RIEMANNIAN MANIFOLDS SATISFYING \([\text{Ric} \wedge g, W] = 0\)

STEVE P. BEAN

Abstract. We study the condition \([\text{Ric} \wedge g, W] = 0\) on 2n-dimensional Riemannian manifolds which also have non-negative curvature on totally isotropic two-planes. We show that if, in addition, certain holomorphic bisectional curvatures are positive, then the manifold is biholomorphically isometric to \(\mathbb{C}^{pn}\).

1. INTRODUCTION

Let \(M\) be a 2n-dimensional Riemannian manifold, \(n \geq 2\), with curvature tensor \(R\). We denote the corresponding curvature operator by \(\hat{R}\). Recall (cf. [1], Chap. 1, Sec. G), that we may decompose \(R\) uniquely into its \(O(2n)\)-irreducible components: \(R = U + Z + W\), where \(U = \frac{1}{4n(2n-1)}g \wedge g, Z = \frac{1}{2(n-1)}[(\text{Ric} - \frac{1}{2n}g) \wedge g]\) (the Ricci-traceless part of \(R\)), and \(W\) is the Weyl tensor.

Given a real vector space \(V\) with inner product \(\langle, \rangle\), let \(V^C = V \otimes_{\mathbb{R}} \mathbb{C}\) denote the complexification of \(V\). Extend \(\langle, \rangle\) to a complex bilinear form on \(V^C\). Finally, let \(\langle \langle, \rangle \rangle\) denote the extension of \(\langle, \rangle\) to \(V^C\) which is linear in the first component and conjugate linear in the second. That is, \(\langle \langle z, w \rangle \rangle = \langle z, \bar{w} \rangle\) for \(z, w \in V^C\).

We now recall the definition of totally isotropic curvature, which we denote by \(K^{iso}\), introduced in [5].

Definition. A complex vector \(z \in V^C\) is isotropic if \(\langle z, z \rangle = 0\). A complex subspace \(W\) of \(V^C\) is totally isotropic if \(z\) is isotropic for all \(z \in W\).

For a complex subspace \(\sigma\) of \(V^C\) spanned by \(z\) and \(w\), let \(K(\sigma) = \frac{\langle \hat{R}(z \wedge w), z \wedge w \rangle}{\|z \wedge w\|^2}\). We note that this number does not depend on the choice of \(z\) and \(w\).

Definition. A curvature tensor \(R\) has non-negative (positive) curvature on totally isotropic 2-planes if \(K(\sigma) \geq 0\) (\(> 0\)) for all totally isotropic complex two-dimensional subspaces of \(V^C\).

We note that two well-studied conditions on curvature imply conditions on isotropic curvatures (cf. [5]):

(1) If the curvature tensor \(R\) is positive (non-negative), that is if \(\langle \hat{R}\alpha, \alpha \rangle > 0\) (\(\geq 0\)) for all non-zero \(\alpha \in \Lambda^2\), then \(K^{iso}\) is positive (non-negative).

(2) If sectional curvature \(K\) is quarter-pinched, that is if \(\frac{\Delta}{4} \leq K \leq \Delta\) for some \(\Delta > 0\), then \(K^{iso} \geq 0\). If \(K\) is strictly quarter-pinched, that is if at least one of the inequalities is strict, then \(K^{iso} > 0\).

In this paper, we examine a condition, \([\text{Ric} \wedge g, W] = 0\), which applies to two important classes of Riemannian manifolds: Einstein and conformally flat. Combined with the concept of curvature on totally isotropic two-planes, this condition invites various diagonalizations
of $\hat{R}$ which can be exploited to a certain extent. In particular, [7] introduces an orthonormal basis for two-forms which allows the expression of eigenvalues of the Weitzenböck operator on 2-forms (denoted $\mathcal{R}_2$) as sums of (non-negative multiples of) totally isotropic curvatures.

A derivation of this formula begins with the choice of an orthonormal basis $\{f_i\}_{i=1}^{2n}$ of a vector space $V$ with respect to which an eigenvector $\alpha$ of $\mathcal{R}_2$ may be written as $\alpha = \sum_{i=1}^{n} \mu_i (f_{2i-1} \wedge f_{2i})$, where $1 = \mu_1 \geq \mu_2 \geq \ldots \geq \mu_n \geq 0$. Let $\mathcal{R}_2 \alpha = q \alpha$. Using the Weitzenböck formula for $\mathcal{R}_2$ (see for example [9], Chapter 2) one computes as in [8] (Eq. 1.4, page 849):

$$q = \langle \mathcal{R}_2 \alpha, f_1 \wedge f_2 \rangle = \sum_{k=2}^{n} \left\{ K_{1,2k} + K_{1,2k-1} + K_{2,2k} + K_{2,2k-1} + 2\mu_k \mathcal{R}_{1,2,2,k-1,2k} \right\}$$

One of the insights of [7], via Lie algebra theory, is that, using the basis $X_{ij} = \frac{1}{\sqrt{2}} [(f_{2i-1} + if_{2i}) \wedge (f_{2j-1} + if_{2j})]$ and $X'_{ij} = \frac{1}{\sqrt{2}} [(f_{2i-1} + if_{2i}) \wedge (f_{2j-1} - if_{2j})]$, the right hand side of this equation is equal to the expression:

$$\sum_{k=2}^{n} \left\{ (1 + \mu_k) \langle \hat{R}X_{1k}, X_{1k} \rangle + (1 - \mu_k) \langle \hat{R}X'_{1k}, X'_{1k} \rangle \right\}$$

which one can confirm by expanding the expression above. Thus we have:

$$q = \langle \mathcal{R}_2 \alpha, f_1 \wedge f_2 \rangle = \frac{1}{2} \sum_{k=2}^{n} \left\{ (1 + \mu_k) \langle \hat{R}X_{1k}, X_{1k} \rangle + (1 - \mu_k) \langle \hat{R}X'_{1k}, X'_{1k} \rangle \right\} \quad (1)$$

where $\langle \langle \hat{R}X_{1k}, X_{1k} \rangle \rangle$ and $\langle \langle \hat{R}X'_{1k}, X'_{1k} \rangle \rangle$ represent sectional curvatures on totally isotropic two-planes.

Using the formula $\hat{R} = \frac{1}{2} (\text{Ric} \wedge g - \mathcal{R}_2)$, we note that $[\text{Ric} \wedge g, W] = 0 \Rightarrow [\text{Ric} \wedge g, \hat{R}] = 0 \Rightarrow [\text{Ric} \wedge g, \mathcal{R}_2] = 0$. This in turn implies the existence of an orthonormal basis of 2-forms simultaneously diagonalizing $\text{Ric} \wedge g$ and $\mathcal{R}_2$. We will use this basis to establish the following consequence of the "sphere theorem" proved in [5]:

**Theorem 1.** Let $M$ be a compact, orientable $2n$-dimensional Riemannian manifold without boundary, $n \geq 2$, with curvature tensor $R$ satisfying $[W, \text{Ric} \wedge g] = 0$. Then if $q_1(x) + q_2(x)$ is positive, where $q_1(x)$ and $q_2(x)$ are the two smallest eigenvectors of $\text{Ric}_x : T_xM \to T_xM$, and if $\exists$ functions $b, \delta : M \to \mathbb{R}$ such that $\delta(x) > 0$ and $\delta(x)b(x) \leq K^{\text{iso}}(T_xM) \leq b(x)$, $\forall x \in M$, $\pi_i(M) = 0$ for $2 \leq i \leq n$. In particular if $M$ is also simply connected, $M$ is homeomorphic to a sphere.

Next, we turn our attention to Kähler manifolds, and again using $[\text{Ric} \wedge g, \mathcal{R}_2] = 0$ as a key step, prove (cf. [8], Theorem B):

**Theorem 2.** Let $n \geq 2$ and suppose $(M, g)$ is a compact, connected $2n$-dimensional Kähler manifold satisfying:
(i) \([\text{Ric} \land g, W] = 0\)
(ii) \(K^{\text{iso}} \geq 0\)
(iii) \(H(X, Y) > 0\) whenever \(X, Y, JX\) are orthonormal vectors in \(T^*M\)

Then \(M\) is biholomorphically isometric to \(\mathbb{CP}^n\), with a multiple of the canonical metric.

Here \(H\) denotes holomorphic bisectional and is given by \(H(X, Y) = \langle R(X, JX)Y, JY \rangle\) (adapting the opposite sign convention of that in [2]).

Finally, we once again exploit the orthonormal basis used to prove Theorem 1 to obtain explicit formulas for eigenvalues of the Weyl tensor in both the Kähler and non-Kähler case.

We note that the condition \([\text{Ric} \land g, W] = 0\) also arises in notions generalizing the Einstein condition. [1] (Chapter 16) gives the following equivalent conditions, all of which (by a result of J.P. Bourguignon-Corollary 16.17, p. 439) imply \([\text{Ric} \land g, W] = 0\) on a manifold \(M\):

1. \(d^\nabla \text{Ric} = 0\) (i.e., Ric is a Codazzi tensor)
2. \(\delta R = 0\) (harmonic curvature)
3. \(n \geq 4 : \delta W = 0\) and scalar curvature constant
4. \(n = 3 : M\) conformally flat and scalar curvature constant

Here \(d^\nabla\) Ric denotes the exterior differential (cf. [1], 1.12) of Ricci tensor considered as an element of \(\wedge^1 \mathcal{M} \otimes TM\) (a one-form with values in the tangent bundle). i.e. \((d^\nabla \text{Ric})(X, Y) = \nabla_X(\text{Ric}Y) - \nabla_Y(\text{Ric}X) - \text{Ric}([X, Y])\). \(\delta W \in \wedge^2 \mathcal{M} \otimes T^*\mathcal{M}\) is given by \((\delta W)(X_1, X_2) = -\text{Tr} [(Y, Z) \rightarrow (\nabla_Y W)(Z, X_1, X_2)]\) (cf. [1], 16.3).

In the same chapter (p. 439), [1] discusses the concept of a pure curvature operator (one for which \(\hat{R}\) is diagonalizable by simple two-forms-see [4]). In particular, if \(R\) is conformally flat, \(\hat{R}\) may be diagonalized by the 2-forms \(e_i \land e_j\), where \(\{e_i\}\) is an orthonormal basis diagonalizing the Ricci operator. It is this basis which is used in [3] and in [6].

2. RESULTS

Proof of Theorem 1. We begin by finding bounds on \(\tilde{\hat{R}}\) based on bounds on isotropic curvatures and on eigenvalues of the Ricci operator.

Proposition 1. Let \(V\) be a \(2n\)-dimensional vector space with curvature tensor \(R\) satisfying:

(i) \([W, \text{Ric} \land g] = 0\)
(ii) \(a \leq K^{\text{iso}} \leq b\)
(iii) \(c \leq \text{Ric} \leq d\)

Then any eigenvalue \(k\) of \(\hat{R} : \wedge^2 V \rightarrow \wedge^2 V\) satisfies \(c - 2b(n - 1) \leq k \leq d - 2a(n - 1)\).

Proof. Let \(\alpha\) be an element of an orthonormal basis which simultaneously diagonalizes \(\text{Ric} \land g\) and \(\mathcal{R}_2\). Then \(\text{Ric} \land g\alpha = (\lambda_i + \lambda_j)\alpha\) (where the \(\lambda_i, \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n\), are eigenvalues of the Ricci operator, and \(\mathcal{R}_2\alpha = q\alpha\), where \(q\) is given in (1). Then \(\hat{R}\alpha = \lambda\alpha\), where

\[
\lambda = \frac{1}{2}(\lambda_i + \lambda_j) - \frac{1}{4} \sum_{k \geq 2} (1 + \mu_k)(\langle \hat{R}X_{1k}, X_{1k} \rangle + (1 - \mu_k)(\langle \hat{R}X'_{1k}, X'_{1k} \rangle)
\]  
(2)

Now apply the bounds on \(\text{Ric}\) and on \(K^{\text{iso}}\) to obtain the desired result. \(\square\)

We note that for \(\mathbb{CP}^n\), equipped with the metric giving constant holomorphic sectional curvature 1, \(0 \leq K^{\text{iso}} \leq \frac{3}{4}\). Hence the above inequality is sharp for \(\mathbb{CP}^2\) (the right hand inequality remains sharp for \(\mathbb{CP}^n\), for any \(n\).
Corollary. Under the same hypotheses as in Proposition 1:

(i) If the smallest two eigenvalues of Ric sum to a positive (non-negative) number, then $b$ must be positive (non-negative).

(ii) If the largest two eigenvalues of Ric sum to a negative (non-positive) number, then $a$ must be negative (non-positive).

Proof.

(i) If the smallest two eigenvalues of Ric sum to a positive number and $b \leq 0$, then $K^{iso} > 0$, which implies $b > 0$, a contradiction.

(ii) If the largest two eigenvalues of Ric sum to a negative number and $a \geq 0$, then $K^{iso} < 0$, which implies $a < 0$, a contradiction.

The cases for non-negative and non-positive are similar.

We now apply that vector space result to a compact manifold to obtain:

Proposition 2. Suppose $M$ is a compact, orientable $2n$-dimensional Riemannian manifold without boundary and curvature tensor $R$ satisfying (pointwise):

(i) $[W, Ric \wedge g] = 0$

(ii) $\exists$ functions $b, \delta : M \to \mathbb{R}$ such that $\delta(x) > 0$ and $\delta(x)b(x) \leq K^{iso}(T_xM) \leq b(x), \forall x \in M.$

(iii) $\forall x, q_1(x) + q_2(x)$ is positive (non-negative), where $q_1(x)$ and $q_2(x)$ are the two smallest eigenvectors of $Ric_x : T_xM \to T_xM$.

Then $K^{iso}$ is pointwise positive (non-negative).

By Proposition 2, the suppositions of Theorem 1 imply $K^{iso} > 0$, precisely the hypothesis needed in the Micallef-Moore sphere theorem.

Proof of Theorem 2.

Lemma 1. Let $V$ be a $2n$-dimensional vector space with orthogonal complex structure $J$ and Kähler curvature $R$ satisfying:

(i) $K^{iso} \geq 0$

(ii) $H(X, Y) > 0$ whenever $X, Y, JX$ are orthonormal vectors in $V$

Then $\omega = \sum e_{2i-1} \wedge e_{2i} \in \ker R_2$ (where $Je_{2i-1} = -e_{2i}$) implies $\ker R_2 = \mathbb{R} \omega$.

Proof. Since the Kähler form may be written as $\omega = \sum e_{2i-1} \wedge e_{2i}$ with respect to any $J$-adapted orthonormal basis of $V$, we may as well assume that $\{e_i, Je_i\}_{i=1}^n$ is an orthonormal basis diagonalizing the Ricci operator. Say $\lambda_i = Ric(e_i, e_i)$. Then:

$$\lambda_i = Ric(e_i, e_i) = \sum_{a=1}^n R_{a^{*}a}e_i e_i = \sum_{a \neq i} R_{a^{*}a} + K_{ii}$$

(where $e_{i^{*}} = Je_i$). But $R_{a^{*}a}e_i e_i = K_{a^{*}i} + K_{ai}$ by the Kählerity of $R$. Therefore we have:

$$\lambda_i + \lambda_j = \sum_{a \neq i} (K_{a^{*}i} + K_{ai}) + \sum_{a \neq j} (K_{a^{*}j} + K_{aj}) + (K_{ii^{*}} + K_{jj^{*}})$$

By assumption (ii), the first two summands above are strictly positive. One then shows (cf. [8], page 849) that under the assumption $K^{iso} \geq 0$, the last summand is non-negative.
for $i \neq j$, and hence that for $i \neq j$, $\lambda_i + \lambda_j > 0$. It is this observation which is needed to conclude (through a computation) that an arbitrary element of $\ker R_2$ has no component in the $-1$ eigenspace of $J : \bigwedge^2 V \to \bigwedge^2$ (cf. [8], Equation 1.12).

Next, using a method found in [2] (Lemma 1, page 229), one shows that any element of the $+1$ eigenspace of $J : \bigwedge^2 V \to \bigwedge^2$ may be written $\sum \beta_i f_i \wedge f_i$, where $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_n$ and $\beta_1 > 0$. Writing an arbitrary element of $\ker R_2$ in this way we conclude through computation (once again using (ii)), that $\beta_i = \beta_1, \forall i$. \hfill \Box

If we assume $[\text{Ric} \wedge g, R_2] = 0$, in addition to the hypotheses of Lemma 1, the Ric $\wedge g$ preserves $\ker R_2 = \mathbb{R}\omega$. Thus $\omega$ is an eigenvector of Ric $\wedge g$. A simple computation now gives:

**Lemma 2.** Let $V$ be a $2n$-dimensional vector space with complex structure $J$ and Kähler curvature $R$. If $\omega = \sum e_{2i-1} \wedge e_{2i}$ is an eigenvector of Ric $\wedge g$, $R$ is Einstein.

**Proof.** Using the same $J$-adapted basis $\{e_i, Je_i\}_{i=1}^n$ diagonalizing the Ricci operator one computes:

$$\sum_{i=1}^n ke_i \wedge Je_i = k\omega = \text{Ric} \wedge g\omega = 2 \sum_{i=1}^n \lambda_i e_i \wedge Je_i$$

Therefore $\lambda_1 = \lambda_2 = \ldots = \lambda_n = \frac{k}{2} = \frac{s}{n}$. \hfill \Box

Finally, to complete the proof of Theorem 2, we note that $M$ is a compact Kähler-Einstein manifold satisfying (iii). In [2], Theorem 5, we observe that in their proof of that theorem the hypothesis of positive holomorphic bisectional curvature is only used to assert that $R_{11*} > 0$, where $i \geq 2$ (p. 232). But $R_{11*} = K_{11} + K_{11*}$, which is positive by our assumption (iii). \hfill \blacksquare

3. **EIGENVALUES OF THE WEYL TENSOR**

Suppose $\alpha$ is any two-form in an orthonormal basis simultaneously diagonalizing Ric $\wedge g$ and $W$. Since eigenvalues of Ric $\wedge g$ have the form $\lambda_i + \lambda_j$ (sums of eigenvalues of the Ricci operator), we assume that Ric $\wedge g\alpha = (\lambda_i + \lambda_j)\alpha$, and that $\hat{W}\alpha = w\alpha$. Then on writing $\alpha = \sum_{k=1}^n \mu_k f_k \wedge f_k$, where $1 = \mu_1 \geq \mu_2 \geq \ldots \geq \mu_n \geq 0$ and $\{f_i\}_{i=1}^n$ is an orthonormal basis of $V$, we note that $\alpha$ is an eigenvector of $\hat{R}$ with eigenvalues given by (2), and also by the formula:

$$\hat{R}\alpha = \left\{ \frac{1}{2n-2} \left[ \text{Ric} - \frac{s}{2(2n-1)} g \right] \wedge g + W \right\} \alpha$$

$$= \left\{ \frac{1}{2n-2} \left[ \lambda_i + \lambda_j - \frac{s}{2(2n-1)} \right] + W \right\} \alpha$$

(3)
Equating (2) and (3) gives:

\[
    w = \frac{n - 2}{2(n - 1)}(\lambda_i + \lambda_j) \\
    + \frac{s}{(2n - 1)(2n - 2)} \\
    - \frac{1}{4} \sum_{k \geq 2} \{(1 + \mu_k)(\langle \hat{R}X_{1k}, X_{1k} \rangle) + (1 - \mu_k)(\langle \hat{R}X_{1k}', X_{1k}' \rangle)\}
\]  

(4)

Exploiting this equation, we obtain (cf. [7], Proposition 2.5):

**Proposition 3.** Let \( V \) be a 2n-dimensional vector space with curvature tensor \( R \) satisfying:

(i) \( \text{Ric} \wedge g, W \) = 0

(ii) \( a \leq K^{iso} \leq b \)

(iii) \( c \leq \text{Ric} \leq d \)

Then if \( d \leq (2n - 1)a \), or if \( (2n - 1)b \leq c \), \( R \) is conformally flat. In particular, if \( a = d = 0 \), or if \( b = c = 0 \), \( R \) is flat.

**Proof.** By (4) and the hypotheses above:

\[
    w = \frac{n - 2}{2(n - 1)}(r_i + r_j) \\
    + \frac{s}{(2n - 1)(2n - 2)} \\
    - \frac{1}{4} \sum_{i \geq 2} \{(1 + \mu_i)(\langle \hat{R}X_{1i}, X_{1i} \rangle) + (1 - \mu_i)(\langle \hat{R}X_{1i}', X_{1i}' \rangle)\} \\
    \leq \frac{n - 2}{n - 1}d + \frac{2nd}{(2n - 1)(2n - 2)} - 2a(n - 1) \\
    = \frac{2(n - 1)}{2n - 1}d - 2a(n - 1) \leq 0
\]

Thus since \( W \) has trace zero, we must have \( W = 0 \). In the second case we obtain \( w \geq 0 \), and so similarly, \( W = 0 \).

If \( a = 0 \), then \( s \geq 0 \) by [7], Proposition 2.5. But \( d = 0 \) implies eigenvalues of \( \text{Ric} \) are non-positive. Hence \( \text{Ric} \equiv 0 \) and \( s = 0 \), and so \( R = 0 \).

\( \square \)

Note that in the four-dimensional case (4) gives:

\[
    w = \frac{s}{6} - \frac{1}{4} \{(1 + \mu_i)(\langle \hat{R}X_{12}, X_{12} \rangle) + (1 - \mu_i)(\langle \hat{R}X_{12}', X_{12}' \rangle)\}
\]

We use this formula to show that the signature integrand of a compact, oriented 4-manifold without boundary is determined by curvatures on totally isotropic two-planes. Recall that the signature of a compact, oriented 4-manifold is given by \( \tau(M) = \frac{1}{12\pi^2} \sum_M (|W^+|^2 - |W^-|^2)\mu_g \).

Here \( W^+ \) and \( W^- \) are the parts of \( W \) acting on the +1 and -1 eigenspaces of the Hodge *-operator, respectively (cf. [1]). We will list the eigenvalues of \( W \) as follows:
Eigenvalues of $W^+$

$\frac{s}{6} - q_1$
$\frac{s}{6} - q_2$
$\frac{s}{6} - q_3$

Note that $\frac{s}{2} - (q_1 + q_2 + q_3) = \text{Tr}W^+ = 0$ and $\frac{s}{2} - (q_4 + q_5 + q_6) = \text{Tr}W^- = 0$.

Proposition 4. Let $(M, g)$ be a compact oriented Riemannian 4-manifold without boundary. Then:

(i) The 4-form representing the signature class of $M$ is completely determined by curvature on isotropic two-planes.

(ii) If $a \leq k^{iso} \leq b$, $\forall x \in M$, then $|\tau(M)| \leq \frac{2}{\pi^2} (b - a) \max(|a|, |b|) \text{Vol}(M)$.

(iii) The upper bounds in (ii) are invariant under dilations of the metric.

Proof. (i) For each point $x \in M$:

$$|W^+(x)|^2 - |W^-(x)|^2 = \sum_{i=1}^{3} \left( \frac{s(x)}{6} - q_i(x) \right)^2 - \sum_{i=4}^{6} \left( \frac{s(x)}{6} - q_i(x) \right)^2$$

$$= \sum_{i=1}^{3} (q_i(x))^2 - \frac{s(x)}{3} \left( \sum_{i=1}^{3} q_i(x) - \sum_{i=4}^{6} q_i(x) \right) - \sum_{i=4}^{6} (q_i(x))^2$$

$$= \sum_{i=1}^{3} (q_i(x) - q_{i+3}(x))(q_i(x) + q_{i+3}(x))$$

Since each $q_i(x)$ is made up of sums of multiples of isotropic curvatures, (i) follows from the formula $\tau(M) = \frac{1}{12\pi^2} \sum_M (|W^+|^2 - |W^-|^2)\mu_g$.

(ii) Note that as in the proof of Proposition 1, we have $2a \leq q_i(x) \leq 2b$, $\forall x \in M$. Thus $|q_i(x) - q_j(x)| \leq 2(b - a)$ and $|q_i(x) + q_j(x)| \leq 4 \max(|a|, |b|)$, and the final line in (5) is bounded above in absolute value by $24 (b - a) \max(|a|, |b|)$. Then on integrating we obtain:

$$\tau(M) = \frac{1}{12\pi^2} \int_M (|W^+(x)|^2 - |W^-(x)|^2)\mu_g \leq \frac{2}{\pi^2} (b - a) \max(|a|, |b|) \text{Vol}(M)$$

(iii) Given $t > 0$, and an $m$-dimensional manifold $(M, g)$, replacement of $g$ by $tg$ has the following effects on the volume and curvatures on totally isotropic two-planes of $M$:

(i) $\text{Vol}_{tg}(M) = t^m/2 \text{Vol}_g(M)$.

(ii) $K_{tg}^{iso} = \frac{1}{t} K_g^{iso}$

Since $M$, is a compact manifold, we may assume that the bounds $a$ and $b$ on $K^{iso}$ are actually achieved, and so for $(M, tg)$, the right hand side of the inequalities in (ii) change to (a constant multiple of) $\frac{1}{t} (b - a) \frac{1}{t} \max(|a|, |b|) t^2 \text{Vol}(M) = (b - a) \max(|a|, |b|) \text{Vol}(M)$.

We note that as a consequence of this theorem, any compact, oriented, Riemannian 4-manifold without boundary having constant curvature on totally isotropic two-planes must have signature zero.
REFERENCES


Received December 17, 1996
Stephen P. Bean
Department of Mathematics and Statistics
University of Nebraska at Kearney
Kearney Ne 68849-1110
USA