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GEOMETRIC-COMBINATORIAL CHARACTERISTICS OF CONES

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Abstract. *It is shown for a proper closed locally compact subset S of a real normed linear space X that* $\ker_R S = \bigcap\{\mathrm{cl\,aff} B_z^R : z \in \mathrm{reg} S\}$, *where* $\ker_R S$ *is the R-kernel of* S, $\mathrm{reg} S$ *denotes the set of regular points of* S *and* $B_z^R = \{s \in S : z$ is R-visible from s *via* $S\}$. *Furthermore, it is shown for a closed connected nonconvex subset* S *of* X *that* $\ker_R S = \bigcap\{\mathrm{conv} B_z^R : z \in D\}$, *where* D *is a relatively open subset of* S *containing the set* $\mathrm{lnc} S$ *of local nonconvexity points of* S. *If* X *is a uniformly convex and uniformly smooth real Banach space, then the first of these formulae is shown to hold with the set* $\mathrm{sph} S$ *of spherical points of* S *in place of* $\mathrm{reg} S$, *and the second one for a closed connected nonconvex set* S. *For a connected subset* S *of a real topological linear space* L *with nonempty* $\mathrm{slnc} S$, *the set of strong local nonconvexity points of* S, *it is shown that* $\bigcap\{\mathrm{aff} A_z^{R^\circ} : z \in \mathrm{slnc} S\} \subseteq \mathrm{qker}_{R^\circ} S$, *where* $\mathrm{qker}_{R^\circ} S$ *is the quasi-*R°*-kernel of* S *and* $A_z^{R^\circ} = \{s \in \mathrm{cl} S : z$ is clearly R°*-visible from* s *via* $S\}$, *and that the equality holds provided, in addition,* S *is open. In conjunction with an infinite-dimensional version of Helly's theorem for flats, these intersection formulae generate Krasnosel'skii-type characterizations of cones and quasi-cones. All this parallels the research done recently by the author for starshaped and quasi-starshaped sets.*

1. INTRODUCTION

We start with some definitions and terminology. Let S be a nonempty set in a real topological linear space L. For points y and x in $\mathrm{cl} S$, y is *visible* from x via S if and only if the open line segment (x, y) lies in S. S is *starshaped* if and only if every point of S is visible via S from a common point q of S, and the set of all such points q is called the *kernel* of S and denoted by $\ker S$. S is a cone if it is the union of a nonempty set of closed halflines having the common endpoint called the apex of the cone and the set of all such apices, denoted by $\ker_R S$, is called the *R-kernel* of S. For distinct points x and y in $\mathrm{cl} S$, y is *R-visible* (*R°-visible*) from x via S if and only if there is in S a closed halfline $R(x, y)$ (an open halfline $R^\circ(x, y)$) emanating from x through y. We extend these definitions to the case when y and x coincide and require then that S contain some closed (open) halfline emanating from x. Similarly, y in $\mathrm{cl} S$ is *clearly R-visible (clearly R°-visible)* via S from x in $\mathrm{cl} S$ if and only if there is some neighbourhood N of y such that each point of $S \cap N$ is R-visible (R°-visible) from x via S. Following [11], a nonempty set S in L is said to be *quasi-starshaped* if and only if there is some point q in $\mathrm{cl} S$ such that the subset of points of S visible via S from q is everywhere dense in S and contains $\mathrm{int} S$, and the set of all such points q is called the *quasi-kernel* of S and denoted by $\mathrm{qker} S$. Furthermore, S is called a *quasi-cone* if and only if there is some point q in $\mathrm{cl} S$ such that the set of points of S which are R°-visible via S from q is everywhere dense in S and contains $\mathrm{int} S$, and the set of all such points q is called the *quasi-R°-kernel* of S and denoted by $\mathrm{qker}_{R^\circ} S$. It is easily seen that for any S in L, $\ker_R S \subseteq \mathrm{qker}_{R^\circ} S \subseteq \mathrm{qker} S$ and $\ker_R S \neq \varnothing$, $\mathrm{qker}_{R^\circ} S \neq \varnothing$

are flats in L, for $S \neq L$, contained in bdry$S$. Moreover, if S is closed, then $\ker_R S = \mathrm{qker}_{R^\circ} S$. Following [22, Def. 4.2], a point s in cl$S$ is said to be a *point of weak local convexity* (wlc point) of S if and only if there is some neighbourhood N of s such that for each pair of points x, y in $S \cap N$, the closed line segment $[x, y]$ lies in S. If S fails to be weakly locally convex at q in cl$S$, then q is called a *point of strong local nonconvexity* (slnc point) of S. Moreover [22, Def. 4.3], a point s in cl$S$ is said to be a *point of strong local convexity* (slc point) of S if and only if $S \cap N$ is convex for some neighbourhood N of s. If S fails to be strongly locally convex at q in cl$S$, then q is called a *point of mild local nonconvexity* (mlnc point) of S. wlc$S$, slnc$S$, slc$S$ and mlnc$S$ will denote sets of wlc, slnc, slc and mlnc points of S, respectively. By [22, Th. 1.5], int$S \subseteq$ wlc$S$ and slnc$S \subseteq$ bdry$S$. If H is a subset of L, then a point s in cl$(S \cap H)$ is called an *H-wlc* (*H-slnc*) *point* of S if and only if $S \cap H$ is weakly locally convex (strongly locally nonconvex) at s. For L locally convex, wlc$S$ = slc$S$ and slnc$S$ = mlnc$S$ are called for simplicity the sets of *local convexity points* (lc points) and *local nonconvexity points* (lnc points) of S, and denoted by lc$S$ and lnc$S$, respectively. A point s in S is called a *regular point* (reg point) [22, Def. 6.4] or a *cone point* of S [14] if and only if there exists a closed halfspace in L which has s in its bounding closed hyperplane and contains all points visible from s via S. A real normed linear space X is said to be *smooth* if its closed unit ball is smooth [22, Def. 7.5]. X is said to be *uniformly convex* if for each $0 < \epsilon \leq 2$ there exists $\delta > 0$ such that $1 - \| \frac{1}{2}(x+y) \| \geq \delta$ whenever $\| x \| = \| y \| = 1$ and $\| x-y \| \geq \epsilon$, and *uniformly smooth* if for each $\epsilon > 0$ there exists $\eta > 0$ such that $1 - \| \frac{1}{2}(x+y) \| \leq \epsilon \| x - y \|$ whenever $\| x \| = \| y \| = 1$ and $\| x - y \| \leq \eta$. If X is smooth, then, following [3], [4] and [9], a point s is called a *spherical point* (sph point) of S if and only if there exists in the complement of S, $\sim$ S, some open ball with s lying on its boundary. reg$S$ and sph$S$ will denote sets of all reg and sph points of S, respectively. By the dimension (codimension) of a subset A of L, we mean the dimension (codimension) of aff$A$, the affine hull of A. (xy), (xyz), $(xy)_z$ and $(xyz)_u$ will represent respectively: a straight line determined by two distinct points x, y, a two-dimensional flat determined by three noncollinear points x, y, z, a two-dimensional closed half-flat determined by (xy) and containing $z \notin (xy)$ and a closed halfspace in aff$\{x, y, z, u\}$ determined by (xyz) and containing $u \notin (xyz)$. For z in $S \subseteq L$, we denote $S_z = \{s \in S : z$ is visible from s via S$\}$ and $B_z^R = \{s \in S : z$ is R-visible from s via S$\}$, and for z in cl$S$, $A_z^R = \{s \in S : z$ is clearly R-visible from s via S$\}$ and $A_z^{R^\circ} = \{s \in \mathrm{cl}S : z$ is clearly R°-visible from s via S$\}$. Observe that $B_z^R \subseteq S_z$ and $A_z^R \subseteq A_z^{R^\circ}$.

A central theorem of combinatorial geometry due to Krasnosel'skii [19], [12, E2], [22, Th. 6.17] states that a compact subset S of R^d is starshaped, i.e. ker$S \neq \emptyset$ or dim ker$S \geq 0$ if and only if every $d+1$ boundary points of S are visible via straight line segments from a common point in S. Since its discovery in 1946, various relatives and generalizations of this criterion have been investigated in detail by many authors (cf. [2], [12, E2], [8], [20]). Recently, making use of closed halflines in place of line segments, the author has extended Krasnosel'skii's theorem to cones in R^d [3]-[7] and exhibited a geometric similarity between cones and starshaped sets (cf. [7] and [8]). In the present paper, we investigate various geometric and combinatorial characterizations involving boundary points and local nonconvexity points for cones in topological linear spaces. This parallels a research done very recently for starshaped sets [8]-[11]. The geometric characteristics of cones are representations of $\ker_R S$ ($\mathrm{qker}_{R^\circ} S$) in the form of intersections of affine hulls of R-visibility (clear R°-visibility) sets of selected boundary points of S. Two such formulae have been previously established in [7]. The desired combinatoria1 characterizations of the codimension of $\ker_R S$ or $\mathrm{qker}_{R^\circ} S$ follow then immediately from Helly's theorem for flats. The reader is referred to [3]-[1 1] for details concerning notation and terminology.

2. PRELIMINARIES

We collect here severa1 lemmas useful in proofs of the main theorems in the next sections. The first of them is a variation of Helly's theorem for flats.

Lemma 2.1 *A nonempty family $\mathcal{G}$ of flats in a linear space has a nonempty intersection of codimension at most* c, *where* $0 < g = \max_{\mathfrak{g}\in\mathcal{G}}\{\mathrm{codim}\mathfrak{g}\} \leq c < \infty$, *if and only if every subfamily of* $c - g + 2$ *orfewer members of* $\mathcal{G}$ *has a nonempty intersection of codimension at most* c.

Proof. We establish the sufficiency of the condition. Fix thus an integer c and let $\mathfrak{g}_0 \in \mathcal{G}$ be such that codimgo = g. We proceed by induction on g. First let g = c. For every g $\in \mathcal{G}$, the imposed intersection condition implies c = codimgo $\leq \mathrm{codim}(\mathfrak{g}_0 \cap \mathfrak{g}) \leq$ c, whence $\mathfrak{g}_0 = \mathfrak{g}_0 \cap \mathfrak{g}$, so $\mathfrak{g}_0 = \bigcap_{\mathfrak{g}\in\mathcal{G}}$ g and, in consequence, $\mathrm{codim}\bigcap_{\mathfrak{g}\in\mathcal{G}}$ g = $\mathrm{codim}\mathfrak{g}_0$ = c, as required. Let further c > 1. Assume the truth for all numbers greater than $g \geq 1$ and consider the given family $\mathcal{G}$ satisfying the intersection condition. Define a family $\mathcal{G}' = \{\mathfrak{g}_0 \cap$ g : g $\in \mathcal{G}\}$ and a number g' = $\max_{\mathfrak{g}'\in\mathcal{G}'}\{$ codimg'}. Of course g' $\geq g$ and the imposed condition implies g' $\leq c$. If $g' = g$, then the inequalities g' $\geq \mathrm{codim}(\mathfrak{g}_0 \cap$ g) $\geq$ codimgo = g imply $\mathfrak{g}_0 = \mathfrak{g}_0 \cap$ g for any g $\in \mathcal{G}$, whence $\mathrm{codim}\bigcap_{\mathfrak{g}\in\mathfrak{G}}$ g = codimg $_0 = g \leq$ c, as required. If g' > g, then, by assumption about $\mathcal{G}$, every subfamily of c $- g + 1$ or fewer members of $\mathcal{G}'$ has a nonempty intersection of codimension at most c and c $- g + 1 \geq c - g' + 2$ so that, by the induction hypothesis, $\mathrm{codim}\bigcap_{\mathfrak{g}\in\mathcal{G}}$ g = $\mathrm{codim}\bigcap_{\mathfrak{g}'\in\mathcal{G}'}$ g' $\leq$ c which proves the assertion for g.

Lemma *2.2. Let S be a closed locally compact subset of a real normed linear space X and p, s distinct points in X, $s \in S$. If s is a relative boundary point of $S \cap R(p,s)$, then in every neighbourhood of s there is a regular point z of S such that $p \notin \mathrm{cl}\,\mathrm{aff}B_z^R$.*

Proof. By assumption, there exists a number $\gamma > 0$ such that $S \cap \mathrm{cl}B(s,\epsilon)$ is compact for all $0 < \epsilon \leq$ y. We fix such $\epsilon > 0$ arbitrarily and assume that p is the origin of X. The subspace Y = cl $\mathrm{aff}((S\cap\mathrm{cl}B(s,\epsilon))$ u *{p})*IS compactly generated, so that arguing as in [9, Lemma 2.21, it can be smoothly renormed.

Suppose first that s is a relative boundary point of $S \cap [p,s]$. As in [9, Lemma 2.3], we produce then in Y an open ν-ball $B_Y(\lambda_0 s,\nu)$ with respect to a new smooth norm with the closure contained in $B(s,\frac{\epsilon}{2})$ and with $0 < \nu < \frac{\epsilon}{2}$ disjoint from S and such that $z \in S \cap \mathrm{cl}B_Y(\lambda_0 s,\nu) \neq \varnothing$ and the unique closed hyperplane $\mathfrak{h}$ in Y supporting $B_Y(\lambda_0 s,\nu)$ at z does not contain *p*. However, the smoothness of $B_Y(\lambda_0 s,\nu)$ at z implies easily that cl aff(Y $\cap B_z^R) \subseteq \mathfrak{h}$, so that $p \notin$ cl aff(Y $\cap B_z^R$). On the other hand, easily $B_z^R \subseteq S_z \subseteq$ Y, so that $p \notin \mathrm{cl}\,\mathrm{aff}B_z^R$. By the Hahn-Banach theorem [18, §17.6.(l)], there is in X a closed hyperplane $\mathfrak{H}$ containing $\mathfrak{h}$ and S_z lies in this closed halfspace determined by $\mathfrak{H}$ in X which does not contain p, so that $z \in \mathrm{reg}S$, and the proof in this case is finished.

Now suppose that s is a relative boundary point of S $\cap R(p, s) \sim [p, s)$ and let $\epsilon' > 0$ be a number for which $B_Y(s,$ E') $\subseteq B(s,$ e). Hence, there is a point $u \in B_Y(s, \epsilon') \cap \sim S \cap (R(p,s) \sim [p, s])$ and a number 6 $>$ 0 such that $\mathrm{cl}B_Y(u,$ 6) $\subseteq B_Y(s, \epsilon') \sim$ S. Fix a point $p_0 \in (p, s) \cap B_Y(s,$ E'). Since $S \cap \mathrm{cl}B_Y(s, \epsilon')$ is compact and $s \in (p_0, u)$, there exists a largest $0 < \lambda_0 < 1$ such that the closure of $B_Y(p_0 + \lambda_0(u \quad p_0), \lambda_0\delta)$, the homothetic image of $B_Y(u,$ 6) with respect to p_0, intersects S. Let z belong to this intersection. Then if $C = \bigcup_{\lambda_0 \leq \lambda \leq 1} B_Y(p_0 + \lambda(u - p_0), \lambda_0\delta) \subseteq \sim$ S, then z belongs to the boundary of C and let $\mathfrak{H}$ be a closed hyperplane in X not containing (ps) and supporting C, and also $B_Y(p_0 + \lambda_0(u - p_0), \lambda_0\delta)$ at z. The smoothness of Y implies easily that $\mathfrak{H}$ **n** $R(p, s) \subseteq [p_0, u)$, and that cl $\mathrm{aff}B_z^R \subseteq \mathfrak{H}$. Consequently, $p \notin$ cl $\mathrm{aff}B_z^R$ and $z \in B(s,$ e). The Hahn-Banach theorem implies that $z \in \mathrm{reg}S$ and the argument is finished.

Lemma 2.3. *Let S be a closed subset of a real Banach space X which is uniformly convex und uniformly smooth, and p, s distinct points in $X, s \in S$. If s is a relative boundary point of $S \cap R(p, s)$, then in every neighbourhood of S there is a spherical point z* ***of*** *S such that $p \notin$* cl $\mathrm{aff}B_z^R$.

Proof. Select in X an arbitrarily small open μ-ball $B(s, \mu)$ centered at s and assume for simplicity that p is the origin in X.

Let first s be a relative boundary point of S $\cap [p,$ s]. Then the argument in [9, Lemma 2.5] produces an open ball $B(x, \|z - $ x$\|) \subseteq \sim$ S with x close enough to $[p, s]$ such that p and x lie in the same open halfspace determined by the closed hyperplane $\mathfrak{H}_z$ supporting $B(x, \|z - x\|)$ at z. The smoothness of X implies that cl $\mathrm{aff}B_z^R \subseteq \mathfrak{H}_z$, whence $p \notin$ cl $\mathrm{aff}B_z^R$, as desired.

Now suppose that s is a relative boundary point of $S \cap R(p, s) \sim [p, s)$. Hence, there is a point $u \in B(s, \mu) \cap \sim S \cap (R(p,s) \sim [p, s\ ))$and a number 6 $>$ 0 such that $\mathrm{cl}B(u, 6) \subseteq B(s, \mu) \sim S$. Fix a point $p_0 \in (p,$ s) $\cap B(s, \mu)$. Since $s \in (p_0, u)$, there exists a smallest $0 < \lambda_0 < 1$ such that S $\cap B((1 - \lambda)p_0 + \lambda u, \lambda\delta) = \varnothing$ for all $\lambda_0 \leq \lambda \leq 1$, where $B((1 \quad \lambda)p_0 + \lambda u, \lambda\delta)$ is the homothetic image of $B(u,$ 6) with respect to p_0. As in [14, Th.] and [9, Lemma 2.51, for any nonzero y $\in$ X let f_y denote the linear functional of norm one which supports the ball $B(\circ, \|$ y $\|)$ at y. Then $f_y(y) = \|$ y $\|$ and the hyperplane supporting $B(x, \| z - x \|)$ at z is given by $\mathfrak{H}_z$ = x $+ f_{z-x}^{-1}(\|$ z $-$ x $\|)$. By construction, for any $0 < \lambda < \lambda_0$ close enough to λ_0, we have S $\cap$ *B((* 1 $\quad \lambda)p_0 +$ Au, $\lambda\delta) \neq \varnothing$. Since X is uniformly convex, [21, Cor.] (cf. [13, Th.]) implies that for any fixed $\theta > 0$ we can find a point x with $\|$x $-$ ((1 $- \lambda)p_0$ + *Au)* $\| < \theta$ such that x has the nearest point z in S. By choosing θ small enough, we can also assume that $\| z - ((1 - \lambda)p_0 + \lambda u) \| < \lambda\delta$. *Since* $\| z - ((1 - \lambda_0)p_0 + \lambda_0 u) \| > \lambda_0\delta$ and $\| z - ((1 - \lambda)p_0 + \lambda u) \| < \lambda\delta$ is equivalent to $\| z + (\lambda_0 - \lambda)(u - p_0) -$ *((* $1 - \lambda_0)p_0 + \lambda_0 u) \| < \lambda\delta < \lambda_0\delta$, we have $\| (z + \lambda_1(u - p_0)) - ((1 - \lambda_0)p_0 + \lambda_0 u) \| = \lambda_0\delta$ for some $0 < \lambda_1 < \lambda_0 - \lambda$, i.e. w = $z + \lambda_1(u - p_0) \in \mathrm{bdry}B((1 - \lambda_0)p_0 + \lambda_0 u, \lambda_0\delta)$.

Furthermore, let $B(\circ, \chi)$ be the homothetic image of $B(u,$ 6) with respect to p_0, where χ = δ **‖** *PO* **‖** / **‖** u *-PO* $\|$. The smoothness of X implies that $B(\circ, \chi)$ and *B((* 1 $\quad \lambda_0)p_0 + \lambda_0 u, \lambda_0\delta)$ lie in the different open halfspaces determined by the hyperplane $\mathfrak{H}_u = (1 - \lambda_0)p_0 + \lambda_0 u + f_{w-((1-\lambda_0)p_0+\lambda_0 u)}^{-1}(\lambda_0\delta)$, so that $\inf_{\|t\| \leq \chi} f_{w-((1-\lambda_0)p_0+\lambda_0 u)}(t)$

$\geq f_{w-((1-\lambda_0)p_0+\lambda_0 u)}(p_0) \geq \lambda_0\delta + f_{w-((1-\lambda_0)p_0+\lambda_0 u)}((1 - \lambda_0)p_0 + \lambda_0 u)$. But

$\inf_{\|t\| \leq \chi} f_{w-((1-\lambda_0)p_0+\lambda_0 u)}(t) = -\sup_{\|t\| \leq \chi} |f_{w-((1-\lambda_0)p_0+\lambda_0 u)}(t)| = -\chi$, so that

-x $\geq \lambda_0\delta + f_{w-((1-\lambda_0)p_0+\lambda_0 u)}((1 \quad \lambda_0)p_0 + \lambda_0 u)$.

Finally, since X is smooth, $z \in \operatorname{sph} S$ and cl $\operatorname{aff} B_z^R \subseteq \mathfrak{H}_z$. If $p \notin \mathfrak{H}_z$, then we are done, assume the contrary, i.e. $\mathfrak{H}_z$ passes through the origin for every point z arising in the way described above. This means that $o \in x + f_{z-x}^{-1}(\| z - x \|)$, i.e. $0 = \| z - x \| + f_{z-x}(x)$. Since X is uniformly smooth, it is uniformly strongly differentiable [18, §26.10.(6)], so that for $\lambda \to \lambda_0$ and $\theta \to 0$, implying $z \to w$ and $x \to (1 - \lambda_0)p_0 + \lambda_0 u$, $0 = \| z - x \| + f_{z-x}(x)$ becomes arbitrarily close to $\lambda_0 \delta + f_{w-((1-\lambda_0)p_0+\lambda_0 u)}((1 - \lambda_0)p_0 + \lambda_0 u) \leq -\mu$, a contradiction finishing the argument.

Lemma 2.4. *Let S be a closed connected nonconvex und locally compact subset of a real normed linear space or a closed connected nonconvex subset of a real Banach space which is uniformly convex und uniformly smooth. Let $x \in \bigcap_{z \in T}$ cl $\operatorname{aff} B_z^R$, where T is the set* ***of*** *all regular or spherical, respectively, points of S in a relatively open subset D* ***of*** *S containing* lnc*S, nnd let $[a, b] \subseteq S$. If points x, a, b are noncollinear, then there exists a number 6 > 0 such that $S \cap \operatorname{conv}(R(x,a) \cup R(x,b)) \cap \bigcup_{y \in [a,b]} B(y, 6)$ consists exclusively* ***of*** conv(R(x, *a)* ∪ *R(x, b))* *-lc points of S. Hence, if, in addition, [u, a] ∪ [a, v] ⊆ S* ***for*** *some points $u \in [x, a]$ and $v \in R(x,a) \sim [x, a)$, then* conv([*u*, v] ∪ [*a*, *b*]) ⊆ *S*.

Proof. By Tietze's theorem [22, Th. 4.4], lnc$S$ is nonempty. Suppose, to reach a contradiction, that no such 6 > 0 exists. Since the segment $[a, b]$ is compact and the set of conv(R(x, a) ∪ $R(x, b)$)-lnc points of S is closed, there must be a conv(R(x, a) ∪ $R(x$, b))-lnc point y of S in $[a, b]$. Obviously, $y \in \operatorname{lnc} S \subseteq D$, so that there exists a number $\epsilon > 0$ such that $B(y, \epsilon) \subseteq D$ and $x \notin \operatorname{cl} B(y, \epsilon)$. Since y is a conv(R(x, a) ∪ $R(x$, b))-lnc point of S, there exist points r, $s \in S \cap$ conv(R(x, a) ∪ $R(x, b)) \cap B(y, \epsilon)$ such that $[r, s] \nsubseteq S$ and additionally, $R(x, r) \cap [a, b] \subseteq B(y, \epsilon)$ and $R(x$, s) $\cap [a, b] \subseteq B(y, \epsilon)$. Select points $t \in [r, s] \sim S$ and $t_0 \in R(x, t) \cap [a, b]$. Let w be a point of S lying on $[t_0, t]$ as close as possible to t. Then w is a relative boundary point of $S \cap R(x, t)$, so that Lemma 2.2 or Lemma 2.3 implies that there exists in $B(y, \epsilon)$ a regular or spherical, respectively, point z of S for which $x \notin$ cl $\operatorname{aff} B_z^R$, a contradiction. This establishes the first assertion. Now suppose that [u, a] ∪ $[a$, v] $\subseteq S$ for some points $u \in$ [x, a] and $v \in R(x, a) \sim$ [x, a). Define a set $P = \{d \in$ [u, v] : conv{d, a, $b\} \subseteq$ S}. It follows easily from the first part of the proof above and [9,Lemma 2.7] that P is relatively open in [u, v]. By closedness of S, it is also relatively closed in [u, v], whence $P \equiv [u, v]$, i.e. conv([u, v] ∪ [a, b]) $\subseteq S$, as required.

Lemma 2.5. *Let S be a nonempty subset* ***of*** *a real topologica1 linear space L. For distinct points x and p in* cl*S, if p is any limit point of the nonempty intersection* wlc$S \cap R^{\circ}(x, p)$ *and is clearly R°-visible from x via S, then it is a wlc point of S.*

Proof. The result is simply a reformulation of [7,Lemma 2.2] for *R*- visibility and its proof is therefore omitted.

Lemma 2.6. *Let S be a nonempty subset* ***of*** *a real topologica1 linear space L. If the point $q \in$* slnc*S is clearly R°-visible via S from every point* ***of*** *a finite subset K ≠ {q}* ***of*** cl*S, then $q \in$* rel int(slnc*S* ∩ aff(K ∪ *{q})). A parallel statement holds with clear R-visibility in place* ***of*** *clear R°-visibility.*

Proof. The argument, included for completeness, is the same for both kinds of visibility. We apply the induction on dim(KU $\{q\}$) = m. The case $m = 1$ follows immediately from Lemma 2.5. Assume the truth for all sets $K' \subseteq \mathrm{cl}S$ with dim(K' U $\{q\}) = m - 1 \geq 1$ and consider a set $K \subseteq \mathrm{cl}S$ such that dim(K U $\{q\}$) = m. There exist in K points $x_1, \ldots, x_m$ such that $\dim\{q, x_1, \ldots, x_m\} = m$. Observe that points $q, x_1, \ldots, x_m$ are pairwise distinct and affinely independent, so that $\dim\{q, x_1, \ldots, x_{m-1}\} = m - 1$ with $x_m \notin \mathrm{aff}\{q, x_1, \ldots, x_{m-1}\}$. By the induction hypothesis for the set $\{x_1, \ldots, x_{m-1}\}$, $q \in \mathrm{rel\,int}(\mathrm{slnc}S \cap \mathrm{aff}\{q, x_1, \ldots, x_{m-1}\})$. Since dim(K ∪ $\{q\}$) = m, we have $\mathrm{aff}\{q, x_1, \ldots, x_m\} \equiv$ aff(K ∪ $\{q\}$). By [22, Th. 1.81, identify $\mathrm{aff}\{q, x_1, \ldots, x_m\}$ in the topology induced from L with R^m and select a relatively open ball B in aff(K ∪ $\{q\}$) centered at q such that $x_m \notin \mathrm{cl}B$, $\widetilde{B} = B \cap \mathrm{aff}\{q, x_1, \ldots, x_{m-1}\} \subseteq \mathrm{slnc}S$ and each point of S ∩ B is clearly R°-visible from x_m via S. Lemma 2.5 and the closedness of slnc$S$ in cl$S$ imply easily that $B \cap \bigcup_{p \in \widetilde{B}} R(x_m, p)$ contains a relative neighbourhood of q in aff(K ∪ $\{q\}$), whence $q \in \mathrm{rel\,int}(\mathrm{slnc}S \cap$ aff(K ∪ $\{q\}$)), as required.

Lemma 2.7. *Let S be a nonempty subset of a real topological linear space L, $z \in \mathrm{cl}S$ and $x \in \mathrm{aff}A_z^{R^\circ} \sim \{z\} \neq \emptyset$. If $z \in S$ or z is a limit point of the nonempty intersection $S \cap R^\circ(x, z)$, then $z \in$ relint(S ∩ $R^\circ(x, z)$). The same holds for $x \in \mathrm{aff}A_z^R \sim \{z\} \neq \emptyset$.*

Proof. Since $A_z^R \subseteq A_z^{R^\circ}$, it is enough to justify the first assertion. Let $x \in \mathrm{aff}A_z^{R^\circ}$, $x \neq z$. By [7, Lemma 2.1], there is a smallest subset of $n+1$ affinely independent points $u_1, \ldots, u_{n+1}$ in $A_z^{R^\circ}$ such that $x \in \mathrm{aff}\{u_1, \ldots, u_{n+1}\}$. If $n = 0$, then $x = u_1$, whence $R^\circ(x, z) \subseteq$ S and we are done. Hence, $n \geq 1$ and $x \notin \{u_1, \ldots, u_{n+1}\}$. If $n = 1$, then, in view of $x \in (u_1u_2)$, a simple planar argument reveals that $z \in$ rel int(S ∩ $R^\circ(x, z)$) and we are done, so let $n \geq 2$ in the sequel. By [22, Th. 1.8], we identify $\mathrm{aff}\{u_1, \ldots, u_{n+1}, z\}$ in the topology induced from L with R^n or R^{n+1}, depending on whether or not $z \in \mathrm{aff}\{u_1, \ldots, u_{n+1}\}$. Let $\overline{B}$ be a full-dimensional closed ball in $\mathrm{aff}\{u_1, \ldots, u_{n+1}, z\}$ centered at z such that each point of $S \cap \overline{B}$ is clearly R°-visible from $u_1, \ldots, u_{n+1}$ via S and $x \notin \overline{B}$. Now if $v \in S \cap \overline{B} \cap R^\circ(x, z)$, then the inductive argument of [3, Lemma 2.8] or Lemma 2.6 shows that $v \in \mathrm{rel\,int}(S \cap \mathrm{aff}\{u_1, \ldots, u_{n+1}, z\})$, whence $v \in$ rel int(S ∩ $R^\circ(x, z)$), i.e. S is relatively open in $\overline{B} \cap R^\circ(x, z)$. On the other hand, if (v', v'') is a nonempty open segment in $S \cap \overline{B} \cap R^\circ(x, z)$, then the argument in [4, p. 366] reveals that $[v', v''] \subseteq S \cap \overline{B} \cap R^\circ(x, z)$. Consequently, $\overline{B} \cap R^\circ(x, z) \subseteq S$, as desired.

Lemma 2.8. *Let S be a subset **of** a real topological linear space L with* slnc$S$ *nonempty, $x \in \bigcap_{z \in \mathrm{slnc}S} \mathrm{aff}A_z^{R^\circ}$ and $[a, b] \subseteq S$. If points x, a, b are noncollinear, then [a, b] consists exclusively **of*** conv(R(x, a) ∪ $R(x, b)$) *- lcpoints **of*** **S.** *If, moreover,* **[u, a]** ∪ *$[a, v] \subseteq S$ for some points $u \in [x, a]$ and $v \in R(x, a) \sim [x, a)$, then* conv((u, v) ∪ *[a, b)) $\subseteq S$.*

Proof. Suppose that there is in *[a, b]* a conv(R(x, a) ∪ $R(x, b)$) -lnc point y of S. Obviously, $y \in \mathrm{slnc}S$, so that $x \in \mathrm{aff}A_y^{R^\circ}$, $x \neq y$. By [7, Lemma 2.11, there is a smallest subset $\{u_1, \ldots, u_{n+1}\}$ of affinely independent points in $A_y^{R^\circ}$ such that $x \in \mathrm{aff}\{u_1, \ldots, u_{n+1}\}$. Let N be a neighbourhood of y in L such that each point of $S \cap$ N is clearly R°-visible via S from points $u_i (i = 1, \ldots, n+1)$. Hence, for every point z in $S \cap N$ we have $x \in \mathrm{aff}\{u_1, \ldots, u_{n+1}\} \subseteq \mathrm{aff}A_z^{R^\circ}$. By [22, Th. 1.8], identify $\mathrm{aff}\{x, a, b\}$ in the topology induced from L with R^2 and select a relatively open ball $B \subseteq$ N in $\mathrm{aff}\{x, a, b\}$ centered at y such that $x \notin \mathrm{cl}B$. A simple application of Lemma 2.7 yields $B\cap\bigcup_{t\in B\cap[a,b]} R(x,t)\subseteq S$, whence y is a $\mathrm{conv}(R(x,a)\cup R(x,b))$ -lc point of S, a contradiction which establishes the first statement.

Now, let in addition $[u,a]\cup[a,v]\subseteq S$ for some points $u\in[x,a]$ and $v\in R(x,a)\sim[x,a)$. Define a set $T=\{t\in[u,v]:\mathrm{conv}((t,a]\cup[a,b))\subseteq S\}$. Since $a\in T$, T is nonempty. It is immediate that T is closed in $[u,v]$ and it remains to show that T is open in $[u,v]$. To fix attention, suppose that $t\in(u,a]\neq\varnothing$. The argument as in the first statement above shows that $[t,b]$ consists exclusively of $\mathrm{conv}(R(x,a)\cup R(x,b))$ -lc points of S. Since $[t,b]$ is compact, there must exist a point $t'\in[u,t)$ such that no $\mathrm{conv}(R(x,a)\cup R(x,b))$ -lnc point of S lies in $\mathrm{conv}\{t',t,b\}\sim[t',b]$. Thus, by [10, Lemma 2.21], $\mathrm{conv}\{t',t,b\}\sim(t',b)\subseteq S$ implying $(t',a]\subseteq T$. Consequently T is open in $[u,v]$, so that $T\equiv[u,v]$, i.e. $\mathrm{conv}((u,v)\cup[a,b))\subseteq S$, as required.

Lemma 2.9. *Let S be a proper subset of R^d. If $q\in\mathrm{bdry}S$ is clearly R°-visible via S from a d- element set $\{x_1,..,x_d\}\subseteq\mathrm{cl}S$ of affinely independent points, then $q\in\mathrm{aff}\{x_1,\ldots,x_d\}$.*

Proof. The statement is simply a reformulation of [5, Lemma 2] for clear R-visibility and its proof is therefore omitted.

Lemma 2.10. *If S is a connected set in $R^d(d\geq 2)$ with lncS nonempty, then $g=\max_{z\in\mathrm{lnc}S}\{\mathrm{codim}A_z^{R^\circ}\}\geq 2$.*

Proof. By Lemma 2.9, $\mathrm{codim}A_z^{R^\circ}\geq 1$ for all $z\in\mathrm{lnc}S$. Assume, to reach a contradiction, that $\mathrm{codim}A_z^{R^\circ}=1$ for all $z\in\mathrm{lnc}S$. Fix any point $z\in\mathrm{lnc}S$. By assumption, it is clearly R° -visible via S from some affinely independent points $x_1,\ldots,x_d$ in $\mathrm{cl}S$. By Lemma 2.9, $z\in\mathrm{aff}\{x_1,\ldots,x_d\}$. Let B_z be an open ball in R^d centered at z such that each point of $S\cap B_z$ is clearly R° -visible via S from $x_1,\ldots,x_d$. By Lemma 2.9, there are no boundary points of S in $H_1\cup H_2$, where H_1 and H_2 are open halfballs in $B_z\sim\mathrm{aff}\{x_1,\ldots,x_d\}$. If some $H_i(i=1,2)$ contains a point of S, then $H_i\cap\mathrm{bdry}S=\varnothing$ implies $H_i\subseteq S$. Besides, Lemma 2.9 applied to $\mathrm{aff}\{x_1,\ldots,x_d\}$ in place of R^d implies that there are no relative boundary points of S in $B_z\cap\mathrm{aff}\{x_1,\ldots,x_d\}$. Hence, if $B_z\cap\mathrm{aff}\{x_1,\ldots,x_d\}$ contains a point of S, then $B_z\cap\mathrm{aff}\{x_1,\ldots,x_d\}\subseteq S$. But $z\in\mathrm{lnc}S$, i.e. there are points r,s of S in $B_z=H_1\cup H_2\cup(B_z\cap\mathrm{aff}\{x_1,\ldots,x_d\})$ such that $[r,s]\nsubseteq S$, which by what has just been established, can happen only when one of points r,s lies in H_1 while the other in H_2, whence $H_1\cup H_2\subseteq S$ and $B_z\cap\mathrm{aff}\{x_1,\ldots,x_d\}\subseteq\sim S$. Now let us define a subset K of $\mathrm{aff}\{x_1,\ldots,x_d\}$ as follows. A point q belongs to K if and only if $q\in\mathrm{bdry}S\sim S$ and for some open ball B_q in R^d centered at q, $B_q\sim\mathrm{aff}\{x_1,\ldots,x_d\}\subseteq S$ and $B_q\cap\mathrm{aff}\{x_1,\ldots,x_d\}\subseteq\sim S$. Since $z\in K$, K is nonempty. Of course, K is relatively open in $\mathrm{aff}\{x_1,\ldots,x_d\}$. Now suppose that a point $u\in\mathrm{cl}K$. Since $K\subseteq\mathrm{lnc}S$ and $\mathrm{lnc}S$ is closed in bdryS, $u\in\mathrm{lnc}S$, whence, by initial assumption, there is in $\mathrm{cl}S$ a d- element set $\{y_1,\ldots,y_d\}$ of affinely independent points from which u is clearly R° -visible via S. Let B_u be an open ball in R^d centered at u such that each point of $S\cap B_u$ is clearly R°- visible from $y_1,\ldots,y_d$ via S. Since $u\in\mathrm{cl}K$, there is in $B_u\cap\mathrm{aff}\{x_1,\ldots,x_d\}$ a point $w\in K$ and an open ball B_w in R^d centered at w such that $B_w\sim\mathrm{aff}\{x_1,\ldots,x_d\}\subseteq S$. Let us say that $B_w\subseteq B_u$. If some of points $y_1,\ldots,y_d$ lay beyond $\mathrm{aff}\{x_1,..,x_d\}$, then the clear R° -visiblity of $S\cap B_u$ from it via S would imply $w\in S$, a contradiction. Hence, $\mathrm{aff}\{x_1,..,x_d\}\equiv\mathrm{aff}\{y_1,..,y_d\}$, whence in the same way as at the beginning, $B_u\sim\mathrm{aff}\{x_1,\ldots,x_d\}\subseteq S$ and

$B_u \cap \operatorname{aff}\{x_1, \ldots, x_d\} \subseteq \sim S$, i.e. $u \in K$. Hence, K is a nonempty open-closed subset of $\operatorname{aff}\{x_1, \ldots, x_d\}$, so that $K \equiv \operatorname{aff}\{x_1, \ldots, x_d\}$. Then however $\operatorname{aff}\{x_1, \ldots, x_d\}$ would separate S, contradictory to the connectedness of S. Consequently, $g = \max_{z \in \mathrm{lnc} S}\{\operatorname{codim} A_z^{R^\circ}\} \geq 2$, as desired.

Lemma 2.11. *Let S be a set in R^d and, for noncollinear points a, b and q, let $R^\circ(a, q) \cup R^\circ(b, q) \subseteq S$. If each lnc point of S is clearly R°-visible via S from some point of (ab), then $\operatorname{conv}(R^\circ(a, q) \cup R^\circ(b, q)) \subseteq S$.*

Proof. An easy argument involving Lemma 2.5 reveals that $R^\circ(a, q)$ and $R^\circ(b, q)$ are composed exclusively of (qab)-lc points of S. Select arbitrarily points $a' \in R^\circ(a, q)$ and $b' \in R^\circ(b, q)$. Suppose that there are (qab)-lnc points of S in $\operatorname{conv}\{q, a', b'\}$ and let h be such a point lying in $\operatorname{conv}\{q, a', b'\}$ as far as possible from $(a'b')$. By above, $h \notin [a', q] \cup [q, b']$. Next, let h' be a (qab)-lnc point of S lying in $\operatorname{conv}\{q, a', b'\}$ on the straight line σ parallel to $(a'b')$ and passing through h, h' as close as possible to (aq). By a variant of Tietze's theorem [10, Cor. 2.31, $\operatorname{conv}((a'', q] \cup [q, b'')) \subseteq S$, where $\{a''\} = \sigma \cap [a', q)$, $\{b''\} = \sigma \cap [b', q)$. By the imposed condition, h' is clearly R°-visible via S from some point of (ab). Since $(h, u'']$ consists of (qab)-lc points too, the situation described in Lemma 2.5 arises and S fails to be (qab)-locally nonconvex at h', a contradiction. Consequently, no (qab)-lnc points of S are present in $\operatorname{conv}\{q, a', b'\}$, whence [10, Cor. 2.3] implies $\operatorname{conv}((a', q] \cup [q, b')) \subseteq S$. An arbitrary choice of a', b' implies the required inclusion $\operatorname{conv}(R^\circ(a, q) \cup R^\circ(b, q)) \subseteq S$.

Corollary 2.12. *Let S be a set in R^d, q its lnc point, a and b distinct points in $\operatorname{cl} S$ collinear with q from which q is clearly R°-visible via S. Then q is clearly R°-visible via S from each point of $[a, q) \cup [b, q)$.*

Proof. Let, e.g., $a \neq q$ and denote by $\overline{B}_q$ a closed nondegenerate ball in R^d centered at q such that $a \not\subseteq \overline{B}_q$ and each point of $S \cap \overline{B}_q$ is clearly R°-visible via S from a and b. For each $z \in S \cap \overline{B}_q \sim (ab)$, by Lemma 2.6, $z \in \operatorname{rel} \operatorname{int}(S \cap (zab))$ and there are no relative boundary points of S in $\overline{B}_q \cap \operatorname{rel} \operatorname{int}(zab)_z$, whence $\overline{B}_q \cap \operatorname{rel} \operatorname{int}(zab)_z \subseteq S$. Identify a with the origin and, for $v \in [a, q)$, denote by $\overline{B}'_q = \overline{B}_q \frac{\operatorname{dist}(v,q)}{\operatorname{dist}(a,q)}$. As easily seen, each point of $S \cap \overline{B}'_q$ is clearly R°-visible from v via S. This reveals that $[a, q) \subseteq A_q^{R^\circ}$ and the assertion is established.

3. GEOMETRIC CHARACTERISTICS

The materia1 of this section plays the key role in the paper. Geometric results below should be compared with their analogues for starshaped sets [8]-[10] as well as with those established previously for cones [7, Th. 3.11.

Theorem 3.1. *Let S be a proper closed locally compact subset of a real normed linear space X. Then*

$$\ker_R S = \bigcap_{z \in \operatorname{reg} S} B_z^R = \bigcap_{z \in \operatorname{reg} S} \operatorname{conv} B_z^R = \bigcap_{z \in \operatorname{reg} S} \operatorname{cl} \operatorname{aff} B_z^R \tag{1}$$

If, moreover, S is closed, connected and nonconvex, then

$$\ker_R S = \bigcap_{z \in D} B_z^R = \bigcap_{z \in D} \operatorname{conv} B_z^R \tag{2}$$

where D is a relatively open subset of S containing lnc$S$. *If X is a real Banach space which is uniformly convex and uniformly smooth, then* (1) *holds with* reg$S$ *replaced by* sph$S$ *for a proper closed set S und (2) holds for a closed connected nonconvex set S.*

Proof. To prove (1) select any point x $\in \bigcap_{z\in W}$ cl affB_z^R, where W stands for reg$S$ or sph$S$, respectively, and any point $s \in S$, $s \neq$ x. It follows from Lemma 2.2 or Lemma 2.3 that in any case S is relatively open in $R(x, s) \subseteq S$, i.e. $x \in \ker_R S$. Consequently, $\bigcap_{z\in W}$ cl aff$B_z^R \subseteq \ker_R S$. Since the sequence of inclusions $\ker_R S \subseteq \bigcap_{z\in W} B_z^R \subseteq \bigcap_{z\in W}$ conv$B_z^R \subseteq \bigcap_{z\in W}$ cl affB_z^R is clear, the proof of (1) is finirhed.

To prove (2) select any point x $\in \bigcap_{z\in D}$ convB_z^R and any point $s \in D$, $s \neq$ x. We claim that $R(x, s) \subseteq$ S. Hence, $x \in$ convB_s^R and, by Carathéodory's theorem [22, Th. 1.21], there is a smallest subset of $n + 1$ affinely independent points $u_1, \ldots, u_{n+1}$ in B_s^R such that x $\in$ conv$\{u_1, \ldots, u_{n+1}\}$. Assume, without loss of generality, that $\{u_1, \ldots, u_{k+1}\}$ is the minima1 subset of $\{u_1, \ldots, u_{n+1}\}$ such that x $\in$ conv$\{u_1, \ldots, u_{k+1}, s\}$. If k = 0, then x $= u_1 \in B_s^R$, whence $R(x, s) \subseteq S$ and we are done. Let thus $k \geq 1$. If $s \in$ aff$\{u_1, \ldots, u_{k+1}\}$, i.e. dim aff$\{u_1, \ldots, u_{k+1}, s\} = k$, then Carathéodory's theorem in R^k and the minimality of $\{u_1, \ldots, u_{k+1}\}$ would imply $x \in$ rel int conv$\{u_1, \ldots, u_{k+1}\} \subseteq$ conv$\{u_1, \ldots, u_{k+1}, s\} = \bigcup_{i=1}^{k+}$ conv$(\{u_1, \ldots, u_{k+1}, s\} \sim \{u_i\})$, whence x $\in$ conv$(\{u_1, \ldots, u_{k+1}, s\} \sim \{u_{i_0}\})$ for some index $1 \leq i_0 \leq k+1$, contradictory to the choice of $\{u_1, \ldots, u_{k+1}\}$. Hence, $s \notin$ aff$\{u_1, \ldots, u_{k+1}\}$. Now observe that $B_z^R \subseteq S_z$ for every $z \in S$, so that x $\in \bigcap_{z\in D}$ conv$B_z^R \subseteq \bigcap_{z\in T}$ convS_z, where T stands for $D\cap$ reg$S$ or $D\cap$ sph$S$, respectively. But, by [9, Th. 3.11, $\bigcap_{z\in T}$ convS_z = ker$S$, i.e. x $\in$ ker$S$ and S is starshaped relative to x. In particular, [x, s] $\subseteq S$ and it remains to show that $R(x, s) \sim [x, s] \subseteq$ S. Since D is a relatively open subset of S and $s \in D$, Lemmas 2.2 and 2.3 imply that in any case s cannot be a relative boundary point of $S \cap R(x, s)$. Suppose that $R(x, s) \not\subseteq S$ and let v be the point of $R(x, s) \sim$ [x, s] closest to s for which [s, v] $\subseteq$ S. Points x, s, u_i are noncollinear for each $1 \leq i \leq k+1$, so that Lemma 2.4 and the closedness of S imply that cl conv([s, v] ∪ $R(u_i$, s) $\sim [u_i$, s)) $\subseteq S$ for $i = 1, \ldots, k+1$. Denote by R_i = v $- u_i + R(u_i$, s) a closed halfline parallel to $R(u_i$, s) emanating from v via S. It must be v $\in$ lc$S$, since otherwise v$\in D$ and Lemmas 2.2 and 2.3 would contradict the choice of v. By [22, Th. 1 .8], we can identify G $=$ aff$\{u_1, \ldots, u_{k+1}$, v$\}$ with R^{k+1}. Since v is a G- lc point of S, one can choose a closed nondegenerate ball $\overline{B}$ in G centered at v such that $S \cap \overline{B}$ is convex and $s \notin \overline{B}$. Denoting $[v, r_i] = \overline{B} \cap R_i$ for $i = 1, \ldots, k+1$, we have conv$\{r_1, \ldots, r_{k+1}, v\} \subseteq S$, whence easily [v, $v_0] \subseteq S$, where $\{v_0\} = (xs) \cap$ aff$\{r_1, \ldots, r_{k+1}\}$, $v_0 \in R(s, v) \sim [s, v]$, contradictory to the choice of v. Hence, $R(x, s) \subseteq S$, as desired. Consequently, x $\in \bigcap_{z\in D} B_z^R$. Now choose any point c in S, c $\neq$ x, to prove that $R(x$, c$) \subseteq$ S. We already know that x $\in$ ker$S$, so that in particular [x, c] $\subseteq S$. By [9, Lemma 2.9] (cf. [7, Lemma 2.3]), [c, $q] \subseteq S$ for some lnc point q of S. If $q = x$, then as established above, points of $D \cap$ (c, $q) \neq \emptyset$ are R -visible from x via S, so that $R(x$, c$) \subseteq S$, as required. Thus let $q \neq$ x. We know that $R(x, q) \subseteq S$. Suppose, to reach a contradiction, that x $\in$ lc$S$ and let $q_0 \in$ (x, $q]$ be an lnc point of S closest to x. But $R(x, t) \subseteq S$ for all points $t \in D$, in other words q_0 is clearly R-visible from x via S. But [x, $q_0) \subseteq$ lc$S$, so that by [7, Lemma 2.21, it must be $q_0 \in$ lc$S$, a contradiction. Consequently, x $\in$ lnc$S$ and again points of $D \cap (c$, x$) \neq 0$ are R-visible from x via S, implying $R(x, c) \subseteq S$, as desired. Hence, x $\in \ker_R S$ and $\bigcap_{z\in D}$ conv$B_z^R \subseteq \ker_R S \subseteq \bigcap_{z\in D} B_z^R \subseteq \bigcap_{z\in D}$ convB_z^R implying (2).

The proof is complete.

Theorem 3.2. *If S is a connected subset of a topological linear space L with* slncS *nonempty, then*

$$\bigcap_{z\in \mathrm{slnc}S} \mathrm{aff}A_z^{R^\circ} \subseteq \mathrm{qker}_{R^\circ}S \tag{3}$$

and

$$\bigcap_{z\in \mathrm{mlnc}S} \mathrm{aff}A_z^{R^\circ} \subseteq \mathrm{qker}_{R^\circ}S \tag{3'}$$

Proof. Since slncS $\subseteq$ mlnc$S$, we have mlnc$S \neq \varnothing$ and it is enough to justify the inclusion (3). Select any point x $\in \bigcap_{z\in \mathrm{slnc}S} \mathrm{aff}A_z^{R^\circ}$. The argument proceeds in three steps.

Firstly, we show that $R(x, y) \subseteq$ slncS for every slnc point of S different from x. By assumption, x $\in \mathrm{aff}A_y^{R^\circ}$, so in virtue of [7, Lemma 2.11, there is a finite subset $K \subseteq A_y^{R^\circ}$ of affinely independent points such that x $\in \mathrm{aff}K$. By Lemma 2.6, y$\in$ rel int (slnc$S\cap$ aff(KU {y}). Since x $\in \mathrm{aff}K$, this implies that $y \in$ rel int (slncS $\cap R^\circ(x,$ y)). Hence, the set of strong loca1 nonconvexity points of S is open in $R^\circ(x,$ y). But as closed in cl$S$, slncS is also closed in $R^\circ(x, y)$, whence $R(x, y) \subseteq$ slnc$S$, as desired.

Secondly, we select a point b of S different from x to show that, for $b \in$ int$S$, b is visible via S from x and, for (x, $b] \not\subseteq$ S, b is a limit of points of S visible via S from x. In other words, we claim that x $\in$ qker$S$. Reca11 that qker$_{R^\circ}S \subseteq$ qker$S$. By [11, Lemma 2.1], slnc$S\cap$ cl$S_b \neq \varnothing$. Select a point $d \in$ slnc$S\cap$ clS_b. For x $\neq d$, it follows from the first step above that $R(x, d) \subseteq$ slnc$S$. Now select an arbitrary point $t \in [d, x]$. Since $t \in$ slnc$S$, we have x $\in$ aff$A_t^{R^\circ}$ and, by [7, Lemma 2.11, there exists a smallest finite subset of affinely independent points in $A_t^{R^\circ}$ containing x in its affine hull. For each $t \in [d, x]$, fix exactly one $(n_t + 1)$-tuple $(u_{1,t}, \ldots, u_{n_t,t}, U_t)$ consisting of n_t affinely independent points $u_{1,t}, \ldots, u_{n_t,t}$ in $A_t^{R^\circ}$ containing x in its affine hull and a neighbourhood U_t of t such that each point of S $\cap U_t$ is clearly R°-visible from $u_{1,t}, \ldots, u_{n_t,t}$ via S. Since $[d,x]$ is compact, it can be covered by finitely many sets $U_{t_j}(j = 1, \ldots,$ k), $t_j \in [d,x]$. $\bigcup_{j=1}^k U_{t_j}$ is an open set containing $[d,x]$, so that, by [22, Th. 1 .10], there exists a starshaped neighbourhood V of the origin in L such that $[d,x] + V \subseteq \bigcup_{j=1}^k U_{t_j}$. Since $d \in$ clS_b, there is a point g $\in S_b \cap (d + V)$. If g = x, then [x, $b] \subseteq$ S and the.argument is finished, so that let g $\neq$ x. We claim that $(x, g] \subseteq$ S. By construction, g $\in d + V \subseteq U_{t_{j_0}}$ for some $1 \leq j_0 \leq k$, $t_{j_0} \in [d,$ x], so that Lemma 2.7 implies that $[g, x) \cap (d + V) \subseteq$ S. Suppose thus that (x, g] $\not\subseteq$ S and let h be a point in [g, x) closest to x such that $[g, h) \subseteq$ S. Obviously, $h \in [d,x] + V \subseteq \bigcup_{j=1}^k U_{t_j}$, so that $h \in U_{t_m}$ for some index $1 \leq m \leq k$ which means that h is clearly R°-visible via S from $u_{1,t_m}, \ldots, u_{n_{t_m},t_m}$ via S and x $\in \mathrm{aff}\{u_{1,t_m}, \ldots, u_{n_{t_m},t_m}\} \subseteq \mathrm{aff}A_h^{R^\circ}$. The situation considered in Lemma 2.7 arises and we conclude that $h \in$ rel int(S $\cap R^\circ(x,$ g)), contradictory to the choice of h. Consequently, (x, g]$\subseteq$ S. If points b, g, x are collinear, then easily $(x, b] \subseteq$ S and the argument is finished, so that in the sequel let b, g, x be noncollinear. Suppose first that $b \in$ int$S$. Then there is a point $b' \in R(g, b) \sim$ [g, $b]$ such that (x, g] ∪ [g, $b'] \subseteq$ S and Lemma 2.8 implies that (x, $b] \subseteq$ conv((x, g] ∪ [g, $b')) \subseteq$ S, as required. If (x, $b] \not\subseteq$ S, then still (x, g] ∪ [g, $b] \subseteq$ S and again by Lemma 2.8, conv((x, g] ∪ [g, $b)) \subseteq$ S, i.e. all points of $[g, b)$ are visible via S from x. If

$x = d$, then $x \in \mathrm{aff}A_x^{R^\circ}$, so that we can select a smallest finite subset of affinely independent points $u_{1,x}, \ldots, u_{n_x,x}$ in $A_x^{R^\circ}$ and a starshaped neighbourhood U_x of x such that each point of $S \cap U_x$ is clearly R°-visible via S from $u_{i,x}(i = 1, \ldots, n_x)$. We select a point $g \in S_b \cap U_x$. Then Lemma 2.7 implies that $(x, g] \cup [g, b] \subseteq S$ and the argument proceeds as above. Hence, the step two of the proof is finished.

Thirdly, we prove that $x \in A_x^{R^\circ}$, i.e. x is clearly R°-visible from itself via S. Since $\mathrm{slnc}S \neq \varnothing$ by assumption, the first step above implies easily that $x \in \mathrm{slnc}S$, whence $x \in \mathrm{aff}A_x^{R^\circ}$. By [7, Lemma 2.1], there is a smallest subset of $n + 1$ affinely independent points $u_1, \ldots, u_{n+1}$ in $A_x^{R^\circ}$ such that $x \in \mathrm{aff}\{u_1, \ldots, u_{n+1}\}$. If x coincides with one of points $u_i (i = 1, \ldots, n+1)$, then obviously $x \in A_x^{R^\circ}$, so that suppose further that $x \notin \{u_1, \ldots, u_{n+1}\}$, whence $n \geq 1$. By Lemma 2.6, $x \in \mathrm{rel\,int}\,(\mathrm{slnc}S \cap \mathrm{aff}\{u_1, \ldots, u_{n+1}\}) \subseteq \mathrm{slnc}S$. Let N be a closed starshaped neighbourhood of x in L such that each point of $S \cap N$ is clearly R°-visible via S from $u_1, \ldots, u_{n+1}$. Select any point $b \in S \cap N$, $b \neq x$, to show that $R^\circ(x, b) \subseteq S$. Of course, $\{u_1, \ldots, u_{n+1}\} \subseteq A_b^{R^\circ}$, so that Lemma 2.7 implies that $b \in \mathrm{rel\,int}(S \cap R^\circ(x, h))$. Immediately also $b \in \mathrm{slnc}S$. An easy application of Lemma 2.7 gives next $R^\circ(x, b) \cap N \subseteq S$. Consider first the case $b \in (xu_1)$. Suppose that $R^\circ(x, b) \nsubseteq S$ and let (x, c) be a largest open line segment in $S \cap R^\circ(x, h)$. As observed in the first step, $c \in \mathrm{slnc}S$, whence $x \in \mathrm{aff}A_c^{R^\circ} \sim \{c\} \neq 0$. Then however, by Lemma 2.7, $c \in \mathrm{rel\,int}(S \cap R^\circ(x, b))$, contradictory to the choice of (x, c). Consequently, $R^\circ(x, b) \subseteq S$ and we are done. Hence in the sequel let $b \notin (xu_1)$. Since $(x, b] \subseteq S$ is clearly R°-visible from u_1 via S, we have $\mathrm{conv}(R^\circ(u_1, x) \cup (x, b]) \sim R^\circ(u_1, x) \subseteq S$. Let us identify, by [22, Th. 1.8], $\mathrm{aff}\{x, u_1, b\}$ in the topology induced from L with R^2 and define a subset K of $R^\circ(x, u_1)$ as follows. A point w belongs to K if and only if there is in $S \cap (xu_1)_b$ a relatively open halfball centered at w. Obviously, $(x, u_1) \subseteq K$ and K is open in $R^\circ(x, u_1)$. To prove that K is closed in $R^\circ(x, u_1)$ select any point $q \in \mathrm{cl}K \cap (R^\circ(x, u_1) \sim (x, u_1))$. We know that $q \in \mathrm{slnc}S$, so that $x \in \mathrm{aff}A_q^{R^\circ}$ and, again by [7, Lemma 2.1], there is a smallest subset of $m + 1$ affinely independent points $v_1, .., v_{m+}$ in $A_q^{R^\circ}$ such that $x \in \mathrm{aff}\{v_1, \ldots, v_{m+1}\}$. Thus there exists in L a neighbourhood M of q such that all points of $S \cap M$ are R°-visible via S from $v_1, , v_{m+1}$. Let us select a relatively open ball B_q in $M \cap \mathrm{aff}\{x, u_1, b\}$ centered at q such that $x \notin \mathrm{cl}B_q$. Since $q \in \mathrm{cl}K$, there exists a point $w \in K$ together with a relatively open halfball $Q_w \subseteq S \cap (xu_1)_b$. Say, $Q_w \subseteq B_q$. For each point $d \in B_q \cap \mathrm{cl}(S \cap B_q)$, we have $\{v_1, \ldots, v_{m+1}\} \subseteq A_d^{R^\circ}$, so that $x \in \mathrm{aff}A_d^{R^\circ}$ and an easy application of Lemma 2.7 yields $B_q \cap \bigcup_{d \in Q_w} R^\circ(x, d) \subseteq S$, whence $q \in K$, as desired. Thus K is nonempty, simultaneously open and closed in $R^\circ(x, u_1)$, implying $K = R^\circ(x, u_1)$. Now select arbitrarily a point $r \in R^\circ(x, u_1) \sim (x, u_1]$. Since $[u_1, r] \subseteq K$ is compact, an easy argument reveals that there exists a point $t \in (u_1, b)$ such that $\mathrm{conv}((u_1, t] \cup [t, Y)) \subseteq S$. The situation described in Lemma 2.8 arises and an easy reasoning yields $\mathrm{conv}(R^\circ(u_1, b) \cup R^\circ(r, u_1)) \sim R^\circ(r, u_1) \subseteq S$. Since r has been chosen arbitrarily in $R^\circ(x, u_1) \sim (x, u\ ]$, we conclude that $(xu_1)_b \subseteq S$, so that $R^\circ(x, b) \subseteq S$. Since b has been chosen arbitrarily in $S \cap N$, we conclude that $x \in A_x^{R^\circ}$, as required.

Finally, steps two and three of the argument imply that every interior point of S is R°-visible via S from x and that every point of S which is not R°-visible via S from x is a limit of points R°-visible via S from x. This means that $x \in \mathrm{qker}_{R^\circ}S$ which completes the proof.

Easy planar examples reveal that in general the inclusion (3) or (3') cannot be replaced by the equality. This is however possible if S is open as the following corollary shows.

Corollary *3.3. If S is un open connected nonconvex subset of a real topologica1 linear space L, then*

$$\mathrm{qker}_{R^\circ}S = \bigcap_{z\in \mathrm{slnc}S} A_z^{R^\circ} = \bigcap_{z\in \mathrm{slnc}S} \mathrm{conv}A_z^{R^\circ} = \bigcap_{z\in \mathrm{slnc}S} \mathrm{aff}A_z^{R^\circ} \tag{4}$$

Proof. By a variant of Tietze's theorem [10, Cor. 2.31, $\mathrm{slnc}S \neq \varnothing$. Since S is open, the sequence of inclusions $\mathrm{qker}_{R^\circ}S \subseteq \bigcap_{z\in\mathrm{slnc}S} A_z^{R^\circ} \subseteq \bigcap_{z\in\mathrm{slnc}S} \mathrm{conv}A_z^{R^\circ} \subseteq \bigcap_{z\in\mathrm{slnc}S} \mathrm{aff}A_z^{R^\circ}$ is clear. On the other hand, if $x \in \bigcap_{z\in\mathrm{slnc}S} \mathrm{aff}A_z^{R^\circ}$, then in view of (3), $x \in \mathrm{qker}_{R^\circ}S$ and (4) is established.

4. COMBINATORIAL RESULTS

Here, a combination of Helly-type theorems with geometric formulae derived in the preceding section will produce a variety of Krasnosel'skii-type criteria for cones.

Corollary 4.1. *Let S be a proper closed locally compact subset of a real normed linear space $X(\dim X \geq 2)$, and $1 \leq c \leq \dim X$ a natural number. Then* codim $\ker_R \leq c$ *if and only if every c + 1 or fewer regulur points of S are R-visible via S from each point of a common c- codimensional subset* ***of*** *S. If S is a proper closed set und X is a uniformly convex and uniformly smooth real Bunuch space, then this holds true with regulur points replaced by spherical points* ***of*** **S.**

Proof. The necessity of the condition is obvious. To establish its sufficiency, consider the nonempty family of flats $\mathcal{G} = \{\mathrm{cl\ aff}B_z^R : z \in \mathrm{reg}S\}$. If all flats in $\mathcal{G}$ coincide with X, then (1) implies that S $\supseteq \ker_R S = \bigcap_{z\in\mathrm{reg}S} \mathrm{cl\ aff}B_z^R \equiv X$, a contradiction. Hence, $g = \max_{\mathfrak{g}\in\mathcal{G}}\{$ codimg$\} > 0$ and, by the imposed condition, every c $+ 1 \geq$ c $-$ g + 2 or fewer members of $\mathcal{G}$ have a nonempty intersection of codimension at most c. Hence, by Lemma 2.1 and the formula (1), codim $\ker_R S = \mathrm{codim}\bigcap_{\mathfrak{g}\in\mathcal{G}} \mathfrak{g} \leq$ c, as required. The proof of the parallel statement for spherical points of S proceeds in the same way.

Corollary 4.2. *Let S be a connected subset* ***of*** *a complete separable metric linear space with* $\mathrm{slnc}S$ *nonempty und α a curdinul number. If* $\mathrm{aff}A_z^{R^\circ}$ *is closedfor every* $z \in \mathrm{slnc}S$, *then* $\dim \mathrm{qker}_{R^\circ}S \geq \alpha$ *provided every countuble subset* ***of*** $\mathrm{slnc}S$ *is clearly R°-visible* ***via*** *S* ***from*** *each point* ***of*** *a common α -dimensiona1 subset* ***of*** cl*S*.

Proof. By assumption, every countable subfamily of the family $\mathcal{H} = \{\mathrm{aff}A_z^{R^\circ} : z \in \mathrm{slnc}S\}$ has at least an a-dimensiona1 intersection and each member of $\mathcal{H}$ is closed, so that, in virtue of an infinite-dimensiona1 version of Helly's theorem [17, 1.8],[16, Th. 4.11, there exists a countable subset C of $\mathrm{slnc}S$ such that $\dim\bigcap_{z\in\mathrm{slnc}S} \mathrm{aff}A_z^{R^\circ} = \dim\bigcap_{z\in C} \mathrm{aff}A_z^{R^\circ} \geq \alpha$, as desired.

Corollary 4.3. *Let S be a connected subset of a real topologica1 lineur space* $L(\dim L \geq 2)$ *with* $\mathrm{slnc}S$ *nonempty, und* $1 \leq$ c $\leq \dim L$ *a natura1 number. Then* codim $\mathrm{qker}_{R^\circ}S \leq$ *cprovided every c* $+ 1$ *orfewer slnc points* ***of*** *S are cleurly R°-visible via Sfrom euch point of a common c-codimensionul subset of* cl*S*. *If L isjìnite dimensionul, then the number* $c+1$ *can be repluced* *by c for a stronger result.*

If S is a proper subset of L, then the first statement holds with $\ker_R S$, clear R-visibility and bdry points in place of $\mathrm{qker}_{R^\circ} S$, clear R°-visibility and slnc points, respectively.

Proof. The argument in the infinite-dimensiona1 case proceeds as in Corollary 4.1. If L is finite dimensional, then, by [22, Th. 1.8], it can be given a topology in which it is topologically isomorphic to R^d. Then by Lemma 2.10, $g = \max_{\mathfrak{f}\in\mathcal{H}}\{\mathrm{codim}\mathfrak{f}\} \geq 2$, where $\mathcal{H} = \{\mathrm{aff}A_z^{R^\circ} : z \in \mathrm{slnc}S\}$. If every c $\geq$ c $g+2$ or fewer members of $\mathcal{H}$ have a nonempty intersection of codimension at most c, then Lemma 2.1 and the formula (3) yield the required inequality $\mathrm{codim}\ \mathrm{qker}_{R^\circ} S \leq$ c.

If S is a proper subset of L, then, by [7,Th.3.1 (l)], $\ker_R S = \bigcap_{z\in \mathrm{bdry}S} \mathrm{aff}A_z^R$ and the argument is as in Corollary 4.1.

The proof is complete.

The proof of the following fina1 theorem focuses most of results of this paper and shows that the combinatoria1 constant in Corollary 4.3 is not optimal. For S closed, connected and nonconvex it yields immediately [3, Th. 1.3], [6].

Theorem 4.4. *Let S be a connected set in R^3 with $\mathrm{lnc}S$ nonempty. If every two lnc points of S are clearly R°-visible via S from a common point of $\mathrm{cl}S$, then S is a quasi-cone. In particular, this holds for an open connected nonconvex set S in R^3.*

Proof. The argument employs the idea of [6], where the case of S closed has been considered, but is technically more complicated. Let us consider the first statement. If there is an lnc point s of S for which $A_s^{R^\circ}$ is a single point p, then, by the imposed combinatoria1 condition, all lnc points of S are clearly R°-visible from p via S, so that Theorem 3.2 implies that $p \in \mathrm{qker}_{R^\circ} S$, i.e. S is a quasi-cone with apex p. Suppose in the sequel that $A_s^{\prime R^\circ}$ is one- or two-dimensiona1 for every $s \in \mathrm{lnc}S$. In view of Lemma 2.9, $A_s^{R^\circ}$ cannot be three-dimensional. Select any lnc point q of S for which $\dim A_q^{R^\circ} = 1$ and choose arbitrarily distinct points u, v $\in A_q^{R^\circ}$ to show that $[u, q] \subseteq \mathrm{lnc}S$ and $\dim A_z^{R^\circ} = 1$ for all $z \in [u, q]$.

Firstly, suppose that $q \notin \mathrm{aff}A_q^{R^\circ}$. Let $\overline{B}_q$ be a closed nondegenerate ball with center at q which is disjoint from $\mathrm{aff}A_q^{R^\circ}$ and such that each point of $S \cap \overline{B}_q$ is clearly R°-visible via S from both u and v. By combinatoria1 condition, each lnc point of S is clearly R°-visible via S from some point of (UV), so that, by Lemma 2.11, $\mathrm{conv}(R^\circ(u,t) \cup R^\circ(v,t)) \subseteq S$ for every point $t \in S \cap \overline{B}_q$. Easily, no points of rel bdry(S $\cap$ *(uvt))* are present in $\overline{B}_q \cap (uvt)$, whence $\overline{B}_q \cap (uvt) \subseteq$ S. Let H and H' be open halfspaces determined by *(uvq)* in R^3. We claim that q is an H-lnc or H'-lnc point of S. Suppose not, i.e. q is both an H-lc and an H'-lnc point of S. Since $q \in \mathrm{lnc}S$, and the condition $S \cap \overline{B}_q \cap (uvq) \neq \varnothing$ implies $\overline{B}_q \cap (uvq) \subseteq S$, there must be a sequence $\{q_n\}_{n=1}^{\infty}$ of pairwise distinct points in $S \cap \overline{B}_q \sim (uvq)$ convergent to q. Easily, such a sequence must exist in at least one of halfspaces H, H'. If, for example, $\{q_n\}_{n=1}^{\infty} \subseteq H$, then, since q is an H-lc point of S, $\mathrm{conv}(\bigcup_{n=n_0}^{\infty}(\overline{B}_q \cap (uvq,)))$ $\subseteq S$ for some index $n_0 \geq 1$, hence S contains some open halfball in H centered at q. For q to be an lnc point of S, it is necessary that such a sequence exist in H' too, whence for some open ball D_q centered at q, $D_q \sim (uvq) \subseteq S$ and $S \cap D_q \cap (uvq) = \varnothing$. Then however the clear R°-visibility of q from u via S implies that q is clearly R°-visible via S from each point of $[u, q)$, whence $\dim A_q^{R^\circ} = 2$, a contradiction.

Hence, assume further that q is an H-lnc point of S. A small variation of Lemma 2.5 (cf. [3, Lemma 2.4]) shows that q cannot be the limit point of H- lc points of S lying on $[u, q]$, i.e. $\overline{B}_q \cap [u, q]$ does not contain H-lc points of S. Let $r \in [u, q)$ be the point closest to u such that $[r, q]$ consists exclusively of H-lnc points of S. Suppose, to reach a contradiction, that $r \neq u$. There exists a sequence $\{r_n\}_{n=1}^{\infty} \subseteq [u, r]$ of H-lc points of S convergent to r. By combinatoria1 condition, r is clearly R°-visible via S from some point $w \in A_q^{R^\circ}$. Again, by Lemma 2.5, $w \neq u$. Since $\dim A_r^{R^\circ} \geq 1$, there is another point $x \in A_r^{R^\circ}$ different from w. Let $\overline{B}_r$ be a closed nondegenerate ball with centre at r such that $\overline{B}_r \cap (uv) = \varnothing$, $\overline{B}_r \subseteq \mathrm{conv}(B_q \cup \{u\})$ and each point of S $\cap$ $\overline{B}_r$ is clearly R° -visible from x and w via S. If $x \notin (uvq)$, then the condition $\overline{B}_r \cap (uvq_n)_{q_n} \subseteq$ S implies $r \in \mathrm{int}S \subseteq \mathrm{lc}S$, a contradiction. Hence, $x \in (mq)$. Next, let $x \notin (rw)$. By Lemma 2.5, $x \notin (ur)$. Diminishing, if necessary, $\overline{B}_r$, we can assume for the moment that $x \notin \overline{B}_r$. Let r_n be a point on [u, r] sufficiently close to r such that $R^\circ(x, r,,)$ and $R^\circ(w, r)$ intersect inside $\overline{B}_r$. The argument as in [3, Lemma 2.4], [7, Lemma 2.2] reveals that all points of $R(x, r,,) \cap \overline{B}_r$ are H-lc points of S and, furthermore, that r is an H-lc point of S too, a contradiction. Hence, $x \in (rw)$. Let first $x = r$, i.e. r is clearly R°-visible from itself via S. Since $\overline{B}_r$ $\cap$ $(uvq_n) \subseteq S$ for $n = 1, 2,$ converges to $\overline{B}_r \cap (uvq)$, immediately $H \subseteq S$, an easy contradiction with the fact that q is an H-lnc point of S. So $x \in (rw) \sim \{r, w\}$. Again, let $x \notin \overline{B}_r$. Select an arbitrary point $a \in$ S $\cap$ $\overline{B}_r \sim (rw)$. It is easily seen that there are no relative boundary points in $S \cap \overline{B}_r \cap$ rel int$(rw)_a$, whence $\overline{B}_r \cap R^\circ(r, a) \subseteq S$. Together with $\overline{B}_r \cap (uvq_n) \subseteq$ S, this implies that $\overline{B}_r \cap H \subseteq$ S, i.e. r is an H- lc point of S, a contradiction. Consequently, $[u, q]$ consists exclusively of H-lnc points of S. An appropriate part of the above argument can be repeated to show that $\dim A_z^{R^\circ} = 1$ for $z \in (u, q)$. This in turn implies that $\dim A_z^{R^\circ} = 1$ for all $z \in [u, q]$.

Secondly, suppose that $q \in \mathrm{aff} A_q^{R^\circ}$ and let, for example, $u \neq q$. By Corollary 2.12, $[u, q) \subseteq A_q^{R^\circ}$, so that one can assume without loss of generality that v $\in (u, q)$. Let $\overline{B}_q$ be a closed nondegenerate ball with center at q not containing points u, v and such that each point of S $\cap$ B_q is clearly R°-visible via S from u and v. Denote by $r \in [u, q]$ an lnc point of S lying farthest from q and such that $[q, r] \subseteq \mathrm{lnc}S$. By Lemma 2.5, $r \neq q$. Suppose, to rcach a contradiction, that $r \neq u$. By the imposed combinatoria1 condition, q and r arc clearly R°-visible via S from some point $y \in$ (UV). Since $r \in \mathrm{lnc}S$ is the limit point of a set $(r, u] \cap \mathrm{lc}S$, Lemma 2.5 implies that it must be $y = r$, i.e. r is clearly R°-visible from itself via S. Since $\dim A_r^{R^\circ} \geq 1$, r is clearly R°-visible also from some point $w \neq r$ which, again by Lemma 2.5, does not lie on (UV). Now let H and H' be open halfspaces determined by (uvw) in R". Suppose that there is a point $z \in S \cap \overline{B}_q \sim$ (UV). It is clear that there are no relative boundary points of S in $\overline{B}_q \cap$ rel int$(uvz)_z$, whence $\overline{B}_q \cap$ rel int$(uvz)_z \subseteq$ S. Since $q \in \mathrm{lnc}S$, there are points $a, b \in$ S $\cap$ $\overline{B}_q$ such that $[a, b] \nsubseteq$ S. If a, b lay in distinct halfspaces, then according to the observation made, conv($\{u\} \cup (B_q \cap$ rel int$(uva)_a)) \subseteq$ S and conv$(\{u\} \cup (\overline{B}_q \cap$ rel int$(uvb)_b)) \subseteq$ S, so that the clear R° -visiblity of r from r and w via S would yield $H \cup H' \subseteq S$. Now if S $\cap$ $\overline{B}_q \cap (uvw) = \varnothing$, then easily q is clearly R°-visible via S from any point of (uvw), so that $\dim A_q^{R^\circ} = 2$ which is contradictory. On the other hand, if $S \cap \overline{B}_q \cap (uvw) \neq \varnothing$, then an easy argument yields S $= R^3$, a contradiction. Hence, suppose that, e.g., a, $b \in \mathrm{cl}H$. If $a, b \in H$, then as above $[a, b] \subseteq H \subseteq$ S, a contradiction. If a, $b \in \mathrm{cl}H \sim H$, then easily $[a, b] \subseteq (uvw) \subseteq S$, a contradiction. If,e.g., $a \in H$ and $b \in \mathrm{cl}H \sim H$, then $H \subseteq$ S and (uvw) $\subseteq$ S, whence again a contradiction. We conclude that $r = u$, i.e. $[u, q] \subseteq \mathrm{lnc}S$, as desired. Now fix any point

$z_0 \in (u, q)$ to show that the assumption $\dim A_{z_0}^{R^\circ} = 2$ leads a contradiction. Applying Lemma 2.5 in the same way as in Lemma 2.6, we obtain that there exists a closed nondegenerate ball $\overline{B}_{z_0}$ centered at z_0 such that $\overline{B}_{z_0} \sim \mathrm{aff}A_{z_0}^{R^\circ} \subseteq S$ and $\overline{B}_{z_0} \cap \mathrm{aff}A_{z_0}^{R^\circ} \subseteq\sim S$. Assume that $(uq) \nsubseteq \mathrm{aff}A_{z_0}^{R^\circ}$. Since $z_0 \in \mathrm{lnc}S$, the combinatorial condition imposed on S implies that q and z_0 are clearly R°-visible via S from some point $t \in (uq)$. If $t \neq z_0$, then immediately $z_0 \in \mathrm{int}S$, a contradiction. If $t = z_0$, then immediately $q \in \mathrm{int}S$, again a contradiction. Hence, it must be $(uq) \subseteq \mathrm{aff}A_{z_0}^{R^\circ}$. Denote $Z = \{z \in (u, q) : \dim A_z^{R^\circ} = 2$ and $\mathrm{aff}A_z^{R^\circ} \equiv \mathrm{aff}A_{z_0}^{R^\circ}\}$. By above assumption and discussion, Z is nonempty and relatively open in (u, q). Let h be a point of $\mathrm{cl}Z$ lying closest to q. It may happen that $h = q$. Of course, $h \in \mathrm{lnc}S$. Assume first that $h = q$. Since q is clearly R°-visible via S from u and $q \in \mathrm{cl}Z$, easily there is a closed nondegenerate ball $\overline{B}_q$ centered at q such that $u \notin B_q, S \cap B_q = B_q \sim \mathrm{aff}A_{z_0}^{R^\circ}$ and each point of $S \cap B_q$ is clearly R°-visible from u via S. Then however it is easily seen that each point in (u, q) has a small relatively open neighbourhood in $\mathrm{aff}A_{z_0}^{R^\circ}$ from every point of which q is clearly R°-visible, i.e. $\dim A_{z_0}^{R^\circ} = 2$, a contradiction. Hence, it must be $h \neq q$. Since $h \in \mathrm{cl}Z$, it is clear that $A_h^{R^\circ} \subseteq \mathrm{aff}A_{z_0}^{R^\circ}$. An easy argument reveals that $A_h^{R^\circ} \subseteq A_q^{R^\circ} \neq o$. If there is a point of $A_h^{R^\circ}$ on (uq) beyond h, then we conclude as just above that $\dim A_h^{R^\circ} = 2$ and $\mathrm{aff}A_h^{R^\circ} \equiv \mathrm{aff}A_{z_0}^{R^\circ}$, i.e. $h \in Z$, a contradiction. Hence, $h \in A_h^{R^\circ}$, i.e. h is clearly R°-visible from itself via S. Besides, there is another point $y \in A_h^{R^\circ} \sim (uq) \subseteq \mathrm{aff}A_{z_0}^{R^\circ}$. If there were points of S in $\mathrm{aff}A_{z_0}^{R^\circ}$ sufficiently close to q, then the conditions $u \in A_q^{R^\circ}$, $h, y \in A_h^{R^\circ}$ would imply $\mathrm{aff}A_{z_0}^{R^\circ} \subseteq S$, a contradiction. Hence, there is some closed nondegenerate circle $C_q \subseteq \mathrm{aff}A_{z_0}^{R^\circ} \sim S$, centered at q and $u, h \notin C_q$. Let $\overline{B}_q$ be a closed nondegenerate ball centered at q such that each point of $S \cap \overline{B}_q$ is clearly R°-visible via S from u and v and u, v, $h \notin \overline{B}_q$ and $\overline{B}_q \cap \mathrm{aff}A_{z_0}^{R^\circ} \subseteq C_q$. Since $q \in \mathrm{lnc}S$, there are points $a, b \in S \cap \overline{B}_q$ with $[a, b] \nsubseteq S$. By established above, $a, b \notin \mathrm{aff}A_{z_0}^{R^\circ}$. By Lemma 2.6, $\overline{B}_q \cap \mathrm{rel\,int}(uv)_a \subseteq S$ and $\overline{B}_q \cap \mathrm{rel\,int}(uv)_b \subseteq S$. If $b \in (uvy)_a$, then these two inclusions together with $u \in A_q^{R^\circ}$, $y \in A_h^{R^\circ}$ and $h \in A_h^{R^\circ}$ imply $[a, b] \subseteq \mathrm{int}(uvy)_a \subseteq S$, a contradiction. If $b \notin (uvy)_a$, then the same reasoning implies $\overline{B}_q \sim (uvy) \subseteq S$, whence $C_q \subseteq\sim S$ yields $\dim A_q^{R^\circ} = 2$, again a contradiction. Hence, the initial assumption about the existence of $z_0 \in (u, q)$ with $\dim A_{z_0}^{R^\circ} = 2$ is contradictory, so that for all $z \in (u, q]$ we have $\dim A_z^{R^\circ} = 1$. If $\dim A_u^{R^\circ} = 2$, then also $\dim A_z^{R^\circ} = 2$ for all z in some neighbourhood of u, a contradiction. Hence, moreover, $\dim A_u^{R^\circ} = 1$, as desired.

Now we are ready to finish the proof. By Lemma 2.10, fix in $\mathrm{cl}S$ an lnc point q of S for which $A_q^{R^\circ}$ is one-dimensional. If q' is another lnc point of S for which $\dim A_{q'}^{R^\circ} = 1$, then, by assumption, there is a point $t \in A_q^{R^\circ} \cup A_{q'}^{R^\circ}$. It follows from the above discussion that $[q, t] \cup [t, q'] \subseteq \mathrm{lnc}S$ and $\dim A_z^{R^\circ} = 1$ for every $z \in [q, t] \cup [t, q']$. Consider a mapping $[q, t] \cup [t, q'] \ni z \mapsto \mathrm{aff}A_z^{R^\circ}$. As easily seen, it is locally constant on $[q, t] \cup [t, q']$, whence all $\mathrm{aff}A_z^{R^\circ}$ coincide for all $z \in [q, t] \cup [t, q']$. In particular, $\mathrm{aff}A_{q'}^{R^\circ} \equiv \mathrm{aff}A_q^{R^\circ}$. Now select an arbitrary point $g \in \mathrm{lnc}S$ for which $\dim A_g^{R^\circ} = 2$ if it exists at all, and let $p_1, p_2, p_3 \in A_z^{R^\circ}$ for all lnc points z in some relative neighbourhood of g in $(p_1p_2p_3)$. Let $\overline{B}_g$ denote a closed nondegenerate ball centered at g such that for each lnc point z in $\overline{B}_g \cap (p_1p_2p_3)$, $\dim A_z^{R^\circ} = 2$. Then, the argument in Lemma 2.10 implies that $\overline{B}_g \cap (p_1p_2p_3) \subseteq\sim S \cap \mathrm{lnc}S$ and for every $z \in \mathrm{int}\overline{B}_g \cap (p_1p_2p_3)$ there is a closed nondegenerate ball $B_z \subseteq B_g$ centered at z such that

$\overline{B}_z \sim (p_1p_2p_3) \subseteq S$. Now if there were no lnc points z of S in $(p_1p_2p_3)$ for which $A_z^{R^\circ}$ is one-dimensional, then this would contradict the connectedness of S, because $(p_1p_2p_3) \subseteq \sim S$, as in Lemma 2.10. Hence, there must be an lnc point r in $(p_1p_2p_3)$ for which $\dim A_r^{R^\circ} = 1$ and since the set of such points r is closed in $(p_1p_2p_3)$, let r be chosen closest to g. Then $[g, r) \subseteq S$ and for any $z \in [g, r)$, $\dim A_z^{R^\circ} = 2$. Again consider the mapping $[g, r) \ni z \mapsto \operatorname{aff} A_z^{R^\circ}$. As easily seen, it is locally constant on $[g, r)$, whence all $\operatorname{aff} A_z^{R^\circ}$ coincide for all $z \in [g, r)$, i.e. $\operatorname{aff} A_g^{R^\circ} \equiv \operatorname{aff} A_z^{R^\circ}$. Now let $k_i, k_2 \in A_r^{R^\circ}$ be affinely independent points. Of course, $k_1, k_2 \in \operatorname{aff} A_g^{R^\circ}$, i.e. $\operatorname{aff} A_r^{R^\circ} \subseteq \operatorname{aff} A_g^{R^\circ}$. But $\operatorname{aff} A_r^{R^\circ} \equiv \operatorname{aff} A_q^{R^\circ}$, by what has been established above. Consequently, $\operatorname{aff} A_q^{R^\circ} \subseteq \bigcap_{z \in \operatorname{lnc} S} \operatorname{aff} A_z^{R^\circ}$, so that by Theorem 3.2, $\operatorname{aff} A_q^{R^\circ} \subseteq \operatorname{qker}_{R^\circ} S \neq \varnothing$, i.e. S is a quasi-cone, as desired.

5. REMARKS

Let us consider the set $S_1 = \{(x, y, z) : x \geq 0, y \geq 0, z \geq 0\} \cup \{(1, y, 0) : y \leq 0\} \subseteq R^3$. It is not a cone, i.e. $\ker_R S_1 = \varnothing$, while, for $D = S_1 \cap B((1, 0, 0), \epsilon)$ with $\frac{1}{2} > \epsilon > 0$, we have $\bigcap_{z \in D} \operatorname{aff} B_z^R = \bigcap_{z \in D \cap \operatorname{reg} S_1} \operatorname{aff} B_z^R = (1, 0, 0)$, so that neither the formula (1) can be strengthened by restricting the family of intersecting flats nor the formula (2) can be extended to the intersection of flats (cf. [7,Th.3.1]), and thus produce an analogue of a formula existing for the kernel of a starshaped set [20, Th. 1], [9, Th. 3.1].

Next, consider the set $S_2 = \{(x, 0) : x \geq 0\} \cup \{(x, y) : 0 \geq x \geq -1\} \subseteq R^2$ and $D = S_2 \cap B((0, 0), \epsilon)$ with $\frac{1}{2} > \epsilon > 0$. Every 3 regular points of D are R-visible from a common point via S_2, yet S_2 is not a cone. This shows that in Corollary 4.1 the combinatorial condition has to be applied to the whole set and not to a vicinity containing its lnc points to ensure that it is a cone.

In this paper, sequel to [3]-[7], we continue to study combinatorial characterizations of cones involving concepts of R-visibility and clear R-visibility. As for Krasnosel'skii-type theorems for starshaped sets [2], [12, E2] certain intersection formulae [4, Th. 2.2], [7, Th. 3.1] representing the R-kernel of a set S as the intersection of a family of flats associated with selected boundary points of S play the key role. The rest is, to some extent, an application of a variant of Helly's theorem for flats formulated in general form in Lemma 2.1. The only thing which remains to be done is to eventually diminish the combinatorial constant c + 1 appearing in Corollary 4.3. In the finite-dimensional setting, as [5] and Corollary 4.3 show, it can be replaced by c for a broad classes of sets S. Nevertheless, the constant c is not optimal even in R^3. Since the arguments in the finite-dimensional setting proceed essentially by induction on the dimension of the surrounding space, they cannot be directly adapted to the case of infinite-dimensional topological linear spaces. Let us illustrate these difficulties in more detail.

Suppose for simplicity that S is a nonempty closed subset of L, $S \neq L$, and let q be a boundary point of S. We saw above that estimating $\operatorname{codim} A_q^R$ is one of crucial steps in finding the combinatorial constant in Corollary 4.3. As a starting point, it would be good even to have the inequality $\operatorname{codim} A_q^R \geq 1$ or, somewhat stronger, $\operatorname{codim}(A_q^R \cup \{q\}) \geq 1$. With hope for a contradiction, assume that $\operatorname{codim}(A_q^R \cup \{q\}) = 0$ or, equivalently, $\operatorname{aff}(A_q^R \cup \{q\}) = L$. Consider in L any finite-dimensional flat $\mathfrak{F}$ through q, $\dim \mathfrak{F} \geq 1$. Hence, $\mathfrak{F} \subseteq \operatorname{aff}(A_q^R \cup \{q\})$ and [7,Lemma 2.1] implies easily that $\mathfrak{F} \subseteq \operatorname{aff}\{y_1, \ldots, y_d, q\}$, where $\{y_1, \ldots, y_d, q\}$ is a subset of affinely independent points of $A_q^R \cup \{q\}$. By [22,Th.1.8], aff$\{y_1, \ldots, y_d, q\}$ can be identified with R^d and [3,Lemma 2.8] implies further that $q \in$ rel int$(S \cap$ aff$\{y_1, \ldots, y_d, q\})$, so that, in particular, $q \in$ rel int$(S \cap \mathfrak{F})$. Hence, the boundary point q of S is an interior point of S relative to any finite-dimensiona1 flat in L through q. If L itself is finite dimensional, then this already leads to a contradiction (cf. [3,Cor.2.9]) and one would like to have examples of some particular infinite-dimensiona1 topologica1 linear spaces L in which this idea works similarly. A tempting example is a rea1 linear space $\mathcal{L}$ endowed with the *topology of finitely open sets* τ in which open is exactly this set whose intersection with every finite-dimensiona1 flat $\mathfrak{F}$ is open in the natura1 topology of $\mathfrak{F}$ [15],[1]. It has been known for a long time that τ transforms $\mathcal{L}$ into a topologica1 linear space L if and only if $\mathcal{L}$ is countable dimensiona1 [15] (cf. also [1], where this has been reestablished), and in the latter case τ coincides with the finest locally convex topology on C. Unfortunately, even in case of such L we cannot claim that conditions $q \in$ bdry$S$ and $q \in$ rel int$(S \cap \mathfrak{F})$ for every finite-dimensiona1 flat $\mathfrak{F}$ through q are mutually exclusive, as an introductory counterexample in [15] reveals. Observe, however, that any other combinatoria1 condition, e.g., with the number 2 instead of c + 1, imposes its own restrictions on the geometry of S, independent of those resulting from estimates of codimA_q^R. Both these factors will surely interplay in any possible improvement of Corollary 4.3.

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