal{L}$ with $l_n \longrightarrow l \in \mathcal{L}$. Let $|E \wedge l| = 1$. Then for sufficiently large $n \in N$ $|E_n \wedge l_n| = 1$, and $E_n \wedge l_n \longrightarrow E \wedge l$.*

Proof. By lemma 2.2, let $a, b \in l$ be two points on different sides of E. Let $a \in S_a, b \in S_b$. By lemma 3.5, there exist open neighborhoods $U(a)$ and $U(b)$ of a and b, respectively, such that for all $n \geq m$ $U(a)$ and $U(b)$ are on different sides of E_n for some $m \in N$. Since $l_n \longrightarrow l$, there exist $a_n \in l_n \cap S_a, b_n \in l_n \cap S_b (n \in N)$ with $a_n \longrightarrow a$ and $b_n \longrightarrow b$. Thus there exists $m' \in N$ such that for all $n \geq m'$ $a_n \in U(a)$ and $b_n \in U(b)$. Let $M := max\{m, m'\}$. Then for all $n \geq M$ there exist $E_n \wedge l_n = E_n \wedge (a_n \vee b_n)$, and $E_n \wedge l_n \in [a_n, b_n]$. By the assumption $H = COT$, the sequence $(E_n \wedge l_n)_{n \in N}$ is bounded. If x is a limit of a subsequence of $(E_n \wedge l_n)_{n \in N}$, then $x \in \limsup E_n \wedge \limsup l_n = E \wedge l$, i.e., $x = E \wedge l$. Therefore, $E_n \wedge l_n \longrightarrow E \wedge l$. □

Corollary 3.7. *Let $D_1 := \{(E, l) \in \mathcal{E} \times l : |E \wedge l| = 1\} \subseteq \mathcal{E} \times \mathcal{L}$. Then D_1 is open in $\mathcal{E} \times \mathcal{L}$, and the mapping $\wedge : D_1 \longrightarrow \mathcal{P}^3 : (E, l) \longrightarrow E \wedge l$ is continuous.*

Theorem 3.8. *Let $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ hold the axiom (Exc) and let $H = COT$. Let $E, F \in \mathcal{E}$, and $(E_n)_{n \in N}, (F_n)_{n \in N}$ be two sequences of planes. If $E_n \longrightarrow E, F_n \longrightarrow F$ and $E \wedge F \in \mathcal{L}$, then for* *sufficiently large $n \in N$ $E_n \wedge F_n \in \mathcal{L}$, and $E_n \wedge F_n \longrightarrow E \wedge F$.*

Proof. Let $l = x \vee y = E \wedge F$ with $x \in S_i, y \in S_j, i \neq j$. Choose a point $a \in E \setminus (E \wedge F)$ with $a \notin S_i \cup S_j$, and let $g := a \vee x$ and $h := a \vee y$. Since E_n converges to E, there are sequences $(x_n), (y_n), (a_n)$ with $x_n, y_n, a_n \in E_n$, which converge to x, y and a, respectively. It is clear that for sufficiently large $n \in N$ a_n is not contained $S_{i_n} \cup S_{j_n}$, where $x_n \in S_{i_n}$, $y_n \in S_{j_n}$. Hence let $g_n := a_n \vee x_n$, $h_n := a_n \vee y_n$, and then $g_n \longrightarrow g, h_n \longrightarrow h$. Since $a \notin F$, $g \not\subseteq F$, it is $F \wedge g = x$. By theorem 3.6, for sufficiently large $n \in N$ $F_n \wedge g_n$ is not empty set, and $b_n := F_n \wedge g_n \longrightarrow F \wedge g = x$. Similiary, it is $c_n := F_n \wedge h_n \longrightarrow F \wedge h = y$. Since $x \in S_i$, $y \in S_j$ with $i \neq j$, b_n and c_n are not contained in the same partition set. Therefore, there exists $l_n \in \mathcal{L}$ with $l_n = b_n \vee c_n = E_n \wedge F_n \longrightarrow E \wedge F = l$. Consequently, $E_n \wedge F_n \longrightarrow E \wedge F$. □

Corollary 3.9. *Let $D_2 := \{(E,F) \in \mathcal{E} \times \mathcal{E} : E \wedge F \in \mathcal{L}\} \subseteq \mathcal{E} \times \mathcal{E}$. Then D_2 is open in $\mathcal{E} \times \mathcal{E}$, and the mapping $\wedge : D_2 \longrightarrow \mathcal{L} : (E,F) \longrightarrow E \wedge F$ is continuous.*

4. SOME TOPOLOGICAL PROPERTIES OF $(R^3, \mathcal{L}, \Lambda)$

In this section, we study R^2-divisible R^3-spaces which have the following partition Λ. Let $\mathcal{P}^3 = R^3 = \{(y,u,v) : y,u,v \in R\}$ and $\Lambda := \{S_y : y \in R\}$, where $S_y := \{y\} \times R^2$. In this case we write $(R^3, \mathcal{L}, \Lambda)$ instead of $(\mathcal{P}^3, \mathcal{L}, \Lambda)$, and $(R^3, \mathcal{L}, \Lambda)$ implies above given conditions. For $p \in R^3$ p_i denote the i-th coordinate of p.

Lemma 4.1. *Let $(R^3, \mathcal{L}, \Lambda)$ be a topological R^2-divisible R^3-space. Then each line $l \in \mathcal{L}$ is written as the graph (f) of a continuous mapping $I : R \longrightarrow R^2$, where I is an interval homeomorphic to R.*

Proof. Let $l \in \mathcal{L}$ with $l = (y_1,u_1,v_1) \vee (y_2,u_2,v_2), y_1 < y_2$. For all $y \in R$ let $S_y = \{y\} \times R^2$, and let $P_y : R^3 \longrightarrow R$ be the projection on y-coordinate and $P_{(u,v)} : R^3 \longrightarrow R^2$ the projection on $<u,v>$-coordinate plane, respectively. Since l is connected, $I := P_y(l)$ is an interval. Since the mapping $y \longrightarrow S_y$ is continuous, hence $f : I \subseteq R \longrightarrow R^2 : y \longrightarrow P_{(u,v)}(l \cap S_y)$ is continuous and graph$(f) = l$. Since $f : I \subseteq R \longrightarrow R^2 : y \longrightarrow P_{(u,v)}(l \cap S_y)$ is continuous, $I \longrightarrow$ graph $(f) : x \longrightarrow (x, f(x))$ is a homeomorphism, and so, I homeomorphic to R. □

Corollary 4.2. *Let $(R^3, \mathcal{L}, \Lambda)$ be a topological R^2-divisible R^3-space, and let $p = (y_1,u_1,v_1)$, $q = (y_1, u_2, v_2) \in R^3$ with $y_1 < y_2$. Then $[p,q] = (p \vee q) \cap ([y_1,y_2] \times R^2)$.*

Definition 4.3. Let $(R^3, \mathcal{L}, \Lambda)$ be an R^2-divisible R^3-space. Let $a,b,c \in R^3$ be three non-collinear points. The *generated triangle* with vertices a,b,c is the following set:

$$[c,[a,b]] := \{x \in R^3 : \exists p \in [a,b] \, such \, that \, x \in [c,p]\}.$$

Let E be a plane with $a,b,c \in E$, and let $d \in R^3 \setminus E$. The *generated pyramid* with vertices a,b,c,d is the following set:

$$[d,[c,[a,b]]] := \{x \in R^3 : \exists p \in [c,[a,b]] \, such \, that \, x \in [d,p]\}.$$

Remark 4.4. In the definitions the following cases contain: for example two points $p \in [a,b]$, $c \in R^3 \setminus [a,b]$ lie on a same S_y, i.e., the joining line of p and c is not defined. In the following lemmas and theorems we assume that such cases are not considered.

Lemma 4.5. *Let X be compact, Y be Hausdorff, and $f : X \longrightarrow Y$ be a continuous bijection. Then f is a homeomorphism.*

Theorem 4.6. *Let $(R^3, \mathcal{L}, \Lambda)$ be a topological R^2-divisible R^3-space. Let each line be a graph(f) of a continuous mapping $f : R \longrightarrow R^2 : y \longrightarrow f(y)$. Let $[c, [a, b]_1]_1$ be the classical triangle with the vertices a, b, c of the classical R^2-divisible R^3-space and let $[c, [a, b]_2]_2$ be the generated triangle with the vertices a, b, c in $(R^3, \mathcal{L}, \Lambda)$. Then there is a homeomorphism Φ between $[c, [a, b]_1]_1$ and $[c, [a, b]_2]_2$.*

Proof. Let $f : R \longrightarrow R^2 : y \longrightarrow f(y)$ and $g : R \longrightarrow R^2 : y \longrightarrow g(y)$ be given two continuous mapping. For $a, b \in R$ with $a < b$ let $(a, f(a)) = (a, g(a))$ and $(b, f(b)) = (b, g(b))$. Let $[a, b]_f := \{(y, f(y)) : a \leq y \leq b\}$ and $[a, b]_g := \{(y, g(y)) : a \leq y \leq b\}$.

Define $\alpha : [a, b]_f \longrightarrow [a, b]_g : (y, f(y)) = S_y \cap [a, b]_f \longrightarrow (y, g(y)) = S_y \cap [a, b]_g$. Then α is evidently a homeomorphism. Let $\alpha : [a, b]_1 \longrightarrow [a, b]_2$ be the above defined homeomorphism.

Define the following mapping: $\Phi : [c, [a, b]_1]_1 \longrightarrow [c, [a, b]_2]_2$:

$$(x \in [c, [a, b]_1]_1 \iff \exists p \in [a, b]_1 \, with \, x \in [c, p]_1 \iff x = S_{x_1} \cap [c, p]_1)$$

$$x \longrightarrow \Phi(x) = S_{x_1} \cap [c, \alpha(p)]_2.$$

Then we show that Φ is a homeomorphism. At first Φ is injective; let $x, y \in [c, [a, b]_1]_1$ with $x \neq y$. Then there exist $p, p' \in [a, b]_1$ with $x \in [c, p]_1$ and $y \in [c, p']_1$. We have the following two cases; case 1; let $p = p'$. Since $x \neq y \iff x_1 \neq y_1$, we have $S_{x_1} \cap S_{y_1} = \emptyset$. Consequently, $\Phi(x) = S_{x_1} \cap [c, \alpha(p)]_2 \neq S_{y_1} \cap [c, \alpha(p')]_2 = \Phi(y)$. Case 2; let $p \neq p'$. Then $[c, p]_1 \cap [c, p']_1 = \{c\}$ and $[c, \alpha(p)]_2 \cap [c, \alpha(p')]_2 = \{c\}$. It is also that $\Phi(c) = c$. We have one of the following cases: $c = x \neq y$, $x \neq y = c$ or $c \neq x \neq y \neq c$. In any case $\Phi(x) \neq \Phi(y)$ is clear. Therefore, Φ is injective.

Φ is surjective; let $y \in [c, [a, b]_2]_2$. Then there is a $p \in [a, b]_2$ with $y \in [c, p]_2$. Therefore, $y = S_{y_1} \cap [c, p]_2$. Let $x = S_{y_1} \cap [c, \alpha^{-1}(p)]_1 \in [c, [a, b]_1]_1$. It is also that $\Phi(x) = S_{y_1} \cap [c, \alpha(\alpha^{-1}(p))]_2 = S_{y_1} \cap [c, p] = y$. Since the classical space and the given space are topological and α is a homeomorphism, Φ is continuous. By lemma 4.5, Φ is a homeomorphism. □

Corollary 4.7. *Let $(R^3, \mathcal{L}, \Lambda)$ be a topological R^2-divisible R^3-space. Let $a, b, c \in R^3$ be non-collinear points, and let $[c, [a, b]_2]_2$ be the generated triangle with the vertices a, b, c. Then $[c, [a, b]_2]_2$ is homeomorphic to the classical triangle $[c, [a, b]_1]_1$ of the classical R^2-divisible R^3-space.*

Proof. By lemma 4.1, the assertion can be proved as in proof in theorem 4.6.

Theorem 4.8. *Let $(R^3, \mathcal{L}, \Lambda)$ be a topological R^2-divisible R^3-space. Let each line be a graph(f) of a continuous mapping $f : R \longrightarrow R^2 : y \longrightarrow f(y)$. Let $[d, [c, [a, b]_1]_1]_1$ be the classical pyramid of the classical R^2-divisible R^3-space and let $[d, [c, [a, b]_2]_2]_2$ be the generated pyramid with the vertices a, b, c, d in $(R^3, \mathcal{L}, \Lambda)$. Then there is a homeomorphism Ψ between $[d, [c, [a, b]_1]_1]_1$ and $[d, [c, [a, b]_2]_2]_2$.*

Proof. The following homeomorphisms are defined in theorem 4.6:

$$\alpha : [a, b]_1 \longrightarrow [a, b]_2 : x \longrightarrow \alpha(x) = S_{x_1} \cap [a, b]_2,$$

$$\Phi : [c,[a,b]_1]_1 \longrightarrow [c,[a,b]_2]_2 : x \longrightarrow \Phi(x) = S_{x_1} \cap [c, \alpha(p)]_2.$$

We define the following mapping: $\Psi : [d,[c,[a,b]_1]_1]_1 \longrightarrow [d,[c,[a,b]_2]_2]_2$:

$$(x \in [d,[c,[a,b]_1]_1]_1 \Leftrightarrow \exists q \in [c,[a,b]_1]_1 \; with \, x \in [d,q]_1$$

$$\Leftrightarrow x \in S_{x_1} \cap [d,q]_1)$$

$$x \longrightarrow \Psi(x) = S_{x_1} \cap [d, \Phi(q)]_2.$$

We show that Ψ is a homeomorphism. At first Ψ is injective; let $x, y \in [d,[c,[a,b]_1]_1]_1$ with $x \neq y$. Then there exist $q, q' \in [c,[a,b]_1]_1$ with $x \in [c,q]_1$ and $y \in [c,q']_1$. Let $q = q'$. Since $x \neq y \iff x_1 \neq y_1$, we have $S_{x_1} \cap S_{y_1} = \emptyset$. Therefore, it holds that $\Psi(x) = S_{x_1} \cap [d,\Phi(p)]_2 \neq S_{y_1} \cap [d, \Phi(p')]_2 = \Psi(y)$.

Let $q \neq q'$. Then $[d,p]_1 \cap [d,p']_1 = \{d\}$ and $[d,\Phi(p)]_2 \cap [d,\Phi(p')]_2 = \{d\}$. It is also that $\Psi(d) = d$. We have one of the following cases: $d = x \neq y$, $x \neq y = d$ or $d \neq x \neq y \neq d$. In any cases $\Psi(x) \neq \Psi(y)$ is clear. Consequently, Ψ is injective. Ψ is surjective; let $y \in [d,[c,[a,b]_2]_2]_2$. Then there is a $p \in [c,[a,b]_2]_2$ with $y \in [d,p]_2$, and so, $y = S_{y_1} \cap [d,p]_2$. Thus we let $x = S_{y_1} \cap [d, \Phi^{-1}(p)]_1 \in [d,[c,[a,b]_1]_1]_1$. It holds that $\Psi(x) = S_{y_1} \cap [d, \Phi(\Phi^{-1}(p))]_2 = S_{y_1} \cap [d,p] = y$.

Since the classical space and the given space are topological and Φ is a homeomorphism, Ψ is continuous. By lemma 4.5, Ψ is a homeomorphism. □

Corollary 4.9. *Let $(R^3, \mathcal{L}, \Lambda)$ be a topological R^2-divisible R^3-space. Let $[d,[c,[a,b]_1]_1]_1$ be the classical pyramid of the classical R^2-divisible R^3-space and let $[d,[c,[a,b]_2]_2]_2$ be the generated pyramid with the vertices a,b,c,d in $(R^3, \mathcal{L}, \Lambda)$. Then $[d,[c,[a,b]_2]_2]_2$ is homeomorphic to $[d,[c,[a,b]_1]_1]_1$.*

Definition 4.10. Let $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ be a topological R^2-divisible R^3-space. A subset $P \subseteq \mathcal{P}^3$ is called a *subgeometry* of $(\mathcal{P}^3, \mathcal{L}, \Lambda)$. If $x, y \in P$ with $(x,y) \in \mathcal{P}^3_{S_i} \times \mathcal{P}^3_{S_j}$, then $x \vee y \subseteq P$.

Lemma 4.11. *Let $(R^3, \mathcal{L}, \Lambda)$ be a topological R^2-divisible R^3-space, and let E be a plane of $(R^3, \mathcal{L}, \Lambda)$. Let P be a subgeometry which contains three non-collinear points a,b,c of E and $P \subseteq E$. Then $P = E$.*

Proof. Without loss of generality, let $c_1 < a_1 < b_1$. Let $D := [c,[a,b]] \setminus \partial[c,[a,b]] \subseteq P$. Since E is a plane, D is convex. Assume that $q \in E \setminus P$. Choose a point $p \in D$ with $p \notin S_{q_1}$, then there is a joiningline $p \vee q$. Since D is convex and open, it is $|D \cap (p \vee q)| \geq 2$. Therefore, $p \vee q \subseteq P$, so that $q \in P$, a contradiction. □

Corollary 4.12. *Let $E = (R^2, \mathcal{L})$ be an R^2-plane. Let P be a subgeometry of E (in the sense of definition 4.10), which contains three non-collinear points a,b,c of E.*

Lemma 4.13. *Let $(R^3, \mathcal{L}, \Lambda)$ be a topological R^2-divisible R^3-space, and let $P \subseteq R^3$ be a subgeometry which contains an open subset ($\neq \emptyset$) of R^3. Then $P = R^3$.*

Proof. For $y \in R^3, \varepsilon > 0$ let $B(y,\varepsilon) := \{x \in R^3 : ||y - x|| < \varepsilon\}$ be an open subset with $B(y,\varepsilon) \subseteq P$. Assume that there exists $q \in R^3 \setminus P$. Choose a point $p \in B(y,\varepsilon)$ with $p \notin S_{q_1}$.

Then there is a joining line of p and q. Evidently it is $B(y, \varepsilon) \cap (p \vee q) \geq 2$. therefore, $p \vee q \subseteq P$, so that $q \in P$, a contradiction. □

Lemma 4.14. *Let $(R^3, \mathcal{L}, \Lambda)$ be a topological R^2-divisible R^3-space, and E be a plane. Let $a, b, c \in E$ be three non-collinear points, and let $d \in R^3 \setminus E$. Let $P \subseteq R^3$ be a subgeometry which contains a, b, c and d. Then $P = R^3$.*

Proof. Without loss of generality, let $c_1 < a_1 < b_1$. Then $[c, [a, b]] \subseteq P$. We have the following two cases:
Case 1: $d_1 \notin [c_1, b_1]$. Then $[d, [c, [a, b]]] \subseteq P$. Therefore, $[d, [c, [a, b]]] \setminus \partial[d, [c, [a, b]]] \subseteq P$, i.e., P contains an open subset. By lemma 4.13, it follows that $P = R^3$.
Case 2: $d_1 \in [c_1, b_1]$. Choose $c', a' \in a \vee b$ with $b_1 < c'_1 < a'_1$, and $b' \in c \vee b$ with $c'_1 < b'_1 < a'_1$. Therefore, $[c', [a', b']] \subseteq P$. Then also $[d, [c', [a', b']]] \subseteq P$. By case 1, it is $P = R^3$. □

Definition 4.15. Let $(R^3, \mathcal{L}, \Lambda)$ be a topological R^2-divisible R^3-space. A subset $E \subseteq R^3$ is called an *incidence plane* of $(R^3, \mathcal{L}, \Lambda)$ if it satisfies the following properties:

(1) If $x, y \in E$ with $(x, y) \in R^3_{S_i} \times R^3_{S_j}$, then $x \vee y \subseteq E$.

(2) E is non-trivial, i.e. $E \neq R^3$,and E is not contained in a line.

Lemma 4.16. *Let $(R^3, \mathcal{L}, \Lambda)$ be a topological R^2-divisible R^3-space satisfying the axiom(Exc), and let $E \subseteq R^3$ be an incidence plan. For $a, b, c \in E$ let $[c, [a, b]]$ be the generated triangle with the vertices a, b, c. Then $[c, [a, b]]$ is convex.*

Proof. Without loss of generality, let $c_1 < a_1 < b_1$. If $[c, [a, b]]$ is contained in a plane, then it is obviously convex, hence we may assume that it is not contained in a plane. By lemma 3.1, there exist planes E_1, E_2 and E_3 with $c \vee a \subseteq E_1$, $c \vee b \subseteq E_2$, and $a \vee b \subseteq E_3$. For $i = 1, 2, 3$, let H_i be the components of $R^3 \setminus E_i$ which contains the third point. Let $P := H_1 \cap H_2 \cap H_3$. Then $\bar{P} = \bar{H}_1 \cap \bar{H}_2 \cap \bar{H}_3$ is convex such that $a, b, c \in \bar{P}$. It is also that $[c, [a, b]] \subseteq \bar{P} \cap E$. By the axiom (Exc), there exists a continuous mapping $\varphi : [a, b] \longrightarrow \mathcal{E}$ with $c \vee z \subseteq \varphi(z)$ for all $z \in [a, b]$. Assume that $[c, [a, b]]$ is not convex. Then there exists $p, q \in [c, [a, b]]$ with $[p, q] \nsubseteq [c, [a, b]]$. Let $x \in [p, q]$ such that $x \notin [c, [a, b]]$. Evidently, it is $[p, q] \subseteq \bar{P} \cap E$. Let $z_1, z_2 \in [a, b]$ with $p \in \varphi(z_1)$ and $q \in \varphi(z_2)$. Since φ is continuouis, there exist a $z \in [a, b]$ with $x \in \varphi(z)$. Therefore, $x, c, z \in \varphi(z) \cap E$. Let $c', z' \in c \vee z$ with $c'_1 < x_1 < z'_1$. Since $[c, [a, b]]$ is not contained in a plane, we may choose $d \neq z \in [a, b]$ such that $d \notin \varphi(z)$. Therefore, c', x, z' and $d \in E$. By lemma 4.14, $E = R^3$, a contradiction. □

Theorem 4.17. *Let $(R^3, \mathcal{L}, \Lambda)$ be topological R^2-divisible R^3-space satisfying the axiom (Exc). Then each incidence plane $E \subseteq R^3$ is a plane of $(R^3, \mathcal{L}, \Lambda)$.*

Proof. Let $a, b, c \in E$ be non-collinear and let $D := [c, [a, b]] \setminus \partial[c, [a, b]]$. Since E is an incidence plane, hence $D \subseteq E$. By corollary 4.7, D is homeomorphic to R^2. As in proof of lemma 4.16, it is also D open in E. Let $x \in E$. Since E is non-trivial, there exists a line $l \in \mathcal{L}$ with $x \in l \subseteq E$. Let $p, c \in l$ with $x \in (p, c)$. Since E is not contained in a line, there exists a point $a \in E$ with $a \notin l$. Without loss of generality, let $a_1 \neq p_1$ and $a_1 \neq c_1$. Let $b \in a \vee p$ with $p \in (a, b)$. Then $b \in E$, and so, $[c, [a, b]] \subseteq E$. It holds also $x \in [c, [a, b]] \setminus \partial[c, [a, b]]$. Therefore, the set $\{[c, [a, b]] \mid a, b, c \text{ non-collinear } \}$ is a basis of the topology of E. Hence E is a topological 2-manifold, i.e. a surface, and E is locally compact. Thus E is open in $\overline{E}$. We show that E is homeomorphic to R^2, $E = \overline{E}$. We show that $E = \overline{E}$; choose a point $p \in E$. Let

$x \in \overline{E} \setminus \{p\}$, and without loss of generality, let $l := p \vee x$. There exists a sequence $(x_n)_{n\in N}$ in E such that $x_n \longrightarrow x$. Hence it follows that $l = \limsup(p \vee x_n) \subseteq \overline{E}$. Therefore, $l \wedge E \neq \emptyset$ is open in $\overline{l} = l$, and since $l \approx R$, it implies $| l \wedge E | \geq 2$, so that $x \in l \subseteq E$. Hence E is closed in R^3. We also show that E is homeomorphic to R^2. Let $\mathcal{L}_E := \{l \in \mathcal{L} \mid l \subseteq E\} \cup \{E \cap S_y \mid y \in R\}$. Then $(E, \mathcal{L}_E)$ is a linear space, and the operation $\vee : E \times E \setminus \Delta \longrightarrow \mathcal{L}_E$ is continuous. We have to show that the intersecting of lines is continuous and stable; let $l, g \in \mathcal{L}_E$ with $p := l \wedge g$. Let $U \subseteq E$ be open with $p \in U$. Let $(l_n)_{n\in N}, (g_n)_{n\in N} \in \mathcal{L}_E$ be two sequences of lines with $l_n \longrightarrow l$ and $g_n \longrightarrow g$. There exists $D := [c, [a, b]] \setminus \partial[c, [a, b]]$ such that $p \in D \subseteq U$. Since $D \approx R^2$ and is convex, the linear space $(E, \mathcal{L}_E)$ induces a Salzmann-plane on D. Since the topology of line space in D is the induced topology of Hausdorff-convergence, it holds also that $l_n \cap D \longrightarrow l \cap D$ and $g_n \cap D \longrightarrow g \cap D$. Since the intersection of lines on D is continuous and stable, there exists $l_n \wedge g_n$ for sufficiently large $n \in N$ with $l_n \wedge g_n \in D \subseteq U$. Since each line $l \subseteq \mathcal{L}_E$ is homeomorphic to R, by **[Sal69]**, E is homeomorphic to R^2. □

Theorem 4.18. *Let $(R^3, \mathcal{L}, \Lambda)$ hold the axiom (Exc) and let $H = COT$. Let $(x_n)_{n\in N}$, $(y_n)_{n\in N}$ and $(z_n)_{n\in N}$ be three sequences of coplanar non-collinear points, and let $x, y, z \in R^3$ be not collinear such that $x_n \longrightarrow x, y_n \longrightarrow y$ and $z_n \longrightarrow z$. Then the points x, y, z are coplanar, and $x_n \vee y_n \vee z_n \longrightarrow x \vee y \vee z$.*

Proof. Let for all $n \in N$ $E_n := x_n \vee y_n \vee z_n$, and let $E := \liminf E_n$. Then it is clear that $x, y, z \in E$. At first we show that $E \neq R^3$; assume that $E = R^3$. Choose a plane F, and $a, b, c \in F$ which are not collinear. Then there exist sequences $a_n, b_n, c_n \in E_n$ such that $a_n \longrightarrow a, b_n \longrightarrow b$ and $c_n \longrightarrow c$, respectively. Since $(R^3, \mathcal{L}, \Lambda)$ is topological, we may assume that a_n, b_n, c_n are not collinear for all $n \in N$. Since $a \neq c$, there exists a $\epsilon > 0$, so that for sufficiently large $n \in N$ it holds that $||a_n - c_n|| \geq 4\epsilon$. Therefore, choose points $q_n \in (a_n, c_n)$, so that for sufficiently large n it holds that $||q_n - a_n|| \geq \epsilon$ and $||q_n - c_n|| \geq \epsilon$. By the assumption $H = COT$, it holds that $q_n \longrightarrow q \in (a, c)$. Choose a point $p \in R^3 \setminus F$ with $p_1 \neq q_1$, and let $p_n \in E_n$ mit $p_n \longrightarrow p$. Therefore, we may assume that $p_n \notin a_n \vee c_n$ for all $n \in N$. Then for all $n \in N$ it holds $(p_n \vee q_n) \cap ([a_n, b_n] \cup [b_n, c_n]) \neq \emptyset$. Therefore, we may assume that for all $n \in N$ $(p_n \vee q_n) \cap [a_n, b_n] \neq \emptyset$. Let $r_n := (p_n \vee q_n) \cap [a_n, b_n]$. By assumption $H = COT$, it holds also $r_n \longrightarrow r$. Since $(R^3, \mathcal{L}, \Lambda)$ is topological, we have $r \in a \vee b \subseteq F$, and $p \in q \vee r \subset F$, a contradiction. Consequently, $E \neq R^3$. We next will show that E is an incidence plane. Let $a, b \in E$ with $a_1 \neq b_1$. Then there exist sequences $a_n, b_n \in E_n$ with $a_n \longrightarrow a$ and $b_n \longrightarrow b$, and $a_{1n} \neq b_{1n}$ for all $n \in N$. Since $(R^3, \mathcal{L}, \Lambda)$ is topological, it implies $a_n \vee b_n \longrightarrow a \vee b$, it holds also that $a \vee b \subseteq E$, since $a_n \vee b_n \subseteq E_n$ for all $n \in N$. By theorem 4.17, E is a plane, i.e., x, y, z are coplanar with $x \vee y \vee z = E$. We have to show that $E_n \longrightarrow E$, we show that $\limsup E_n \subseteq E$. Let n_k be a subsequence, and let $a_{n_k} \in E_{n_k}$ with $a_{n_k} \longrightarrow a$. Then it follows that $a \in \liminf E_{n_k}$, and as in above $F := \liminf E_{n_k}$ is a plane. Since $x, y, z \in E \wedge F$, and by lemma 2.2, $E = F$, therefore, $a \in E$. We have $\lim E_n = E$, as required. □

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