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GROUPS WITH NEIGHBOURHOOD CONDITIONS FOR CERTAIN LATTICES

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Introduction. In the theory of infinite groups very often subgroups are considered as "very near" if only finite indices are involved. This is background for calling an extension of a soluble group by a finite group *almost suluble* or to use the term *almost equal* [8], [4] for two subgroups that have their intersection as subgroup of finite index in both of them. The scope of this note is nearer to the concept considered by Buckley, Lennox, B.H. Neumann, H. Smith and Wiegold [1]: they considered groups all of whose subgroups contain a subgroup of finite index which is normal in the whole group and called these subgroups *normal-by-finite*. In the sense of the title this is a neighbourhood condition imposed on the lattice of subgroups and the lattice of normal subgroups. There are two more or less obvious generalizations of this: One is to impose that every subnormal subgroup of a group contains a normal subgroup of finite index, and this was examined by Franciosi, deGiovanni and Newell [2]; the other is to consider groups G all of whose subgroups possess a subgroup of finite index which is subnormal in the whole of G. For a short description we adopt the wording of the two papers [1], [2] and speak of a subgroup S of G being *subnormal-by-finite* if there is a subnormal subgroup T of G that is a subgroup of finite index of S. (It is known that T can be chosen normal in S, whereby S becomes a normal extension of T by a finite group).

The purpose of this note is to examine the third named class as well as to add to the discussion of the other classes. The reader will see that the results of the two papers named before are often the key for statements made here. Here are two examples of results obtained.

Proposition A. Assume that A is a subnormal subgroup of the group G and that all subnormal subgroups of G are normal-by-finite. Then A^G/A_G is finite-by-nilpotent.

On the other hand it is shown that a weak solubility condition like that used in [2] is indispensible: an example is given of a group which is residually soluble but not soluble, such that all subnormal subgroups U are finite modulo U_G.

In addition, the defects of the subnormal subgroups are bounded by 4. On the other hand, for the class of groups of our main concern we have

Proposition B. Assume that all subgroups of the locally finite group G are subnormal-by-finite. Then the Hirsch-Plotkin-radical H of G is of finite index in G, and there are only finitely may primes p such that H possesses non-subnormal p-subgroups.

Every group is an extension of its finite residual by a residually finite group, and all conditions mentioned are inherited both by normal subgroups and by quotient groups. For the class considered in [2] (all subnormal subgroups are normal-by-finite) we have a strong restriction for the finite residual: it is a T-group with soluble quotients central. For the class of groups with all subgroups subnormal-by-finite we find

Proposition C. Assume that all subgroups of the group G are subnormal-by-finite. If S is a

subnormal subgroup of the finite residual $G^{\mathcal{F}}$ of G, we have:
(1) If T is a subgroup of finite index of S, then T is subnormal
(2) If S is perfect, it is a normal subgroup of G and it does not possess subgroups of finite index.

1. JOINS OF SUBNORMAL SUBGROUPS

It is well known that the join of two subnormal subgroups need not be subnormal. In our special case, however, we obtain subnormality. The following statement is known, the proof is therefore only sketched.

Lemma 1. *Assume that A and B are two subnormal subgroups of the group G and that there is a subnormal subgroup S of G which is a subgroup of finite index of $\langle A, B\rangle$. Then also $\langle A, B\rangle$ is subnormal in G.*

Proof. We may assume that the subnormal subgroup S is normal in $\langle A, B\rangle$. The proof then proceeds by induction on the defect d of S: If we have

$$G = R_0 \subseteq R_1 \subseteq \ldots \subseteq R_d = S$$

where $R_{i+1} = S^{R_i}$, we obtain that R_{i+1} is normal in $\langle A, B, R_i\rangle$ and therefore $\langle , B, R_{i+1}\rangle$ is subnormal in $\langle A, B, R_i\rangle$ by [6, Theorem 1.6.2] since AR_{i+1}/R_{i+1} and BR_{i+1}/R_{i+1} are finite. □

The next statement is almost obvious.

Lemma 2. *Assume that G is a group such that for every subgroup of G is subnormal by finite. If $\{V_i | i \in \mathcal{I}\}$ is a family of subnormal subgroups of G which is ordered by inclusion, then also $\bigcup_{i\in\mathcal{I}} V_i$ is subnormal in G.*

Proof. Let $Y = \bigcup_{i\in\mathcal{I}} V_i$. By hypothesis there is a subnormal subgroup Z of G which is a subgroup of finite index of Y. Without loss of generality we may assume that Z is a normal subgroup of Y. By [6, Theorem 1.2.1], all subgroups ZV_i are subnormal in G; they form a finite set which is ordered by inclusion. Now $Y = ZV_j$ for some $j \in \mathcal{I}$, and Y is subnormal. □

We combine these two statements and obtain

Theorem 3. *Assume that every subgroup U of the group G is subnormal-by-finite. Denote by M the subgroup of G which is generated by all cyclic subnormal subgroups of G. Then all subgroups of M are subnormal in G and G / M is a torsion group.*

Proof. Consider an element $x \in G$ which is of infinite order. By hypothesis there is a subgroup $\langle x^n\rangle \subseteq \langle x\rangle$ which is subnormal in G and therefore a subgroup of M. This shows that G/M is a torsion group. From Proposition A we deduce that finitely generated subgroups of M are subnormal in G. For every subgroup T of M there is a family $\{R_i | i \in \mathcal{I}\}$ which is ordered by inclusion, with the following properties. The trivial subgroup 1 is member of the family. If $i \in \mathcal{I}$ possesses a predecessor j, then there is an element x_j such that $R_i = \langle R_j, x_j\rangle$, if $i \in \mathcal{I}$ has no predecessor, then $R_i = \bigcap_{j<i} R_j$. Now, if i possesses a predecessor j and R_j is subnormal, then so is R_i by Lemma 1. If, on the other hand, i does not possess a predecessor

and all R_j with $j < i$ are subnormal, then the same is true for R_i by Lemma 2. By obvious transfinite induction, all subgroups of M are subnormal. □

Remark. If all subgroups of G are normal-by-finite, we have the stronger statement that there is an abelian normal subgroup N of G such that G / N is a torsion group, since in any group the subgroup generated by all infinite cyclic normal subgroups is abelian (see [1, Lemma 4.2 p. 394]). This leads to a slightly stronger statement than exhibited in [1]:

Lemma 4. *Assume that all subgroups of the group G are normal-by-finite and all finitely generated subgroups of G are nilpotent-by-finite. Then G is abelian-by-finite.*

Proof. We denote the FC-centre of G by F. The infinite cyclic normal subgroups of G are contained in $Z(F)$. Take a torsionfree subgroup Y of $Z(F)$ such that $Z(F) / Y$ is a torsion group. Then, by hypothesis, there is a subgroup X of finite index in Y such that X is normal in G; also we see that G / X is a torsion group. If S is a finitely generated subgroup of G, also SX / X is finitely generated and possesses by hypothesis a nilpotent normal subgroup TX / X of finite index. But then also TX / X is finitely generated and finite since it is a torsion group. We deduce that SX / X is finite and G / X is locally finite.

By the theorem of [1], GX / X possesses an abelian normal subgroup AX / X of finite index. So A is metabelian. By [1, Lemma 4.3], $G / C(X)$ is either trivial or of order 2. In particular, $A \cap C(X)$ is of finite index in G. It is an extension of an abelian torsionfree group by an abelian torsion group and also nilpotent of class (at most) 2. This yields that $A \cap C(X)$ is abelian. □

2. INTERSECTIONS OF NORMAL SUBGROUPS

The statement of this section is certainly known. A proof is added for the convenience of the reader.

Lemma 5. *Let G be some group and H be a fixed finite group. Denote a set of normal subgroups N of G with $G / N \cong H$ by $\mathcal{M}$. If $D = \bigcap_{N \in \mathcal{M}} N$, then G / N is locally finite, and locally nilpotent subgroups are nilpotent.*

Proof. If the set $\mathcal{M}$ is finite, nothing is to be shown. Assume therefore that $\mathcal{M}$ is infinite. We proceed by induction on the length of normal chains of H, i.e. of sequences $H = K_0 \subseteq K_1 \ldots \subseteq K_m = 1$ where K_{i+1} is a maximal normal subgroup of K_i, for $0 \leq i < m$. As initial step we assume that H is simple. If furthermore H is abelian, G / D is elementary abelian and the statement is true. Assume now that H is simple and non-abelian. Choose a subgroup K / D of G / D which is generated by the finite set $x_1 D, x_2 D, \ldots x_m D$ of elements. There are only $m^{|H|}$ many possibilities to choose m elements from H. For every $N \in \mathcal{M}$ we fix an isomorphism

$$\sigma_N : G / N \to H$$

and we form the equivalence classes of those normal subgroups $K, L \in \mathcal{M}$ satisfying

$$\sigma_K(x_i D) = \sigma_L(x_i D) \text{ for all } i.$$

According to the remark before there are at most $m^{|H|}$ many such classes. We choose from every class occurring one normal subgroup $N_k \in \mathcal{M}$ and consider $M = \cap_k N_k$. Choose any further normal subgroup $T \in \mathcal{M}$, then T will belong to one of the equivalence classes, for instance to that of N_j. Now $G / (T \cap M)$ is isomorphic to the direct product of the quotient groups G / N_i and G / T, and the canonical mapping of the elements $x_i D \in G / D$ onto $x_i N_k \in G / N_k$ resp. onto $x_i T \in G / T$ yields for those $yD \in \langle x_1, \ldots, x_m \rangle D / D$ that satisfy $y \notin T$ the fact that $y \notin N_j$ (since $\sigma_{N_j}(yD) = \sigma_T(yD)$) This shows that

$$\langle x_1, \ldots x_m \rangle (M \cap T) / (M \cap T) \cap T / (M \cap T) = 1$$

for every additional normal subgroup $T \in \mathcal{M}$, and the canonical epimorphism mapping $\langle x_1, \ldots, x_m \rangle D / D$ onto $\langle x_1, \ldots, x_m \rangle M / M$ is an isomorphism. So subgroups of G / D generated by m elements are finite, for every finite m. This finishes the initial step. Assume now as induction hypothesis that the Lemma is shown for all groups H^+ with subnormal chain length smaller than a number $t > 1$, and assume that H has chain lenght t. Choose some maximal normal subgroup K of H. Then H / K is a simple group. Let R be the intersection of all normal subgroups K^+ of H with $H / K^+ \cong H / K$. As before we have the set $\mathcal{M}$ of normal subgroups N of G such that $G / N \cong H$ and put $D = \bigcap_{N \in \mathcal{M}} N$. Let furthermore $\mathcal{N}$ be the set of *all* normal subgroups F / D of G / D such that $(G / D) / (F / D) \cong H / K$, and put $B / D = \bigcap_{F / D \in \mathcal{N}} F / D$. By our initial step we know that $(G / D) / (B / D)$ is locally finite. On the other hand, if $N \in \mathcal{M}$, then by construction $B / (B \cap N) \cong R$, and R has smaller normal chain length than H, and the induction hypothesis can be applied. This means that $B / (\bigcap_{N \in \mathcal{M}} (N \cap B) = B / D$ is locally finite. Now also G / D is locally finite as an extension of a locally finite group by a locally finite group (see for instance [7, Theorem 14.3.1]). If now Y / D is a locally nilpotent subgroup of G / D, we have that for every $N \in \mathcal{M}$, YN / N is nilpotent of class bounded by some integer t, and $Y_{t+1} \subseteq \bigcap_{N \in \mathcal{M}} N = D$. □

3. SUBGROUPS AND SUBNORMAL SUBGROUPS

We will use now our previous results for groups all of whose subgroups are subnormal-by-finite. First we find a normal subgroup with torsion quotient group.

Lemma 6. *Assume that every subgroup of the group G is subnormal-by-finite. Then G possesses a unique normal subgroup H such that H is maximal with respect to the property that all subgroups of N are subnormal. The quotient group G / N is a torsion group.*

Proof. By Theorem 3 we see that the normal subgroup N mentioned here is the subgroup generated by all cyclic subnormal subgroups of G. That G / N is a torsion group now follows from the fact that for every element $x \in G$ of infinite order there is a natural number k such that $\langle x^k \rangle$ is subnormal in G. □

Now we consider locally finite groups of this class. The following is a reformulation of Proposition B.

Theorem 7. *Assume that every subgroup of the locally finite group G is subnormal-by-finite.*
(1) If H is the Hirsch-Plotkin-radical of G, then G / H is finite.
(2) If F is the subgroup generated by all cyclic subnormal subgroups of G, the set of primes p_i such that H / F possesses elements of order p_i is finite.

Proof. To show the first statement we prove first that for every prime p there is a number f_p such that p-subgroups of G / H are of order at most f_p. Choose a maximal p-subgroup P of G.

By hypothesis there is a subnormal subgroup P^+ of G which is a subgroup of finite index of P. Since it is locally nilpotent, P^+ is a subgroup of H, and PH / H is finite, while $P \cap H$ is the maximal normal p-subgroup of G. In turn, if X / H is a finite maximal p-subgroup of G / H, then there is a maximal p-subgroup Y of G such that $YH = X$. Assume that there is a family $\{X_i / H\}$ of finite p-subgroups of G / H of ascending order. Then we have likewise a family $\{Y_i\}$ of p-subgroups, all containing P, such that the sequence of indices $|Y_i : P|$ is monotonic and unlimited. Now $\langle Y_i | i \leq i \rangle / P = Z_j / P$ is finite and possesses a sylow-p-subgroup S_j / P. By the sylow theorems it is possible to choose, for all j, the follower S_{j+1} such that $S_j \subseteq S_{j+1}$. We obtain a sequence of subgroups S_j which is ordered by inclusion, and so $\bigcup_{j=1}^{\infty} S_j$ is a p-subgroup of G. This is a contradiction since the maximal normal p-subgroup P is no longer a subgroup of finite index of this group. So such a family of subgroups does not exist; there is a bound f_p such that all p-subgroups of G / H have order not exceeding f_p. Having shown this preliminary statement, we assume that G / H possesses an infinite abelian subgroup V / H. Since all p-subgroups of G / H are finite, there is an infinite set Γ of primes q such that there are elements of order q in V / H for every $q \in \Gamma$. For every such prime q we choose an element y_q of G with the following properties: $(y_q)^q \in H$ while $y_q \notin H$. Now $\langle \dots, y_q, \dots \rangle = W$ is a locally (finite and soluble) subgroup such that WH / H is infinite; it is also the set-theoretical union of the finite soluble subgroups $W_n = \langle y_q | q < n \rangle$. The infinite set Γ of primes can be written as the set-theoretic union of infinitely many infinite subsets Λ_j such that no two of these subsets have an element in common. In every group W_n there is a Hall Λ_j-subgroup $W_{n,j}$, and we assume that $W_{n,j} \subseteq W_{n+1,j}$ for all n. Now $S_j = \bigcup_{n=1}^{\infty} W_{n,j}$ is locally finite, the orders of elements of S_j are divisible only by primes belonging to Λ_j, and $S_j\, H / H$ is infinite. By hypothesis there is a subnormal subgroup T_j of G which is a subgroup of finite index of S_j; in particular, $T_j \nsubseteq H$. Since the elements have relatively prime orders, the subnormal subgroups T_j and T_k centralize each other for $j \neq k$. Now we choose elements $x_j \in T_j$ for every j such that no x_j belongs to H. The subgroup $\langle \dots, x_j \dots \rangle = A$ is easily seen to be abelian, and AH / H to be infinite. There is a subnormal subgroup B of G which is of finite index in A. By commutativity, $B \subseteq H$, so $|AH : H| = |A : (A \cap H)| A : (A \cap H) \mid \leq |A : B| < \infty$ contrary to construction. This shows that the locally finite group G / H does not possess infinite abelian subgroups. By [5, 2.5 Corollary, p. 72] infinite locally finite groups possess infinite abelian subgroups, so G / H is finite, our first statement. We proceed to prove the second statement. To derive a contradiction, assume that there are infinitely many primes p_i such that there are elements $x_i F \in H / F$ that are of order p_i. Without loss of generality we may assume that each of the elements $x_i \in H$ are of order a power of p_i. Now $\langle x_i \mid i = 1, \dots \rangle = X$ is an infinite abelian subgroup of G, and there is a subgroup $T \subseteq X$ which is subnormal in G and has finite index in X. Since T is abelian, all subgroups of T are subnormal, so $T \subseteq F$ and XF / F is finite contrary to construction of X. □

Remark. If we consider groups with all subgroups subnormal-by-finite and all finitely generated subgroups nilpotent-by-finite, we find that they are extensions of a group with all subgroups subnormal by a locally finite group, and we can apply Proposition B for the quotient group. It seems an open question whether these groups are already (locally nilpotent)-by-finite (compare with Lemma 4).

4. SUBNORMAL AND NORMAL SUBGROUPS

We will prove here Proposition A; we begin by recalling that $Z_1(G) = Z(G) = \{x|x \in G, [x, g] = 1 \text{ for all } g \in G.\}$ and that $Z_k(G)$ is defined inductively by $Z_k(G) / Z_{k-1}(G) = Z(G / Z_{k-1}(G)$. For brevity we also make use of the notation U_G and U^G for the biggest normal subgroup included in resp. the smallest normal subgroup containing U.

Theorem 8. *Assume that every subnormal subgroup of G is normal-by-finite. If S is a finite subnormal subgroup of defect n of G, then* $S^G / Z_{n-1}(S^G)$ *is finite.*

Proof. The theorem is true for $n = 1$ with $Z_0(G) = 1$. Assume now that $n = 2$. Then S is a finite normal subgroup of S^G and therefore $C_{S^G}(S)$ is of finite index in S^G and a subnormal subgroup of G. By hypothesis there is a G-invariant subgroup C of finite index of $C_{S^G}(S)$. By construction, C is of finite index in S^G and $C \subseteq Z(S^G)$. This shows that the theorem is true for $n = 2$. Assume now that $n = k > 2$ and that the theorem is proved for all pairs G^+, S^+ with S^+ subnormal of defect at most $k - 1$ in G^+. We construct the series

$$G = V_0 \subseteq V_1 \subseteq \ldots \subseteq V_k = S,$$

where $V_{i+1} = S^{V_i}$ for $0 \leq i \leq n - 1$. Again, S is a finite normal subgroup of $V_{k=1}$, so $C_{V_{k-1}}(S)$ is a normal subgroup of finite index of V_{k-1}. It follows that $C_{V_{k-1}}(S)$ contains a G-invariant subgroup C of finite index, and $C \subseteq Z(V_1)$. Now V_{k-1} / C is a finite subnormal subgroup of defect $k - 1$ of G / C, so $(V_1, C)/ Z_{k-2}(V_1 / C)$ is finite by induction hypothesis. The statement now follows from $Z_{k-2}(V_1 / C) \subseteq (Z_{k-1}(V_1)) / C$. □

As usual we denote by $H^{\mathcal{N}}$ the intersection of all normal subgroup K of H with H / K nilpotent (the *nilpotent residual*). Now we deduce from Theorem 8:

Corollary 9. *(Proposition A) If every subnormal subgroups of G is normal-by-finite and S is some finite subnormal subgroup of G, then* S^G *is finite-by-nilpotent.*

Proof. Assume that S has defect n. By Theorem 8, $(S^G)/ Z_{n-1}(S^G)$ is finite. So there is a finite set Δ of conjugates $x^{-1}Sx$ of S such that

$$\langle x^{-1}Sx|x^{-1}Sx \in \Delta\rangle Z_{n-1}(S^G) = S^G.$$

Since S is finite, also $T = \langle x^{-1}Sx|x^{-1}Sx \in \Delta\rangle$ is finite (see for instance [6, Theorem 1.3.3]), and for all $k > n$ we have $(S^G)_k = (TZ_{n-1}(S^G))_k = T_k$, because the subgroup $Z_{n-1}(U)$ is the marginal subgroup to the commutator word of length n. In particular, $(S^G)^{\mathcal{N}} = T^{\mathcal{N}}$ is a finite normal subgroup of G, and

$$T^{\mathcal{N}} = \langle (x^{-1}Sx)^{\mathcal{N}}|x^{-1}Sx \in \Delta\rangle$$

(see [6, Theorem 4.4.1]). □

The next statement is now immediate.

Corollary 10. *Assume that all subnormal subgroups of G are normal-by-finite. If S is a perfect finite subnormal subgroup of G, then* S^G *is finite.*

5. LOCALLY SOLUBLE GROUPS

In the context of the groups considered in the previous section, Franciosi, de Giovanni and Newell [2] have paid particular attention to groups all of whose epimorphic images, not 1, possess abelian subnormal subgroups, not 1, and have shown that these groups are metabelian-by-finite (e [2, Theorem 3.4]). As we shall see in this section, this condition can not be weakened to local solubility. The example showing this will be a residually (finite and soluble) group such that the lattice of normal subgroups which are contained in the commutator subgroup is a decending chain of infinite length. The following statement is crucial.

Lemma 11. *Let p, q be two different odd primes. Further, let A be a finite split extension of an extraspecial p-group N by some group B such that N' is the only minimal normal subgroup of A, and $Z(A) = N'$. Then there is an extraspecial q-group M such that A operates faithfully on M / M' and all elements of M' are left invariant. If M is chosen of minimal order with this property, then in the split extension AM of M by A, (canonically defined), every subnormal subgroup S of AM with $S \cap M \neq S \cap NM$ contains M', and every subnormal subgroup T of AM with $T \cap M \neq T \cap MN'$ contains M.*

Proof. The first statement (existence of M) can be done by induction on the order of A, the proof is left to the reader. As to the second part, we find that there is an element $x \neq 1$ of order p in S and by construction $|M : [M,x]M'|^p \leq |M : M'|$. Since $p > 2$, we find that $[M,x] \subseteq M'$, and the statement follows from $S \subseteq [M,x]$ since the orders of M and of $\langle x \rangle$ are relatively prime. Accordingly in T we find an element y of order p which will operate without fixed points on M / M' (by minimality of M), so $T \subseteq [M,y] = M$. □

Example. We modify a method given by Hawkes [3, Theorem, p. 291] and construct a sequence of groups by the following prescription: First we define a sequence $\{p_i\}$ of odd primes such that, for each i, p_{i+1} is not a square modulo p_i. Next we define some extraspecial p_1-group A_1. If A_n is already defined, then A_{n+1} is a split extension of an extraspecial p_{n+1}-group K_{n+1} by a group B_{n+1} isomorphic to A_n such that $K' + n + 1 = Z(A_{n+1})$ and K_{n+1} is of smallest possible order. In particular, we have $[Z(B_{n+1}), K_{n+1}] = K_{n+1}$, and K_{n+1} / K'_{n+1} is a chief factor of A_{n+1} (here the special choice of the primes comes in; if the sequence of primes is only taken such that the follower is different from its predecessor, the interval $[K_{n+1} : K'_{n+1}]$ in the lattice of normal subgroups may also be a diamond). Having constructed these groups A_i we remember the existence of endomorphisms

$$\alpha_i : A_i \to A_{i+1}$$

and of epimorphisms

$$\beta_i : A_{i+1} \to A_i$$

which are obvious by the construction; we define them in such a way that $x^{\alpha_i\beta_i} = x$ for all $x \in A_i$. Consider the (unrestricted) direct product P of all A_i, and define the following subgroups D_n: the n-th component of D_n consists of all elements of A_n, if $x \in A_n$ is the n-th component of some element in D_n, its $(n - i)$-th component is $x^{\beta_{n-1}\cdots\beta_{n-i}}$, while its $(n + i)$-th component is $x^{\alpha_n\cdots\alpha_{n+i-1}}$. By the construction we have $D_n \subseteq D_{n+1}$ for all n, and also $D_n \cong A_n$ for all n. So all D_n are soluble, and $D^* = \bigcup_{n=1}^{\infty} D_n$ is a locally soluble group. By the construction of this set-theoretical union we see that normal subgroups of D^* are either some term $(D^*)^{(n)}$ of the derived series of D^* or contain $(D^*)'$. Consider now a subnormal subgroup S of D^*, and let m be maximal such that the first m components of every elements of S, consider as element of P, is 1. Then, by the preceding Lemma, $(D^*)^{(2m+3)} \subseteq S$, a normal subgroup of D^* which is of finite index in D^* and in S. The sequence of primes $\{p_i\}$ may be the alternating sequence of the primes 3,5 as is easily checked. Using the results of Hawkes [3] we see that in D^* all subnormal subgroups have defect at most 4.

6. THE FINITE RESIDUAL

For any group G we will put $G^{\mathcal{F}} = \bigcap_{|G:U|<\infty} U$ and call this subgroup of G the *finite residual* of G. In our case the finite residual have a particular structure.

Lemma 12. *Assume that every subgroup of G is subnormal-by-finite. Then if S is a perfect subnormal subgroup of $G^{\mathcal{F}}$, it is a normal subgroup without subgroups of finite index.*

Proof. We proceed by induction on the defect of S, showing first that S does not possess subgroups of finite index. For the initial step we assume that S is normal in G and that S possesses subgroup T such that $|S : T|$ is finite. The number of conjugates of T is S is finite, so there is a normal subgroup T^+ of finite index of S. Consider now the set of conjugates of T^+ in G and call D the intersection of all of these conjugates. By Lemma 5 we know that S / D is locally finite. Assume that H / D is the Hirsch-Plotkin-radical of S / D. By Lemma 5, H / D is nilpotent. Since only finitely many primes occur as orders of elements of S / H and by lemma 7 the orders of p-subgroups of S / H are bounded by some number f_p. Now S / H is finite since only finitely many primes occur, and the perfectness of S together with the nilpotency of H / D leads to $S / H \neq 1$. Let $W / H = C_{G/H}(S / H)$. By finiteness of S / H we have that W is a subgroup of finite index in G, and $W \not\subseteq S$ by noncommutativity of S / H, contrary to $S \subseteq G^{\mathcal{F}}$. So the normal subgroup S of G which is contained in $G^{\mathcal{F}}$ has no subgroups of finite index. Assume now that we have shown the same for perfect subnormal subgroups T of defect k that are contained in $G^{\mathcal{F}}$. Let S be a subnormal subgroup of G with the same properties, but of defect $k + 1$. There is the series

$$G = V_0 \subseteq V_1 \subseteq \ldots \subseteq V_k \subseteq V_{k+1} = S$$

such that $V_{i+1} = S^{V_i}$ for all natural i. So the perfect subnormal subgroup V_k does not possess subgroups of finite index, and the same is true for the normal subgroup U of $V_k = V_k^{\mathcal{F}}$. Now all perfect subnormal subgroups of $G^{\mathcal{F}}$ are shown to be without subgroups of finite index.

We still have to show that all perfect subnormal subgroups S of $G^{\mathcal{F}}$ are normal subgroups of G. Using the same series as before for a subnormal subgroup S of defect $k > 1$, we consider S as a subnormal subgroup of defect 2 of V_{k-2} and remember that $V_{k-1} = V_{k-1}^{\mathcal{F}}$. There is an element $x \in V_{k-2}$ that does not normalize S. This element x is either of finite order or some subgroup $\langle x^m \rangle$ is subnormal, and $[[\ldots[y, x^m], \ldots], x^m] = 1$ whenever the number of entries x^m occurring exceeds a fixed number t. Therefore there is a finite number s such that $C = S^{\langle x \rangle}$ $= \langle S^{x^r} \mid 0 \leq r \leq s \rangle$. Let s be minimal with this property. Then $A = \langle S^{x^r} \mid 0 \leq r \leq r - 1 \rangle$ $\neq C$ but $AA^x = C$. Since $C^x = C$, we obtain $C / A \cong C / A^x$. If $B = A \cup A^x$, we have C / B

$= (A/B) \times (A^x/B)$ and $A^x/B \cong C/A \cong C/A^x \cong A/B$. Using this isomorphism of the two direct factors we have a subgroup K/B of C/B such that $KA = KA^x = C$ and $K \cup A = K \cup A^x = B$. By hypothesis there is a subnormal subgroup L of G such that $K \subseteq L \subseteq B$ and $|K : L| < \infty$. The subnormality of L yields that the sequence $A, [A, L], [[A, L], L], \ldots$ will lead to subgroups of $L \cup A = B$ after a finite number of steps, and so L/B is nilpotent. On the other hand, $K/B \cong A/B$ is perfect, so $K \neq L$, and LA is a proper subgroup of finite index of C. Since C is a perfect normal subgroup of $V_{k-1} = V_{k-1}^{\mathcal{F}}$, this is impossible by the first part of our proof. So the defect of S can only be 1 in G and S is normal in G. □

We obtain also the following statement on quotients of perfect subnormal subgroups.

Lemma 13. *Assume that every subgroup of G is subnormal-by-finite and that A and B are two perfect normal subgroups of $G^{\mathcal{F}}$. Then there is no normal subgroup $K \subseteq A \cap B$ of $G^{\mathcal{F}}$ such that $AK/K \cong BK/K \neq 1$.*

Proof. Since $A \cap B \subseteq K$ we have

$$AK/K \cong A/(A \cap K) \cong A(B \cap K)/(A \cap K)(B \cap K)$$

and, by symmetry,

$$A(B \cap K)/(A \cap K)(B \cap K) \cong AK/K \cong BK/K \cong B(A \cap K)/(A \cap K)(B \cap K),$$

furthermore

$$A(B \cap K) \cap B(A \cap K) = (A(B \cap K) \cap B)(A \cap K) = (A \cap B)(B \cap K)(A \cap K)$$

$A \cap B \subseteq K$ now yields

$$A(B \cap K) \cap B(A \cap K) = (B \cap K)(A \cap K).$$

we have seen

$$AB/(A \cap B)(B \cap K) = A(B \cap K)/(A \cap K)(B \cap K) \times B(A \cap K)/(A \cap K)(B \cap K)$$

with both factors isomorphic by an isomorphism, say, σ. The subgroup $U/(A \cap K)(B \cap K)$ of $AB/(A \cap K)(B \cap K)$ consisting of the elements $aa^{\sigma}(A \cap K)(B \cap K)$ satisfies the conditions $UA = UB = AB$; $U \cap A(B \cap K) = U \cap B(A \cap K) = (A \cap K)(B \cap K)$. The subnormal subgroup V of G which is of finite index in U can be chosen such that $(A \cap K)(B \cap K) \subseteq U$, and $U/(A \cap K)(B \cap K)$ is nilpotent by the same argument as in Lemma 12. So $V \neq U$; but then also $VA \cap B$ is a (proper) subgroup of finite index of B, a contradiction to Lemma 12. □

For more explicit statements we would for instance need more facts on the normal extensions (automorphism groups) of the so-called Tarski monsters than seem to be known just now.

In the non-perfect case we find

Lemma 14. *Assume that every subgroup of the group G is subnormal-by-finite.*
If $T \subseteq S \subseteq G^{\mathcal{F}}$ with S subnormal in G and $|S : T| < \infty$, then also T is subnormal in G.

Proof. We will proceed by induction on the defect of S. For our initial step we assume that S is a normal subgroup of G. Since T is of finite index in S, there is a subgroup T^+ of finite index of S such that $T^+ \subseteq T \subseteq S \subseteq N(T^+)$. To derive a contradiction, we assume further that T is not subnormal in S; in particular, S / T^+ is not nilpotent. Denote by D the intersection of all normal subgroups N of S with $S / N \cong S / T^+$. By Lemma 5 we know that S / D is locally finite and that its Hirsch-Plotkin-radical H / D is nilpotent, furthermore we deduce from Lemma 7, that S / H is finite. The centralizer $C_{G / H} (S / H)$ is of finite index in G / H, and so $G^{\mathcal{F}} / H$ centralizes S / H showing that S / H is abelian. We have already seen that S / T^+ is non-nilpotent, therefore S / D cannot be nilpotent and $S \neq H$. Pick a prime p dividing the order of S / H and choose a maximal p-subgroup P / D of S / D. Since S / D is locally finite and $(P / D)/((P / D) \cap (H / D))$ is finite, we find $(HP / D)\, N_{G / D} (P / D) = G / D$ by the Frattini Lemma. Put $K / D = N_{G / D} (P / D)$. By hypothesis there is a subgroup K^+ / D of finite index of K / D which is subnormal in G / D. Now $(K^+ / D)\, (HP / D)$ and (K^+ / D) (H / D) are of finite index in G / D. Also $P / D \not\subseteq K^+ / D$ since P / D is not subnormal in S / D. This shows that also $P / D \not\subseteq G^{\mathcal{F}}$ contradiction $P \subseteq S \subseteq G^{\mathcal{F}}$. This contradiction shows that our assumption that S / T^+ is non-nilpotent was wrong. The initial step is proved. Assume now that the lemma is proved for all subnormal subgroups of defect up to $k-1$, where $k > 1$, in other words, finite epimorphic images of such subnormal subgroups are nilpotent. Consider a subnormal subgroup S of defect k with the canonical subnormal sequence

$$G = L_0 \subseteq L_1 \subseteq \ldots \subseteq L_{k-1} \subseteq L_k = S$$

where $L_{i+1} = S^{L_i}$ for $0 \leq i < k$. Again, consider a subgroup T of S of finite index with intersection T^+ of all S-conjugates of T, and denote by D the intersection of all normal subgroups N of S satisfying $S / N \cong S / T^+$. As before we assume taht T is not subnormal in S and so S / T^+ and S / D are not nilpotent. If H / D is the Hirsch-Plotkin-radical of S / D, it is nilpotent and S / H is finite. By a famous theorem of P. Hall (see for instance [7, Theorem .]), S / H' is not nilpotent. We may therefore assume without loss of generality thet $H' = D$ and H / D is abelian. For a prime p dividing the order of S / H we consider the maximal p-subgroup P / D of S / D. Because of the finiteness of S / H we may apply the Frattini Lemma for S / H to obtain $(S / H)\, N_{(L_{k-1} / H)} (PH / H) = L_{k-1}/ H$. Again the finiteness of S / N yields that $N_{L_{k-1} / H} (PH / H)$ is a subgroup of finite index of L_{k-1} / H and therefore subnormal. But then also its intersection with S / H, that is $N_{S / H} (PH / H)$, is subnormal, and PH / H is a normal subgroup of S / H. Now all maximal p-subgroups of S / D are conjugate to P / D, and the Frattini Lemma yields $(S / D)\, N_{L_{k-1} / D} (P / D) = L_{k-1} / D$. By hypothesis there is a subgroup W / D of finite index in $N_{L_{k-1} / D} (P / D)$ which is subnormal in L_{k-1} / D. Denote by K / D the complement of $(H \cap P) / D$ in H / D. Since H / D is commutative and of finite index, the same applies for K / D, and since the elements have relatively prime orders we have $K / D = [K / D, P / D] \times C_{K / D} (P / D)$; again without loss of generality we may assume that $C_{K / D} (P / D) = (W \cap K)/ D$. Furthermore, $R / D = [K / D, P / D]$ is a normal subgroup of L_{k-1} / D and $[R / D, W / D] \neq R / D$ since W / D is subnormal and intersects R / D trivially. Consider a maximal subgroup M / D of R / D which contains $[R / D, W / D]$. Then $(H / D)\, (W / D) \subseteq N_{L_{k-1}(M / D)}$ and M / D has only finitely many conjugates in L_{k-1}. The intersection M^+ / D of all conjugates of M / D under L_{k-1} / D is therefore of finite index in K / D, and S / M^+ is finite and non-nilpotent. Now also $(L_{k-1} / M^+)/ C_{L_{k-1} / M^+} (S / M^+)$

is finite and nilpotent and we obtain that also the subgroup $(S / M^+) / Z(S / M^+)$ is nilpotent. This is a contradiction to the non-nilpotency of S / M^+. This finishes the induction argument. □

Proposition C now follows from Lemma 12 and Lemma 14. Now we have a look at finite residuals of the second class of groups considered in this note.

Lemma 15. *Assume that all subnormal subgroups of the group G are normal-by-finite and that N is a residually finite normal subgroup of G which is contained in $G^{\mathcal{F}}$. Then $N \subseteq Z(G^{\mathcal{F}})$.*

Proof. Assume W is a subgroup of finite index of N. Then there is a normal subgroup W^+ of finite index in N, and this is a subnormal subgroup of G. By hypothesis $W^{++} = (W^+)_G$ is of finite index in W^+. Now N / W^{++} is a finite normal subgroup of G / W^{++} and $C / W^{++} = C_{G / W^{++}} (N / W^{++})$ is of finite index in G / W^{++}. This shows $[N, G^{\mathcal{F}}] \subseteq W^{++} \subseteq W^+ \subseteq W$ for all subgroups of finite index W of N, and $[N, G^{\mathcal{F}}] \subseteq N^{\mathcal{F}} = 1$. □

Lemma 16. *Assume that all subnormal subgroups of the group G are normal-by-finite. If N is an abelian normal subgroup of G which is contained in $G^{\mathcal{F}}$, then $N \subseteq Z(G^{\mathcal{F}})$.*

Proof. Consider an element $x \in N$, we want to show that x belongs to $Z(G^{\mathcal{F}})$. Consider the smallest normal subgroup X of G which contains x. If x is of finite order, the normal subgroup X is of finite exponent and abelian and therefore residually finite. By the preceding Lemma we have that $X \subseteq Z(G^{\mathcal{F}})$ in this case. If x is of infinite order, there is a subgroup $\langle x^n \rangle$ of finite index in $\langle x \rangle$ which is normal in G. Now X is abelian and an extension of an infinite cyclic group by a group of finite exponent, so X is residually finite also in this case, and $X \subseteq Z(G^{\mathcal{F}})$. In particular, $x \in Z(G^{\mathcal{F}})$ for all $x \in N$. □

The last statement can be strengthened considerably; we obtain

Lemma 17. *Assume that every subnormal subgroup of the group G is normal-by-finite and that N is a soluble normal subgroup of G which is contained in $G^{\mathcal{F}}$. Then $N \subseteq Z(G^{\mathcal{F}})$.*

Proof. Since N is soluble, it has a characteristic series with abelian factors. By the preceding Lemma, all these factors are central in $G^{\mathcal{F}}$ and therefore also in N. As a first step we deduce
(1) N is nilpotent.

Now we apply [2, Corollary 3.3] to obtain that N possesses an abelian normal subgroup K of finite index. Certainly K is subnormal in G, and by hypothesis $K_G = L$ is of finite index in N. Now $L \subseteq Z(G^{\mathcal{F}})$ since L is abelian, and finiteness of N / L yields $[N, G^{\mathcal{F}}] \subseteq L$. We summarize
(2) $N / Z(N)$ is finite and abelian.

Choose a maximal torsionfree subgroup F of $Z(N)$. Since F is a subnormal subgroup of G, we know that $F_G = H$ is of finite index in F, and so N / F and N / H are torsion groups and also
(3) $(N / H) / Z(N / H)$ is finite and abelian.

Consider any element xH of N / H. This element is of finite order and $X / H = \langle xH \rangle^{G / H} = \langle x \rangle^G H / H$ is a nilpotent group of class 2, of finite exponent, and with a centre of finite index. It follows that X / H is residually finite, and by Lemma 15, $X / H \subseteq Z(G^{\mathcal{F}} / H)$ and also $xH \in Z(G^{\mathcal{F}} / H)$ for every $xH \in N / H$. So
(4) N / H is abelian,

and since $N / Z(N)$ is an abelian torsion group, also N' is a torsion group, so from $N' \subseteq H$ we obtain

(5) N is abelian.

The result now follows from Lemma 16. □

We are now in a position to prove our statement on $G^{\mathcal{F}}$.

Proposition 18. *Assume that every subnormal subgroup of the group G is normal-by-finite. Then the finite residual $G^{\mathcal{F}}$ is a T-group with the following additional property: If M is a normal subgroup of $G^{\mathcal{F}}$, then $[M, G^{\mathcal{F}}]$ is perfect and does not possess subgroups of finite index.*

Proof. Assume that S is a subnormal subgroup of $G^{\mathcal{F}}$. Clearly S is also subnormal in G, and there is a G-invariant subgroup L of S such that $|S : L|$ is finite. By Theorem 8 there is an integer k such that $(S^G / L)/ Z_k\, (S^G / L)$ is finite. Now by Lemma 15 and Lemma 17 (S^G / L) $/ Z_k\, (S^G / L)$ is abelian and $S^G / L \subseteq Z\, (G^{\mathcal{F}} / L)$. So $[S^G, G^{\mathcal{F}}] \subseteq L$ and S is normal $G^{\mathcal{F}}$. This shows that $G^{\mathcal{F}}$ is a T-group. The additional property follows from the fact that soluble quotients are central, so $([M, G^{\mathcal{F}}])^{\mathcal{F}}$ $[[M, G^{\mathcal{F}}], G^{\mathcal{F}}] \subseteq [M, G^{\mathcal{F}}]$ for all normal subgroups M of G which are contained in $G^{\mathcal{F}}$. □

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