EXAMPLES OF LOCALLY CONFORMAL KÄHLER STRUCTURES

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Dedicated to Professor Haruo Kitahara in occasion of his sixtieth birthday

Abstract. We shall prove that there are locally conformal symplectic structures on products of two even-dimensional (resp. odd-dimensional) manifolds endowed with certain 2-forms of maximal rank. In particular, we shall show that the Riemannian product of a Sasakian manifold and a Kenmotsu manifold carries a locally conformal Kähler structure.

1. INTRODUCTION

Let $V$ be an even-dimensional differentiable manifold endowed with a non-degenerate 2-form $\Omega$, i.e., $(V, \Omega)$ is an almost symplectic manifold. An almost symplectic manifold $(V, \Omega)$ is called a locally conformal symplectic manifold by Vaisman in [9] if there is a global 1-form $\omega$, called the Lee form, on $V$ such that

$$d\Omega = \omega \wedge \Omega, \quad d\omega = 0. \quad (1)$$

If $\dim V \geq 6$, the first equation of (1) implies the second one. In particular, if the Lee form $\omega = 0$ on $V$, then $(V, \Omega)$ is a symplectic manifold. Hermitian manifolds are typical examples of almost symplectic manifolds. If, for the Kähler form $\Omega$ of a Hermitian manifold $V$, there is a 1-form $\omega$ satisfying the equation (1), then $V$ is called a locally conformal Kähler manifold in [9]. Vaisman in [9] gave some examples of locally conformal symplectic manifolds and locally conformal Kähler manifolds.

In this paper, we shall prove the following theorem.

Theorem A. There are locally conformal symplectic structures on the products of the following differentiable manifolds $M$ and $N$.

(a) $M$ is a $2m$-dimensional symplectic manifold with the exact symplectic form $d\theta$, and $N$ is a $2n$-dimensional locally conformal symplectic manifold with a 2-form $\Phi$ of maximal rank and the Lee form $\varphi$, i.e., $d\Phi = \varphi \wedge \Phi$.

(b) $M$ is a $(2m+1)$-dimensional contact manifold with the contact form $\Theta$, i.e., $\Theta \wedge (d\Theta)^m \neq 0$, and $N$ is a $(2n+1)$-dimensional differentiable manifold with a 2-form $\Phi$ and a 1-form $\varphi$ such that $\varphi \wedge \Phi^n \neq 0$ (i.e., almost contact), $d\Phi = \varphi \wedge \Phi$ and $d\varphi = 0$.

For example, the cotangent bundles of $m$-dimensional manifolds and the tangent bundles of $m$-dimensional Riemannian manifolds are $2m$-dimensional symplectic manifolds with the exact symplectic form (see, for instance, [2],[9]). On the other hand, there are some examples of locally conformal symplectic manifolds in [9] as mentioned above. Hence we obtain new examples of locally conformal symplectic manifolds by (a) of Theorem A.

In Section 3, we apply (b) of Theorem A to the products of certain Riemannian manifolds, and give some examples of locally conformal Kähler structures.
Throughout this paper, we shall be in $C^\infty$-category and deal with paracompact, connected differentiable manifolds without boundary. We assume that the dimensions of manifolds are at least two.

2. PROOF OF THEOREM A

We define a 2-form $\Omega$ on $M \times N$ as follows:

$$\Omega = d\theta + \Phi + \theta \wedge \varphi.$$  \hspace{1cm} (2)

First we prove that $\Omega$ is non-degenerate on $M \times N$. Let $X$ be a vector field on $M \times N$, and assume that $\Omega(X, Y) = 0$ for any vector field $Y$ on $M \times N$. Then it is sufficient to show that $X = 0$. By (2),

$$0 = \Omega(X, Y) = d\theta(X_1, Y_1) + \Phi(X_2, Y_2) + \frac{1}{2}\{\theta(X_1)\varphi(Y_2) - \theta(Y_1)\varphi(X_2)\}$$  \hspace{1cm} (3)

for any $Y$, where $X_1$ and $Y_1$ (resp. $X_2$ and $Y_2$) are $M$ (resp. $N$)-components of $X$ and $Y$, respectively. Since $Y$ is arbitrary, by setting $Y = Y_1$ or $Y = Y_2$ in (2), we obtain

$$d\theta(X_1, Y_1) = \frac{1}{2}\theta(Y_1)\varphi(X_2),$$  \hspace{1cm} (4)

$$\Phi(X_2, Y_2) = -\frac{1}{2}\theta(X_1)\varphi(Y_2)$$  \hspace{1cm} (5)

for any $Y_1$ and $Y_2$. Moreover, setting $Y_1 = X_1$ in (4), we obtain $\theta(X_1)\varphi(X_2) = 0$, i.e., $\theta(X_1) = 0$ or $\varphi(X_2) = 0$.

In the case (a) of Theorem A, if $\theta(X_1) = 0$, then, by (5), we have

$$\Phi(X_2, Y_2) = 0 \quad \text{for any } Y_2.$$  

Since $\Phi^n \neq 0$, we then obtain $X_2 = 0$. Thus, by (4), we also have

$$d\theta(X_1, Y_1) = 0 \quad \text{for any } Y_1.$$  

Since $(d\theta)^m \neq 0$, we obtain $X_1 = 0$, and hence $X = 0$. Similarly, if $\varphi(X_2) = 0$, then we conclude $X = 0$.

In the case (b) of Theorem A, since $\theta \wedge (d\theta)^m \neq 0$ (resp. $\theta \wedge \Phi^n \neq 0$), there is a nonvanishing vector field $\xi$ on $M$ (resp. $\xi$ on $N$) such that $\theta(\xi) = 1$ and $d\theta(\xi, Y_1) = 0$ for any $Y_1$ (resp. $\varphi(\xi) = 1$ and $\Phi(\xi, Y_2) = 0$ for any $Y_2$). If $\theta(X_1) = 0$, then, by (5), we have

$$\Phi(X_2, Y_2) = 0 \quad \text{for any } Y_2.$$  

Thus $X_2 = b\xi$, where $b$ is a function on $N$. So, by (4), we have

$$d\theta(X_1, Y_1) = \frac{b}{2}\theta(Y_1) \quad \text{for any } Y_1.$$
Setting $Y_1 = \zeta$, in this equation, we get $b = 0$, i.e., $X_2 = 0$. Thus, by (4), we also have

$$d\theta(X_1, Y_1) = 0 \quad \text{for any } Y_1.$$ 

Thus, $X_1 = a\zeta$, where $a$ is a function on $M$. Since $0 = \theta(X_1) = a$, we obtain $X_1 = 0$, and hence $X = 0$. Similarly if $\varphi(X_2) = 0$, then we conclude $X = 0$.

Next we prove $d\Omega = \varphi \wedge \Omega$. In fact, by (2),

$$d\Omega = d^2\theta + d\Phi + d\theta \wedge \varphi - \theta \wedge d\varphi$$

$$= \varphi \wedge \Phi + d\theta \wedge \varphi$$

$$= \varphi \wedge (d\theta + \Phi + \theta \wedge \varphi)$$

$$= \varphi \wedge \Omega.$$ 

Hence $\Omega$ and $\varphi$ defines a locally conformal symplectic structure on $M \times N$.

3. LOCALLY CONFORMAL KÄHLER STRUCTURES

A Hermitian manifold $V$ is called a locally conformal Kähler manifold if, for the Kähler form $\Omega$ of $V$, there is a 1-form $\omega$ satisfying (1). It is known that there are some examples of such manifolds, for instance, the Hopf manifolds $S^{2m-1} \times S^1$, the Inoue surfaces, locally conformal Kähler nilmanifolds, locally conformal Kähler solvmanifolds and so forth (cf. [5],[9]). In this section, we shall give new examples of locally conformal Kähler structures.

Let $M$ be a $(2m+1)$-dimensional almost contact manifold with the structure tensor $(\phi, \xi, \eta)$, i.e.,

$$\eta(\xi) = 1, \quad \phi^2 = -I + \eta \otimes \xi,$$

where $I$ denotes the identity transformation of the tangent spaces. An almost contact structure $(\phi, \xi, \eta)$ is said to be normal (cf. [3]) if $[\phi, \phi] + 2d\eta \otimes \xi = 0$, where $[\phi, \phi]$ denotes the Nijenhuis tensor of $\phi$. A Riemannian metric $g$ on $M$ is said to be compatible if

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any vector fields $X, Y$ on $M$. An almost contact manifold $M$ with a compatible Riemannian metric $g$ is said to have an almost contact metric structure $(\phi, \xi, \eta, g)$. It is known that there always exists an almost contact metric structure on an almost contact manifold. The fundamental 2-form $\Phi$ on an almost contact metric manifold $M$ is defined by

$$\Phi(X, Y) = g(X, \phi Y)$$

for all vector fields $X, Y$ on $M$. Then we have $\eta \wedge \Phi^n \neq 0$. If $\Phi = d\eta$, then $M$ is, by definition, a contact manifold. Such an almost contact metric structure is called a contact metric structure. Moreover if a contact metric structure is normal, it is called a Sasakian structure (cf. [3]). On the other hand, if there is a closed 1-form $\varphi$ on an almost contact metric manifold $M$ such that $[\phi, \phi] = 0, d\Phi = \varphi \wedge \Phi$ and $d\eta = \frac{1}{2} \varphi \wedge \eta$, then $M$ is called a locally conformal cosymplectic manifold (cf. [4],[5],[8]). If a locally conformal cosymplectic manifold $M$ is normal, then we
obtain $\varphi = f\eta$, where $f$ is a function on $M$ such that $df \wedge \eta = 0$. In particular, if $f = 2$, then $M$ is, by definition (cf. [6],[8]), a Kenmotsu manifold, i.e., $d\Phi = 2\eta \wedge \Phi$ and $d\eta = 0$.

We now recall the result of Morimoto [7] that the product of two normal almost contact manifolds is a complex manifold. Let $M_i$ be an almost contact manifolds with the structure tensor $(\phi_i, \xi_i, \eta_i)$ for each $i = 1, 2$. On the product manifold $V = M_1 \times M_2$, there is an almost complex structure $J$ defined by

$$J = \phi_1 + \eta_2 \otimes \xi_1 + \phi_2 - \eta_1 \otimes \xi_2. \quad (6)$$

Morimoto in [7] proved that this almost complex structure $J$ is integrable if and only if both almost contact structures of $M_1$ and $M_2$ are normal.

Moreover, if $g_1$ and $g_2$ are the compatible Riemannian metrics on $M_1$ and $M_2$ respectively, the Riemannian product metric $g = g_1 + g_2$ on $V$ is compatible with $J$, that is, $g$ is a Hermitian metric on $V$. Thus, if $M_i$ is normal for each $i = 1, 2$, the product manifold $V$ is a Hermitian manifold. Then its Kähler form $\Omega$ is given by

$$\Omega = \Phi_1 + \Phi_2 + 2\eta_1 \wedge \eta_2, \quad (7)$$

where $\Phi_i$ denotes the fundamental 2-form on $M_i$ for each $i = 1, 2$. Especially if $M_1$ and $M_2$ are a $(2m + 1)$-dimensional Sasakian manifold and a $(2n + 1)$-dimensional Kenmotsu manifold respectively, then we have

$$\eta_1 \wedge \Phi_i^m \neq 0, \quad \Phi_i = d\eta_i, \quad (8)$$

$$\eta_2 \wedge \Phi_2^n \neq 0, \quad d\Phi_2 = 2\eta_2 \wedge \Phi_2, \quad d\eta_2 = 0. \quad (9)$$

Thus, by Theorem A, the Kähler form $\Omega$ of $V = M_1 \times M_2$ with the Hermitian structure mentioned above satisfies $d\Omega = 2\eta_2 \wedge \Omega$. Hence we have

**Theorem B.** The Riemannian product of a Sasakian manifold and a Kenmotsu manifold endowed with the complex structure (6) is locally conformal Kähler.

Moreover, the following fact is well-known (for instance, see 1.167 in [1]).

**Proposition.** A Riemannian product $(M_1 \times M_2, g_1 + g_2)$ is conformally flat if and only if either

(a) $(M_i, g_i)$ is one-dimensional, and $(M_j, g_j)$ $(i \neq j)$ is of constant curvature, or

(b) $(M_1, g_1)$ and $(M_2, g_2)$ are of dimension at least two, one with constant curvature 1, and other with constant curvature $-1$.

The Hopf manifolds are Riemannian products of type (a). A conformally flat, locally conformal Kähler manifold is said to be locally conformally Kähler-flat in Vaisman [10]. The following theorem implies the existence of locally conformally Kähler-flat structures which are Riemannian products of type (b).

**Theorem C.** The Riemannian product of a Sasakian manifold $M_1$ and a Kenmotsu manifold $M_2$ endowed with the complex structure (6) is locally conformally Kähler-flat if and only if $M_1$ and $M_2$ have constant curvature 1 and $-1$, respectively.

**Remark.** It is well-known (cf. [3]) that there is the standard Sasakian structure on the unit sphere $S^{2m+1}$ in $\mathbb{C}^{m+1}$, and its compatible Riemannian metric has constant curvature 1. On the
other hand, $\mathbb{R} \times f\mathbb{C}^n$, where $f(t) = ce^t$ (c: positive constant, $t \in \mathbb{R}$), is an example of Kenmotsu manifolds with constant curvature $-1$ (see [6]). Thus, by Theorem C, $S^{2m+1} \times (\mathbb{R} \times f\mathbb{C}^n)$ carries a locally conformally Kähler-flat structure. But its Lee form $\omega$ is exact, i.e., $\omega = 2\eta_2 = 2dt$. Hence its locally conformally Kähler-flat structure is globally conformal Kähler (cf. [9]).
REFERENCES


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