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SOME DUALITY RESULTS FOR THE CAPACITY THEORY RELATED TO NON SYMMETRIC DIRICHLET FORMS

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Abstract. *This paper deals with some dual formulations related to the capacity of a set in the framework of coercive closed forms. Through the theory of dual variational inequalities introduced by Mosco in [14] one gets some suitable relations between potential and capacitary distribution which become equalities for Dirichlet forms.*

0. INTRODUCTION

The recent development of the theory of Dirichlet forms is due to the growing interest in its applications. In particular, the study of fractals and of Dirichlet forms progresses at the same time, since it seems that the latter represent the most appropriate instrument to an adequate variational description of sets without differentiable structure.

The success of the modern theory of Dirichlet forms is also based on the rich interplay between its analytic and probabilistic components. The analytic part of the theory goes back to the pioneering papers of A. Beurling and J. Deny [1] [2] [4], whereas the more recent probabilistic part was really initiated by the fundamental work of M. Fukushima [5] [6] [7] [8] and M.L. Silverstein [15] combining symmetric Markov processes and Dirichlet forms. The link between these two parts is the analytic theory of potential for the Dirichlet forms that, for this reason, become very important. Z.M.Ma and M. Röckner, in their book [11], extend the theory of potential to non symmetric Dirichlet forms; in particular, they give a new definition of capacity that coincides with the classic one when the form is the energy form.

The idea of defining the capacitary potential as the solution of a suitable variational inequality was introduced, in some celebrated papers, by Stampacchia [16] [17] etc., in the framework of non selfadjoint uniformly elliptic operators. More precisely in the present paper, as in [11], the capacitary potential of a compact set E is defined as the solution of the variational inequality:

$$u \in K : \quad \mathcal{E}_1(u, v-u) \geq 0 \quad \forall v \in K \tag{0.1}$$

with $\mathcal{E}_1(\cdot,\cdot) = \mathcal{E}(\cdot,\cdot) + (\cdot,\cdot)$, where $(\cdot,\cdot)$ is the inner product of a Hilbert space H, $\mathcal{E}(\cdot,\cdot)$ is a regular coercive closed form with domain $D \subset H$ and K is a suitable subset of the domain D. On the other hand the capacitary copotential is defined as the solution of the adjoint variational inequality:

$$\hat{u} \in K : \quad \mathcal{E}_1(v-\hat{u}, \hat{u}) \geq 0 \quad \forall v \in K. \tag{0.1$'$}$$

The aim is to generalize a result in [13] for second order not self-adjoint uniformly elliptic operators, to regular coercive closed forms also in case Dirichlet property is not satisfied. In particular the capacitary distribution will be obtained as solution of the variational inequality, dual to (0.1). In our approach four variational inequalities appear: the basic (0.1), its adjoint

(0. 1′), its dual and the adjoint dual. It is interesting to check that the relations between the four solutions are in general expressed by inequalities that become equalities if the form $\mathcal{E}(\cdot,\cdot)$ we consider in (0.1) is a regular Dirichlet form. However potential and copotential, in general, coincide only in the symmetric case, see the Example in section 3.

Finally let us point out the interest of the dual formulation for capacity theory in the framework of numerical approximations. To this purpose we refer to [9] and to [13].

1. NON SYMMETRIC DIRICHLET FORMS

We present here, for reader's convenience, the necessary background from functional analysis and present the analytic framework of the theory of Dirichlet forms on arbitrary measure space.

Let H be a real Hilbert space equipped with the inner product $(\cdot,\cdot)_H$, D a linear subspace of H and let $\mathcal{E}: D\times D\to \mathbf{R}$ be a bilinear map.

Definition 1.1. *A pair $(\mathcal{E},D)$ is called a closed form (on H) if D is a dense linear subspace of H and $\mathcal{E}: D\times D\to\mathbf{R}$ is a positive semidefinite bilinear form which is symmetric and D is complete w.r.t. the norm $\mathcal{E}_1^{\frac{1}{2}}$.*

$$\|u\|_{\mathcal{E}_1^{\frac{1}{2}}}=\mathcal{E}_1(u,u)^{\frac{1}{2}}=(\mathcal{E}(u,u)+(u,u)_H)^{\frac{1}{2}}. \tag{1.1}$$

To give an analogous definition for non symmetric forms, it is necessary to define

the *symmetric part* of $\mathcal{E}$ $\quad\tilde{\mathcal{E}}(u,v)=\frac{1}{2}(\mathcal{E}(u,v)+\mathcal{E}(v,u))$
the *antisymmetric part* of $\mathcal{E}$ $\quad\check{\mathcal{E}}(u,v)=\frac{1}{2}(\mathcal{E}(u,v)-\mathcal{E}(v,u))$

Clearly $(\mathcal{E}(u,v),D)=(\tilde{\mathcal{E}}(u,v)+\check{\mathcal{E}}(u,v),D)$.

So we have

Definition 1.2. *A pair $(\mathcal{E},D)$ is called a coercive closed form (on H) if D is a dense linear subspace of H and $\mathcal{E}: D\times D\to\mathbf{R}$ is a bilinear form such that the following conditions hold:*

i) its symmetric part $(\tilde{\mathcal{E}},D)$ is a closed form on H

ii) $(\mathcal{E},D)$ satisfies the following "weak sector condition":

there exists a constant $K>0$ (called continuity constant) such that

$$|\mathcal{E}_1(u,v)|\le K\mathcal{E}_1(u,u)^{\frac{1}{2}}\mathcal{E}_1(v,v)^{\frac{1}{2}}\qquad \forall u,v\in D \tag{1.2}$$

i.e. $(\mathcal{E}_1,D)$ is continuous w.r.t. the norm $\tilde{\mathcal{E}}_1^{\frac{1}{2}}$ on D.

We recall that a continuous bilinear form $(\mathcal{E},D)$ on some Hilbert space H is called coercive on H if there exists a constant $c>0$ such that $\mathcal{E}(u,u)\ge c(u,u)_H$ for all $u\in D$.

Clearly, $(\mathcal{E},D)$ is then a coercive closed form on H and the following condition holds: there exists a constant $K>0$ such that

$$|\mathcal{E}(u,v)|\le K\mathcal{E}(u,v)^{\frac{1}{2}}\mathcal{E}(v,u)^{\frac{1}{2}}\quad \forall u,v\in D. \tag{1.3}$$

Let us recall now the following basic theorem due to G. Stampacchia ([16]). From now on (unless otherwise stated) we consider D to be equipped with the norm $\tilde{\mathcal{E}}_1^{\frac{1}{2}}$ and we shall call it intrinsic norm.

Proposition 1.3. *Let $(\mathcal{E}, D)$ be a coercive closed form on H and let C be a non-empty closed convex subset of D. Let J be a continuous linear functional on D. Then there exists a unique $v \in C$ such that*

$$\mathcal{E}_1(v, w - v) \geq J(w - v) \qquad \forall w \in C. \tag{1.4}$$

Proof. See, in the framework of Dirichlet forms, [11] Theorem 2.6.

Now we summarize the main definitions and properties of closed forms and coercive closed forms which are necessary in the following sections.

Definition 1.4. *Let $(A, D(A))$ be a linear operator densely defined on H. A is called a generator if:*

i) $(0, \infty) \subseteq \rho(A)$

ii) $\|\alpha(\alpha - A)^{-1}\| \leq 1 \quad \forall \alpha > 0$

where, $\rho(A)$ is defined to be the set of all $\alpha \in \mathbf{R}$ such that $(\alpha - A) : D(A) \to H$ is one-to-one and for its inverse $(\alpha - A)^{-1}$ we have

a) $D((\alpha - A)^{-1}) = H$

b) $(\alpha - A)^{-1}$ *is continuous on H.*

Let us remark that if A is a selfadjoint negative semidefinite operator (*i.e.*, A coincides with its adjoint $\hat{A}$ on H and $(Au, u) \leq 0$ for all $u \in D(A)$) hence i), ii) hold; see e.g. [12].

Definition 1.5. *A family $\{G_\alpha, \alpha > 0\}$ of linear operators on H with $D(G_\alpha) = H$ for all $\alpha \in (0, \infty)$ is called a strongly continuous contraction resolvent on H if:*

i) αG_α *is a contraction on H for all $\alpha > 0$.*

ii) $G_\alpha - G_\beta = (\beta - \alpha) G_\alpha G_\beta$ *for all $\alpha, \beta > 0$.*

iii) $\lim_{\alpha \to \infty} \alpha G_\alpha u = u$ *for all $u \in H$.*

For any given coercive closed form $(\mathcal{E}, D)$, there always exist a generator A and a strongly continuous contraction resolvent $\{G_\alpha, \alpha > 0\}$ associated with $(\mathcal{E}, D)$. They are defined as follows:

$$\begin{cases} D(A) = \{u \in D : w \to \mathcal{E}(u, w) \text{ is continuous w.r.t. } (\cdot, \cdot)_H^{\frac{1}{2}} \text{ on } D\} \\ (-Au, v)_H = \mathcal{E}(u, v) \quad \forall u \in D(A)\ \forall v \in D. \end{cases} \tag{1.5}$$

Moreover one has that $D(A)$ is a dense linear subspace of D w.r.t. the norm $\tilde{\mathcal{E}}_1^{\frac{1}{2}}$.

On the other side, for all $u \in H$, $G_\alpha u$ is the unique element in H such that

$$\mathcal{E}_\alpha(G_\alpha u, v) = \mathcal{E}(G_\alpha u, v) + \alpha(G_\alpha u, v)_H = (u, v)_H \quad \forall v \in D \tag{1.6}$$

and one has that the range $R(G_\alpha) \subseteq D$. Furthermore one can show that the generator and the resolvent are connected by the relation:

$$G_\alpha u = (\alpha - A)^{-1} u \qquad u \in \overline{D(A)} = H \quad \alpha > 0 \tag{1.7}$$

with $R(G_\alpha) = D(A)$.

In addition, the generator and the resolvent are not self-adjoint operators. Let us call the *cogenerator* and *coresolvent* the adjoint operators of A and G_α. Moreover, in this case, the generator and the resolvent satisfy the following properties respectively:

$$|((I-A)u,v)_H| \leq \text{ const.}\,((I-A)u,u)_H^{\frac{1}{2}}((I-A)v,v)_H^{\frac{1}{2}} \tag{1.8}$$

where I is the identity operator and

$$|(G_1u,v)_H| \leq \text{const.}\,(G_1u,u)_H^{\frac{1}{2}}(G_1v,v)_H^{\frac{1}{2}}. \tag{1.9}$$

For further details see [8], [11] and [12].

From now on, we replace H by the concrete space $L^2(X,m)$ with the usual inner product $(\cdot,\cdot)_{L^2}$ where (X,m) is a measure space. As usually we set for all $u,v : X \to \mathbf{R}$

$$u \vee v := \sup(u,v), \quad u \wedge v := \inf(u,v), \quad u^+ : u \vee 0, \quad u^- := -(u \wedge 0).$$

As we mentioned in the Introduction we are interested in considering the dual inequality of (0.1) which involves the inverse of the generator, *i.e.* its resolvent, associated with the initial form $(\mathcal{E},D)$. Let us remark that in this abstract contest it is not quite trivial to give the formulation of the dual problem.

We fix our attention on resolvent G_1 and its generator A. We know that

$$A : D(A) \to L^2$$

so the classical approach consisting of the consideration of the Laplacian as an isomorphism with domain given by H_0^1 and range H^{-1} does not correspond to this point of view.

Actually one has to take into account the three following propositions which yield some suitable extensions of $I-A$ and G_1:

Proposition 1.6. *There exist two injective continuous linear maps with dense range between*

$$j : D \to L^2 \quad i : L^2 \to D'$$

where D' is the dual of D.

Proof. By definition, D is a dense subspace of L^2, therefore the first embedding is obvious. Now let us prove that there exists an injective continuous linear map i between L^2 and D' such that $\overline{i(L^2)} = D'$.

Let u be an element of D, hence $j(u) \in L^2$ and

$$\|j(u)\|_{L^2} \leq \|u\|_{\tilde{\mathcal{E}}_1^{\frac{1}{2}}} = (\tilde{\mathcal{E}}_1(u,u))^{\frac{1}{2}}.$$

So we can consider $i : L^2 \to D'$ defined in the following way:

$$\forall f \in L^2 \quad i(f) = i_f : D \to \mathbf{R} : \quad i_f(u) = (f, j(u))_{L^2} \forall u \in D.$$

Then i is well defined, it is linear and continuous, as

$$|(f, j(u))_{L^2}| \leq \|f\|_{L^2} \|j(u)\|_{L^2} \leq \|f\|_{L^2} \|u\|_{\tilde{\mathcal{E}}_1^{\frac{1}{2}}}$$

which yields

$$\|i(f)\|_{D'} = \sup \frac{|(f, j(u))_{L^2}|}{\|u\|_{\tilde{\mathcal{E}}_1^{\frac{1}{2}}}} \leq \|f\|_{L^2}.$$

Finally one can observe that the adjoint operator of j coincides with i and i is injective so $\overline{L^2} = D'$.

Proposition 1.7.

$$(I - A) : D(A) \to L^2$$

can be extended into an injective continuous linear map from D to D' such that

$$\|(I - A)f\|_{D'} \leq K \|f\|_{\tilde{\mathcal{E}}_1^{\frac{1}{2}}}$$

where K is the continuity constant.

Proof. For every $f \in D(A)$

$$\begin{aligned} \|(I - A)f\|_{D'} &= \sup\{|((I - A)f, g)_{L^2}| : \quad g \in D, \|g\|_{\tilde{\mathcal{E}}_1^{\frac{1}{2}}} \leq 1\} \\ &= \sup\{|\mathcal{E}_1(f, g)|, g \in D, \|g\|_{\tilde{\mathcal{E}}_1^{\frac{1}{2}}} \leq 1\} \\ &\leq K \|f\|_{\tilde{\mathcal{E}}_1^{\frac{1}{2}}} \end{aligned}$$

and by density the relation found holds for all $f \in D$ (see (1.5)).

Proposition 1.8. *The operator*

$$G_1 : L^2 \to D(A)$$

can be extended into an injective linear and continuous map from D' to D.

Proof. For every $u \in L^2$, by (1.6),

$$\|G_1 u\|^2_{\tilde{\mathcal{E}}_1^{\frac{1}{2}}} = \tilde{\mathcal{E}}_1(G_1 u, G_1 u) = (u, G_1 u)_{L^2}$$

hence

$$\|G_1 u\|_{\tilde{\mathcal{E}}_1^{\frac{1}{2}}} = \frac{(u, G_1 u)_{L^2}}{\|G_1 u\|_{\tilde{\mathcal{E}}_1^{\frac{1}{2}}}} \leq \|i(u)\|_{D'}$$

where i is the map defined in Proposition 1.6. The thesis follows by density of $i(L^2)$ in D'.

We want to remark now that given a form $\mathcal{E}$ and its generator A we have

$$\mathcal{E}_1(u, v) = ((I - A)u, v)_{L^2} \quad \forall u \in D(A), v \in D$$

but one can easily prove that

$$\mathcal{E}_1(u,v) = ((I-A)u,v) \quad \forall u,v \in D$$

where $(\cdot,\cdot)$ denotes the duality pairing between D' and D.
Moreover

$$((I-A)u,v) = (u,(I-\hat{A})v) \quad \forall u,v \in D.$$

We are now in a position to give the definitions of Dirichlet form and non symmetric Dirichlet form.

Definition 1.9. *A symmetric form $(\mathcal{E},D)$ is called a Dirichlet form (on $L^2(X,m)$) if it is closed on L^2 (X,m) and if, for all $u \in D$, it results*

$$u^+ \wedge 1 \in D \quad \text{and} \quad \mathcal{E}(u^+ \wedge 1, u^+ \wedge 1) \leq \mathcal{E}(u,u). \tag{1.10}$$

Definition 1.10. *A coercive closed form $(\mathcal{E},D)$ is called a non symmetric Dirichlet form (on $L^2(X,m)$) if, for all $u \in D$, it results*

$$u^+ \wedge 1 \in D \text{ and } \begin{array}{l} \mathcal{E}(u-u^+ \wedge 1, u+u^+ \wedge 1) \geq 0 \\ \mathcal{E}(u+u^+ \wedge 1, u-u^+ \wedge 1) \geq 0. \end{array} \tag{1.11}$$

Remark 1.11. *Let $(\mathcal{E},D)$ be a coercive closed form. Then one has*

$$\begin{array}{lll} \mathcal{E}(u+u^+ \wedge 1, u-u^+ \wedge 1) \geq 0 & \Longleftrightarrow & \mathcal{E}(u^+ \wedge 1, u-u^+ \wedge 1) \geq 0 \\ \mathcal{E}(u-u^+ \wedge 1, u+u^+ \wedge 1) \geq 0 & \Longleftrightarrow & \mathcal{E}(u-u^+ \wedge 1, u^+ \wedge 1) \geq 0. \end{array}$$

Proof. See [11] pag. 32 Theorem 4.4.

2. CAPACITY FOR NON SYMMETRIC DIRICHLET FORMS

Let X be an arbitrary locally compact separable Hausdorff space and let m be a given Radon measure supported on the whole of X. By H we denote the Hilbert space $H = L^2$ (X,m), with inner product $(u,v)_{L^2} = \int_X uvm(dx)$ and norm $\|\cdot\| = (\cdot,\cdot)_{L^2}^{\frac{1}{2}}$ and by $(\mathcal{E},D)$ a coercive closed form. As before we denote by D' the dual space of D and by $(\cdot,\cdot)$ the duality pairing between D' and D. Moreover we denote by $C_0(X)$ the space of all bounded continuous functions on X with compact support. $C_0(X)$ is endowed with the uniform norm $\|u\| = \sup_X |u(x)|$.

In this section we shall consider only regular forms i.e.

Definition 2.1. *A pair $(\mathcal{E},D)$ is called a regular form if it possesses a core, that is a subset F of $D \cap C_0$ (X), which is dense in C_0 (X) with the uniform norm and in D with the intrinsic norm.*

At this point, given a compact subset E of X, we introduce the following closed convex subset of D

$$K := \{v \in D : v \geq 1 \text{ on } E\} = \overline{\{v \in F : v \geq 1 \text{ on } E\}}^D$$

where $\overline{\{\qquad\}}^D$ represents the closure in the space D, and the following variational inequality

$$u \in K : \quad \mathcal{E}_1(u, v-u) \geq 0 \quad \forall v \in K. \tag{2.1}$$

Definition 2.2. *The solution u of (2.1) is called the capacitary potential or the equilibrium potential of the set E.*

In the same way we introduce the following subset of the dual space D' of D,

$$\begin{aligned} C = &\{\tau \in D' : \tau \text{ is a positive Radon measure with} \\ &\operatorname{supp}\tau \subset E, \tau(E) = 1\} \end{aligned}$$

and the following variational inequality

$$\mu \in C : \quad b(\mu, \tau - \mu) \geq 0 \quad \forall \tau \in C \tag{2.2}$$

where $b(\mu, \tau) = (G_1\mu, \tau)$ which is well defined taking into account Proposition 1.8.

Definition 2.3. *The solution μ of (2.2) is called the capacitary distribution or the equilibrium measure of the set E.*

Then we consider the adjoint variational inequalities:

$$\hat{u} \in K : \quad \mathcal{E}_1(v - \hat{u}, \hat{u}) \geq 0 \quad \forall v \in K \tag{2.1$'$}$$

$$\hat{\mu} \in C : \quad b(\tau - \hat{\mu}, \hat{\mu}) \geq 0 \quad \forall \tau \in C. \tag{2.2$'$}$$

Definition 2.4. *The solution $\hat{u}$ of (2.1') is called the capacitary copotential and the solution $\hat{\mu}$ of (2.2') is called the capacitary codistribution.*

Definition 2.5. *The following expression*

$$Cap\ E = \mathcal{E}_1(u, \hat{u}) \tag{2.3}$$

is called the capacity of E associated with the coercive closed form $(\mathcal{E}, D)$.

Remark 2.6. *If $Cap\,E \neq 0$, it follows that $\mathcal{E}_1(u,u) \neq 0$ and $\mathcal{E}_1\ (\hat{u}, \hat{u}) \neq 0$ from (1.2).*

Lemma 2.7. *The inequality (2.1) is equivalent to*

$$u \in K : \mathcal{E}_1(u, v) \geq 0 \quad \forall v \in D, v \geq 0\, on\, E. \tag{2.4}$$

Proof. Let u be the capacitary potential, hence $\mathcal{E}_1(u, w) \geq 0$ for all $w \in K - u$. Since $w \in K - u$ if and only if $w + u \in K$, then $w \in D$ and $w \geq 0$ on E as $u \in K$.

Proposition 2.8. *If $(\mathcal{E}, D)$ is a symmetric form, the solution u of (2.1) coincides with the unique element $u \in K$ minimizing $\mathcal{E}_1\ (\cdot, \cdot)$ on K.*

Proof. Let u be the solution of (2.1), hence for all $w \in K$ one gets

$$\mathcal{E}_1(w,w) = \mathcal{E}_1(u+(w-u), u+(w-u)) =$$
$$\mathcal{E}_1(u,u) + 2\mathcal{E}_1(u, w-u) + \mathcal{E}_1(w-u, w-u) \geq \mathcal{E}_1(u,u).$$

Conversely let v be any element of D, $v \geq 0$ on E, then $u+\varepsilon v \in K$ and $\mathcal{E}_1\ (u+\varepsilon v, u+\varepsilon v) \geq \mathcal{E}_1\ (u,u)$ for any $\varepsilon > 0$. From this we get $\mathcal{E}_1\ (u,v) \geq 0$ and, by Lemma 2.7, the thesis.

We observe that choosing

$$\mathcal{E}_1(u,v) = \int_\Omega \nabla u \nabla v dx \quad \forall u,v \in H_0^1(\Omega)$$

one obtains the classic definition of capacity.

Now we are exclusively concerned with the case Cap $E \neq 0$.

Theorem 2.9. *There exist exactly two paris of elements $u, \hat{u} \in K \quad \mu, \hat{\mu} \in C$ which solve (2.1),(2.1'),(2.2).(2.2') respectively. Moreover the following relations hold:*

$$\mu \geq (CapE)^{-1}(I-A)u \quad \hat{\mu} \geq (CapE)^{-1}(I-\hat{A})\hat{u} \tag{2.5}$$

in the sense of measures and

$$CapE \geq (b(\mu,\hat{\mu}))^{-1}. \tag{2.6}$$

Finally putting $\mu' = -(I-A)u$ and $\hat{\mu}' = -(I-\hat{A})\hat{u}$ one has

$$CapE \geq -\mu'(E) \geq 0 \qquad CapE \geq -\hat{\mu}'(E) \geq 0 \tag{2.7}$$

$$CapE = (-\mu', \hat{u}) = (u, -\hat{\mu}') \tag{2.8}$$

$$b(\mu', \hat{\mu}') = CapE. \tag{2.9}$$

In order to prove the theorem, we have to recall two basic general results, the first concerning duality for variational inequalities, the second concerning an integral representation for positive elements of the space D'.

As for the duality result let A be an injective map from a locally convex Hausdorff topological vector space X to its dual $X', D(A)$ the domain of A and $R(A)$ the range of A. Let $f : X \to (-\infty, +\infty]$ be a lower semicontinuous convex function, not identically ∞, and let us consider the following variational inequality

$$u \in D(A) : (Au, v-u) \geq f(u) - f(v) \quad \forall v \in X. \tag{j}$$

We associate to (j) a dual inequality involving the inverse A^{-1} of A and the Fenchel conjugate f^* of f, that is, the l.s.c. convex function f^* defined on X' by

$$f^*(v^*) = \sup\{(v^*, v) - f(v) : v \in X\}, \quad v^* \in X'.$$

Moreover we define $A' = A^{-1} : R(A) \subseteq X' \to X$. The dual variational inequality associated with (j) can be written as follows:

$$u^* \in D(A') : (A'u^*, v^* - u^*) \geq f^*(u^*) - f^*(v^*) \quad \forall v^* \in X'. \qquad (jj)$$

We have the following

Proposition 2.10. *A vector u of X is a solution of (j) if and only if the vector $u^* = -Au$ of X' is a solution of (jj). Moreover, (j) and (jj) hold if and only if $u^* = -Au$, or $u = -A'u^*$, and the following identity is satisfied*

$$f(u) + f(u^*) = (u, u^*).$$

Proof. See Mosco [14] Theorem 1 pag 203.

As for the integral representation for positive elements of D' we recall the following

Proposition 2.11. *For all $v \in D'$, $v \geq 0$ there exists a unique Radon measure μ such that*

$$(v, \varphi) = \int_X \varphi d\mu \quad \forall \varphi \in D \cap C_0(X). \qquad (2.10)$$

Proof. See Biroli-Tchou [3] Lemma 2.11.

Now we are in a position to state the theorem.

Proof. First of all $(\mathcal{E}, D)$ is a coercive closed form and K is a closed convex subset of D. K is a non-empty set by Uryshon Lemma and by the density of $C_0(X) \cap D$ in $C_0(X)$. Hence, by Proposition 1.3, applied to $J \equiv 0$, there exists a unique $u \in K$ such that $\mathcal{E}_1\ (u, v - u) \geq 0$ for all $v \in K$. The same considerations hold for (2.1').

In order to apply Proposition 2.10, we write the inequality (2.1) as (j) taking into account that u is the unique solution of the problem (2.1), that A is the generator associated with $(\mathcal{E}, D)$ so $(I - A)$ is an injective map and that (Proposition 1.7) $I - A : D \to D'$ with D and D' normed spaces. Then we can write the inequality (2.1) as follows:

$$u \in D : ((I - A)u, v - u) \geq \delta_K(u) - \delta_K(v) \quad \forall v \in D \qquad (2.11)$$

where δ_K is the indicator function of K, that is, $\delta_K(w) = 0$ if $w \in K$, $\delta_K(w) = +\infty$ otherwise. The dual variational inequality of (2.11) is the following:

$$u^* \in D' : (G_1 u^*, v^* - u^*) \geq \delta_K^*(u^*) - \delta_K^*(v^*) \quad \forall v^* \in D' \qquad (2.12)$$

where the conjugate of δ_K is the support function σ_K of K *i.e.*

$$\sigma_K(v^*) := \delta_K^*(v^*) = \sup_D\{(v^*, v) - \delta_K(v)\} = \sup_K (v^*, v).$$

In the following we denote by (2. 11$'$) and (2. 12$'$) the analogous adjoint inequalities of (2.11) and (2.12) respectively.

Now we want to evaluate $\sigma_K(v^*)$. We claim:

$\sigma_K(v^*) < \infty$ *if and only if* v^* *is a negative element of* D' *such that* $\text{supp}v^* \subset E$.

We recall now the definition of the support of an element $v^* \in D'$. The point x_0 belongs to $\text{supp}v^*$ if and only if for every neighbourhood U of x_0 there exists a function $\varphi \in C_0(U) \cap D$ such that $(v^*, \varphi) \neq 0$.

Let us check now that $\text{supp}v^* \not\subset E$ implies that $\sigma_k(v^*) = +\infty$.

Indeed let x_0 be an element of $\text{supp}v^* \not\subset E$, then there exists a neighbourhood V of x_0 such that $V \subset X - E$, hence there exists a function $\varphi_0 \in C_0(V) \cap D$ such that $(v^*, \varphi_0) \neq 0$. Now, being K a non-empty set, we can put $\varphi = \varphi_1 + t\varphi_0$ where $\varphi_1 \in K$, hence $\varphi \in K$ and $\sigma_k(v^*) \geq (v^*, \varphi_1) + t(v^*, \varphi_0)$. If $(v^*, \varphi_0) < 0$, we take $t \to -\infty$, and if $(v^*, \varphi_0) > 0$ we take $t \to \infty$, so in both two cases one concludes $\sigma_K(v^*) \geq +\infty$.

Let now $\text{supp}v^* \subset E$. Let us prove that, if v is a positive element of D', *i.e.* (v^*, φ_0) for every $\varphi_0 \geq 0$, then $\sigma_k(v^*) = \infty$. At this purpose it is enough to choose $\varphi_1 \in K$ and set $\varphi = \varphi_1 + t\varphi_0$ for all $t \geq 0$ to have again $\varphi \in K$ and $\sigma_K(v^*) \geq (v^*, \varphi_1) + t(v^*, \varphi_0)$ from which $\sigma_K(v^*) = \infty$.

Let us evaluate now $\sigma_K(v^*)$ for negative elements $v^* \in D'$ such that $\text{supp}v^* \subset E$. Actually let φ be a positive element of D, and $[\varphi]_1 := \varphi \wedge 1$. We have

$$\begin{aligned}\sigma_K(v^*) &= \sup_K (v^*, \varphi) = \sup\{(v^*, \varphi) : \varphi \in C_0(X) \cap D, \varphi \geq 1 \text{ on } E\}\\ &\leq \sup\{(v^*, [\varphi]_1) : \varphi \in C_0(X) \cap D, \varphi \geq 1 \text{ on } E\} = (v^*, 1).\end{aligned}$$

Therefore, recalling Proposition 2.11, if we set $v^* = \nu$ negative Radon measure such that $\text{supp}\,\nu \subset E$, we have $\sigma_K(v^*) = \sigma_K(\nu) \leq \nu(E)$.

We shall prove that $\sigma_K(\nu) \geq \nu(E)$. We know that E is a compact set of X, then by Uryshon Lemma there exists a sequence $\{\psi_n\} \in C_0(X)$ such that $\psi_n = 1 + \frac{1}{n}$ on E. By regularity of the form, we have that every element of the sequence $\{\psi_n\}$, can be approximated by a sequence $\{g_k^n\} \in C_0(X) \cap D$, then, for each fixed $n \in N$

$$-\frac{1}{k} < g_k^n - \psi_n < \frac{1}{k}$$

and $g_n^n \geq 1$ on E. We consider now the sequence $\{\varphi_n\} = \{g_n^n\}$. This sequence is such that $\varphi_n \in C_0(X) \cap D$ and $\varphi_n \geq 1$ on E, hence $\varphi_n \in K$.

$$\sigma_k(\nu) \geq (\nu, \varphi_n) = \int_E \varphi_n d\nu = -\int_E \varphi_n d\alpha$$

where $\alpha = -\nu$. We consider now

$$\begin{aligned}&|\int_E \varphi_n d\alpha - \alpha(E)| =\\ &|\int_E \varphi_n d\alpha - \int_E d\alpha| = |\int_E (\varphi_n - 1) d\alpha| \leq\\ &\int_E |\varphi_n - 1| d\alpha \leq \int_E (|\varphi_n - \psi_n| + |\psi_n - 1| d\alpha) \leq \frac{2}{n}\alpha(E)\end{aligned}$$

Then $-\frac{2\alpha(E)}{n} < \alpha(E) - \int_E \varphi_n d\alpha < \frac{2\alpha(E)}{n}$ and $-\int_E \varphi_n d\alpha \geq -\alpha(E) - \frac{2\alpha(E)}{n}$. It follows

$$\sigma_K(\nu) \geq \nu(E) - \frac{2\alpha(E)}{n}$$

and finally, taking the limit as $n \to +\infty$, $\sigma_k(\nu) \geq \nu(E)$. Now we have found that

$$\sigma_K(v^*) = \begin{cases} \nu(E) \text{ if } v^* = \nu \text{ negative Radon measure, supp}\nu \subset E \\ +\infty \text{ otherwise.} \end{cases}$$

We have obtained that in the variational inequality (2.12) the set D' can be replaced by the set

$$C' = \{\tau' \in D' \text{: negative Radon measure, supp}\tau' \subset E\}$$

and (2.12) becomes

$$\mu' \in C' : b(\mu', \tau' - \mu') \geq \mu'(E) - \tau'(E) \quad \forall \tau' \in C'.$$

Therefore, by Proposition 2.10 and by the fact the $u[\hat{u}]$ is the unique solution of (2.11) [(2. 11′)] (since $u[\hat{u}]$ is the unique solution of (2.1) [(2. 1′)]), we have

$$\begin{aligned} &\mu' = -(I - A)u \text{ is the unique solution of (2.12) and } (\mu', u) = \mu'(E) \\ &[\hat{\mu}' = -(I - \hat{A})\hat{u} \text{ is the unique solution of (2,12') and } (\hat{\mu}', \hat{u}) = \hat{\mu}'(E)]. \end{aligned} \tag{2. 13}$$

Moreover

$$\begin{aligned} \mu'(E) &= (-(I - A)u, u) = -\mathcal{E}_1(u, u). \\ [\hat{\mu}'(E) &= (-(I - \hat{A})\hat{u}, \hat{u}) = -\mathcal{E}(\hat{u}, \hat{u})]. \end{aligned} \tag{2. 14}$$

Now we have to prove that the solution μ of (2.2) is given by

$$\mu = (\mu'(E))^{-1}\mu', \tag{2. 15}$$

that is well defined by Remark 2.6, since Cap $E \neq 0$. We put $\tau = (\mu'(E))^{-1}\ \tau$ where $\tau' \in C'$ such that $\tau'(E) = \mu'(E)$. The measure τ is an element of C, conversely, if we put $\tau' = \mu'(E)\tau$ for all $\tau \in C$ we have that $\tau' \in C'$ and $\tau'(E) = \mu'(E)$.

We know that μ' is the solution of (2.12), then it is the solution of

$$\mu' \in C' \quad (\mu'(E))^{-2} b(\mu', \tau' - \mu') \geq 0 \quad \forall \tau' \in C' : \tau'(E) = \mu'(E) \tag{2. 16}$$

then $\mu = (\mu'(E))^{-1}\mu'$ solves (2.2). Obviously, by analogous considerations, $\hat{\mu} = (\hat{\mu}'(E))^{-1}\hat{\mu}'$ is the solution of (2. 2′).

We can now point out that, by (2.1) [(2.1')] with $v = \hat{u}[v = u]$

$$\mathcal{E}_1(u, \hat{u}) \geq \mathcal{E}_1(u, u) \quad [\mathcal{E}_1(u, \hat{u}) \geq \mathcal{E}_1(\hat{u}, \hat{u})]$$

which implies the relations (2.7) and (2.8) *i.e.*

$$\begin{aligned} &-\text{Cap } E = -\mathcal{E}_1(u, \hat{u}) \leq -\mathcal{E}_1(u, u) = \mu'(E) \\ &[-\text{Cap } E = -\mathcal{E}_1(u, \hat{u}) \leq -\mathcal{E}_1(\hat{u}, \hat{u}) = \hat{\mu}'(E)] \end{aligned}$$

and

$$\text{Cap } E = \mathcal{E}_1(u, \hat{u}) = ((I - A)u, \hat{u}) = -(\mu', \hat{u}) = (u, (I - \hat{A})\hat{u}) = -(u, \hat{\mu}').$$

Finally (2.5) and (2.6) hold, indeed

$$\begin{aligned} \mu = (\mu'(E))^{-1}\mu' &= -(\mu'(E))^{-1}(I - A)u \geq (\text{Cap } E)^{-1}(I - A)u \\ [\hat{\mu} = (\hat{\mu}'(E))^{-1}\hat{\mu}' &= -(\hat{\mu}'(E))^{-1}(I - \hat{A})\hat{u} \geq (\text{Cap } E)^{-1}(I - \hat{A})\hat{u}] \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} b(\mu, \hat{\mu}) = (\mu'(E))^{-1}(\hat{\mu}'(E))^{-1}((I - A)^{-1}\mu', \hat{\mu}') \geq \\ (\text{Cap } E)^{-1}(\text{Cap } E)^{-1}(-u, \hat{\mu}') = (\text{Cap } E)^{-1} \end{aligned} \tag{2.18}$$

then

$$\text{Cap } E \geq (b(\mu, \hat{\mu}))^{-1}.$$

We can note also that (2.9) hold:

$$b(\mu', \hat{\mu}') = (G_1\mu', \hat{\mu}') = ((I - A)^{-1}\mu', \hat{\mu}') = (-u, \hat{\mu}') = \text{ Cap } E.$$

Corollary 2.12. *The capacitary potential can be characterized as the solution of the following system: $u \in D$ such that*

$$\begin{cases} u \geq 1 & on\, E \\ \mathcal{E}_1(u, v) \geq 0 & \forall v \geq 0 \; supp v \cap E \neq \emptyset \\ \mathcal{E}_1(u, v) = 0 & \forall v \; supp v \subset X - E \\ \mathcal{E}_1(u, u - 1) = 0 \end{cases} \tag{2.19}$$

Proof. Let u be the capacitary potential. Then, since $u \in K$, $u \geq 1$ on E and, by Lemma 2.7 we have that $\mathcal{E}_1\ (u, v) \geq 0$ for all $v \in D$ such that $v \geq 0$ on E so supp$v \cap E \neq \emptyset$. Moreover, by Theorem 2.9, we know that supp$(I - A)\ u \equiv$supp$(-\mu') \subset E$, then if supp$v \subset (X - E)$, it results $\mathcal{E}_1\ (u, v) = 0$. Again by Theorem 2.9, $\mu'(E) = (\mu', u) = (-(I - A)\ u, u)$, but $\mu'(E) = \int_E d\mu' = (\mu', 1) = (-(I - A)\ u, 1)$ and this implies $((I - A)\ u, u - 1) = 0$.

Conversely, if u is a solution of (2.19), $u \in K$ by the first relation of (2.19); furthermore fixed $v \in K$, it results $v - 1 \geq 0$ on E so, by the second relation of (2.19), $((I - A)\ u, v - 1) \geq 0$ and

$$u \in K : ((I - A)u, v - u) = ((I - A)u, v - 1) - ((I - A)u, u - 1) \geq 0 \quad \forall v \in K$$

by the last relation of (2.19).

Corollary 2.13. *If $(\mathcal{E}, D)$ is a non symmetric Dirichlet form, the following relations hold*

$$0 \leq u, \hat{u} \leq 1 \tag{2.20}$$

and

$$\mathcal{E}_1(u, u) = \mathcal{E}_1(u, \hat{u}) = \mathcal{E}_1(\hat{u}, \hat{u}) \tag{2.21}$$

Proof. First of all, we note that, if $(\mathcal{E}, D)$ is a non symmetric Dirichlet form, then $\mathcal{E}_\alpha(\cdot,\cdot) = \mathcal{E}(\cdot,\cdot) + \alpha(\cdot,\cdot)$ is a non symmetric Dirichlet form too.

$$0 \le \mathcal{E}_1(u - u^+ \wedge 1, u - u^+ \wedge 1) = \mathcal{E}_1(u, u - u^+ \wedge 1) - \mathcal{E}_1(u^+ \wedge 1, u - u^+ \wedge 1).$$

Actually, $(\mathcal{E}_1, D)$ is a non symmetric Dirichlet form, so by Remark 1.11 we have $\mathcal{E}_1(u^+ \wedge 1, u - u^+ \wedge 1) \ge 0$, and by the fact that $u^+ \wedge 1 \in K$, $\mathcal{E}_1(u, u - u^+ \wedge 1) \le 0$, hence $0 \le \mathcal{E}_1(u - u^+ \wedge 1, u - u^+ \wedge 1) \le 0$. This implies $u = u^+ \wedge 1$, that is $0 \le u \le 1$ on X and $u = 1$ on E. Interchanging the two entries of $\mathcal{E}$ and replacing u by $\hat{u}$ we obtain $0 \le \hat{u} \le 1$ on X and $\hat{u} = 1$ on E.

Now we have to show that

$$\mathcal{E}_1(u, u) = \mathcal{E}_1(u, \hat{u}) = \mathcal{E}_1(\hat{u}, \hat{u}).$$

Indeed we know that $u = \hat{u} = 1$ on E, so $u - \hat{u} = 0$ on E hence, by Corollary 2.12,

$$\mathcal{E}_1(u, u - \hat{u}) = 0 \text{ and } \mathcal{E}_1(u - \hat{u}, \hat{u}) = 0$$

that is respectively

$$\mathcal{E}_1(u, u) = \mathcal{E}_1(u, \hat{u}) \text{ and } \mathcal{E}_1(\hat{u}, \hat{u}) = \mathcal{E}_1(u, \hat{u}).$$

Corollary 2.14. *Let $(\mathcal{E}, D)$ be a non symmetric regular Dirichlet form. Then the following relations hold:*

$$\mu = (Cap\, E)^{-1}(I - A)u \quad \hat{\mu} = (Cap\, E)^{-1}(I - \hat{A})\hat{u} \tag{2.22}$$

and

$$(b(\mu, \hat{\mu}))^{-1} = Cap\, E. \tag{2.23}$$

Proof. It is an easy consequence of Corollary 2.13 and of inequalities (2.7), (2.17) and (2.18).

3. EXAMPLE

Let us give a simple example related to capacity for a non symmetric Dirichlet form. We shall see that the potential and copotential are indeed different, with values in [0,1] and (2.21) holds.

Let $X = \Omega = (-1, 1)$ and $m = dx$ Lebesgue measure on Ω. Let $a_1, a_2 \in \mathbf{R}^+$ such that

$$a_1 > a_2 \tilde{c} \tag{3.1}$$

where $\tilde{c}$ is Poincaré constant, that is the constant that appears in Poincaré inequality:

$$\|u\|_{H_0^1} \le \tilde{c} \|\dot{u}\|_{L^2} \quad \forall u \in H_0^1 \subset L^2.$$

Let us define the linear operator L on $L^2(\Omega, dx)$ by

$$Lu = -a_1 \ddot{u} - a_2 \dot{u}, \quad u \in D(L) = H_0^1(\Omega) \cap H^2(\Omega). \tag{3.2}$$

We shall show that the form $(\mathcal{E}, D)$ associated with L is a non symmetric regular Dirichlet form.

L is negative semidefinite: for all $u \in H_0^1 \cap H^2$

$$(Lu,u) = -a_1 \int_{-1}^{1} \ddot{u}u - a_2 \int_{-1}^{1} \dot{u}u = a_1 \int_{-1}^{1} \dot{u}^2 - a_2 \int_{-1}^{1} \dot{u}u \tag{3.3}$$

so, by Hölder inequality

$$-a_2 \int_{-1}^{1} \dot{u}u \geq -a_2 \left[\int_{-1}^{1} \dot{u}^2\right]^{\frac{1}{2}} \left[\int_{-1}^{1} u^2\right]^{\frac{1}{2}} \geq -a_2 \|u\|_{H_0^1} \|u\|_{H_0^1} = -a_2 \|u\|_{H_0^1}^2$$

and by (3.1)

$$(Lu,u) \geq \frac{a_1}{\tilde{c}} \|u\|_{H_0^1}^2 - a_2 \|u\|_{H_0^1}^2 = \left(\frac{a_1}{\tilde{c}} - a_2\right) \|u\|_{H_0^1}^2 \geq 0. \tag{3.4}$$

The associated form $(\mathcal{E}, D)$ with $-L$ is defined as

$$\mathcal{E}(u,v) = (Lu,v) = a_1 \int_{-1}^{1} \dot{u}\dot{v} - a_2 \int_{-1}^{1} \dot{u}v \quad D = H_0^1(\Omega). \tag{3.5}$$

We can observe that D is a dense linear subspace of $H = L^2(\Omega, dx)$, that the form satisfies the weak sector condition and that it is not a symmetric form.

By Hölder and by (3.4) we obtain the equivalence between the intrinsic norm and the norm $\|\cdot\|_{H_0^1}$, then D is complete with respect to the intrinsic norm, so $(\tilde{\mathcal{E}}, D)$ is closed and $(\mathcal{E}, D)$ is a coercive closed form on $L^2\,(\Omega, dx)$.

$(\mathcal{E}, D)$ is a non symmetric Dirichlet form:
we know that if $u \in H_0^1$, then $u^+ \wedge 1 \in H_0^1$ (see, for example, G. Stampacchia [16]). By simple calculations one easily checks that $\mathcal{E}\,(u + u^+ \wedge 1, u - u^+ \wedge 1) \geq 0$ and $\mathcal{E}\,(u - u^+ \wedge 1, u + u^+ \wedge 1) \geq 0$.

By Corollary 2.12 we can look for the capacitary potential as the solution of the following system:

$$\begin{cases} (I+L)u = 0 \text{ on } \Omega \\ u = 1 \text{ on } E \\ u = 0 \text{ on } \partial\,\Omega \end{cases}$$

where $E = \left[-\frac{1}{2}, \frac{1}{2}\right]$. We can divide the system in two Dirichlet problems on $\left(-1, -\frac{1}{2}\right)$ and $\left(\frac{1}{2}, 1\right)$, since $u = 1$ on $E = \left[-\frac{1}{2}, \frac{1}{2}\right]$. We have

$$\begin{cases} a_1\ddot{u} + a_2\dot{u} - u = 0 \quad \text{on } \left(-1, -\dfrac{1}{2}\right) \\ u\left(-\dfrac{1}{2}\right) = 1 \\ u(-1) = 0 \end{cases} \tag{3.6}$$

whose solution is given by

$$u(x) = \frac{e^{-\lambda_1 x}}{e^{-\frac{\lambda_1}{2}} - e^{\frac{\lambda_2}{2}-\lambda_1}} + \frac{e^{\lambda_2 x}}{e^{-\frac{\lambda_2}{2}} - e^{\frac{\lambda_1}{2}-\lambda_2}} \tag{3.7}$$

where

$$\lambda_1 = \frac{-a_2 + \sqrt{a_2^2 + 4a_1}}{2a_1} \qquad \lambda_2 = \frac{-a_2 - \sqrt{a_2^2 + 4a_1}}{2a_1} \tag{3.8}$$

so, $\lambda_1, \lambda_2 \in \mathbf{R}$, since $a_1, a_2 > 0$. The second problem is

$$\begin{cases} a_1 \ddot{u} + a_2 \dot{u} - u = 0 \text{ on } \left(\frac{1}{2}, 1\right) \\ u\left(\frac{1}{2}\right) = 1 \\ u(1) = 0 \end{cases} \tag{3.9}$$

whose solution is given by

$$u(x) = \frac{e^{\lambda_1 x}}{e^{\frac{\lambda_1}{2}} - e^{-\frac{\lambda_2}{2}+\lambda_1}} + \frac{e^{\lambda_2 x}}{e^{\frac{\lambda_2}{2}} - e^{-\frac{\lambda_1}{2}+\lambda_2}} \tag{3.10}$$

To evaluate the capacitary copotential we have to consider the adjoint problem:

$$\begin{cases} -a_1 \ddot{u} + a_2 \dot{u} + u = 0 \text{ on } \left(-1, -\frac{1}{2}\right) \\ u\left(-\frac{1}{2}\right) = 1 \\ u(-1) = 0 \end{cases} \tag{3.11}$$

whose solution is given by

$$\hat{u}(x) = \frac{e^{-\lambda_1 x}}{e^{\frac{\lambda_1}{2}} - e^{-\frac{\lambda_2}{2}+\lambda_1}} + \frac{e^{-\lambda_2 x}}{e^{\frac{\lambda_2}{2}} - e^{-\frac{\lambda_1}{2}+\lambda_2}} \tag{3.12}$$

and

$$\begin{cases} -a_1 \ddot{u} + a_2 \dot{u} + u = 0 \quad \text{on } \left(\frac{1}{2}, 1\right) \\ u\left(\frac{1}{2}\right) = 1 \\ u(1) = 0 \end{cases} \tag{3.13}$$

whose solution is given by

$$\hat{u}(x) = \frac{e^{-\lambda_1 x}}{e^{-\frac{\lambda_1}{2}} - e^{\frac{\lambda_2}{2}-\lambda_1}} + \frac{e^{-\lambda_2 x}}{e^{-\frac{\lambda_2}{2}} - e^{\frac{\lambda_1}{2}-\lambda_2}}. \tag{3.14}$$

Finally one gets

$$u(x) = \begin{cases} \dfrac{e^{\lambda_1 x}}{e^{-\frac{\lambda_1}{2}} - e^{\frac{\lambda_2}{2} - \lambda_1}} + \dfrac{e^{\lambda_2 x}}{e^{-\frac{\lambda_2}{2}} - e^{\frac{\lambda_1}{2} - \lambda_2}} & [-1, -\frac{1}{2}) \\ 1 & [-\frac{1}{2}, \frac{1}{2}] \\ \dfrac{e^{\lambda_1 x}}{e^{\frac{\lambda_1}{2}} - e^{-\frac{\lambda_2}{2} + \lambda_1}} + \dfrac{e^{\lambda_2 x}}{e^{\frac{\lambda_2}{2}} - e^{-\frac{\lambda_1}{2} + \lambda_2}} & (\frac{1}{2}, 1] \end{cases} \tag{3. 15}$$

and

$$\hat{u}(x) = \begin{cases} \dfrac{e^{-\lambda_1 x}}{e^{\frac{\lambda_1}{2}} - e^{-\frac{\lambda_2}{2} + \lambda_1}} + \dfrac{e^{-\lambda_2 x}}{e^{\frac{\lambda_2}{2}} - e^{-\frac{\lambda_1}{2} + \lambda_2}} & [-1, -\frac{1}{2}) \\ 1 & [-\frac{1}{2}, \frac{1}{2}] \\ \dfrac{e^{-\lambda_1 x}}{e^{-\frac{\lambda_1}{2}} - e^{\frac{\lambda_2}{2} - \lambda_1}} + \dfrac{e^{-\lambda_2 x}}{e^{-\frac{\lambda_2}{2}} - e^{\frac{\lambda_1}{2} - \lambda_2}} & (\frac{1}{2}, 1]. \end{cases} \tag{3. 16}$$

We can notice that $u \neq \hat{u}$ with $u(x) \in [0, 1]$ $\hat{u}(x) \in [0, 1]$ $\forall x \in [-1, 1]$.

Moreover, we know that $u = \hat{u}$ on E while u is solution of $(I + L)\, u = 0$ and $\hat{u}$ is solution of $(I + \hat{L})\, \hat{u} = 0$ on $\Omega - E$, so $((I + L)\, u, u - \hat{u}) = 0$ and $(u - \hat{u}, (I + \hat{L})\, \hat{u}) = 0$, hence respectively $\mathcal{E}_1\ (u, u - \hat{u}) = 0$ and $\mathcal{E}_1\ (u - \hat{u}, \hat{u}) = 0$ that implies

$$\mathcal{E}_1(u, u) = \mathcal{E}_1(u, \hat{u}) = \mathcal{E}_1(\hat{u}, \hat{u})$$

that is (2.21).

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