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# A NOTE ON BINARY DESIGNS

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**Abstract.** *In this note, we give an upper bound and a lower bound on the number of blocks for binary block designs.*

Consider $v$ treatments arranged in $b$ blocks in a block design with the incidence matrix $N = (n_{ij})$, where $n_{ij}$ denotes the number of experimental units in the $j$-th block getting the $i$-th treatment, such that the $i$-th treatment is replicated $r_i$ times ($i = 1, 2, \ldots, v$) and the $j$-th block is of size $k_j$ ($j = 1, 2, \ldots, b$). If $n_{ij} = 0$ or 1, the design is called *binary*.

Using the Cauchy-Schwarz inequality, it follows that

$$\sum_{i=1}^{v}\sum_{j=1}^{b}(n_{ij}^2 / k_j) \geq [\sum_{i=1}^{v}\sum_{j=1}^{b}(n_{ij} / k_j)]^2 / [\sum_{i=1}^{v}\sum_{j=1}^{b}(1 / k_j)], \tag{1}$$

$$\sum_{i=1}^{v}\sum_{j=1}^{b}(n_{ij}^2 / r_i) \geq [\sum_{i=1}^{v}\sum_{j=1}^{b}(n_{ij} / r_i)]^2 / [\sum_{i=1}^{v}\sum_{j=1}^{b}(1 / r_i)], \tag{2}$$

both equalities holding if and only if the design is a complete block design, that is, $N = cJ_{v\times b}$ for some constant $c$, where $J_{v\times b}$ is a $v \times b$ matrix whose elements are all unity. Throughout this paper, we deal only with a non-complete binary design, that is, $N \neq J_{v\times b}$. This means that the equalities in (1) and (2) are not attained in the present design. Since $n_{ij} = 0$ or 1, then, from (1), we have

$$b < v[\sum_{j=1}^{b}(1 / k_j)](= A, say), \tag{3}$$

which shows that when $A$ is an integer, then $b \leq A - 1$; when $A$ is not an integer, then $B \leq |A|$, where $|x|$ denotes the largest integer not exceeding $x$.

On the other hand, from (2), we have

$$b > v[\sum_{i=1}^{v}(1 / r_i)]^{-1}, \tag{4}$$

which yields that $b \geq |v[\sum_{i=1}^{v}(1 / r_i)]^{-1}| + 1$.

Thus, we have established the following

**Theorem.** *For a binary block design with parameters $v, b, r_i, k_j (i = 1, 2, \ldots, v; j = 1, 2, \ldots, b)$, which is not a complete block design, we have*

(*i*) $\quad b \geq |v[\sum_{i=1}^{v}(1 / r_i)]^{-1}| + 1;$

*(ii)* when $v[\sum_{j=1}^{b}(1/k_j)]$ is an integer, then $b \leq v[\sum_{j=1}^{b}(1/k_j)] - 1$;

*(iii)* when $v[\sum_{j=1}^{b}(1/k_j)]$ is not an integer, then $b \leq \lfloor v[\sum_{j=1}^{b}(1/k_j)]\rfloor$. □

In general, it is known [1] that for a block design, $b \geq v - \alpha$ holds, where $\alpha$ is the multiplicity of the maximum eigenvalue, 1, of the matrix $R^{-1/2}\,CR^{-1/2}$ in which $C = diag\,[r_1, r_2, \ldots, r_v] - Ndiag\,[k_1^{-1}, k_2^{-1}, \ldots, k_b^{-1}]N'$ and $R^{1/2} = diag\,[\sqrt{r_1}, \sqrt{r_2}, \ldots, \sqrt{r_v}]$. However, the statements here are very simple and alternative expressions. This is the main point of this note, together with a representation of an "upper" bound on the number of blocks in a general binary block design. In [2], Saha gave essentially an inequality $b < v\,[\sum_{j=1}^{b}(1/k_j)]$ for an efficiency-balanced block design. Note that the same inequality (3) as the Saha one is valid in general. But, our bounds are still valid for a general binary block design.

**Remark 1.** *For inequalities (ii) and (iii), if we transform them as in the following form*

$$b/[\sum_{j=1}^{b}(1/k_j)] < v, \tag{5}$$

*then, for given values v and b, (5) shows some restrictions on block sizes $k_j$, for $j = 1, 2, \ldots, b$.*

**Remark 2.** *The inequality (i) shows that in a binary block design satisfying $\sum_{i=1}^{v}(1/r_i) < 1$, (i) is stronger than the Fisher inequality, $b \geq v$.*

*But, most block designs satisfy $\sum_{i=1}^{v}(1/r_i) > 1$, which means that, as is well-known, the Fisher inequality does not hold, in general, in any binary block design having some structure, for example, in certain partially balanced incomplete block designs.*

*When $r_1 = r_2 = \ldots = r_v$ (= r, say) and $k_1 = k_2 = \ldots = k_b$, from (i) and (ii) we have $b \geq r + 1$, which is obvious.*

**Numerical example.** Consider the following block designs with incidence matrices

$$\begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{matrix}, \quad \begin{matrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{matrix}.$$

Obviously, both of which attain all the bounds in inequalities (*i*), (*ii*) and (*iii*).

Incidentally, as an upper bound for the number of blocks, in [2] it is also shown that for a general block design, we have

$$b < |nv[\sum_{j=1}^{b}(1/k_j)]|^{1/2},$$

where $n = \sum_{i=1}^{v} v_1 = \sum_{j=1}^{b} k_j$. However, by the Cauchy-Schwarz inequality as

$$[\sum_{j=1}^{b}(1/k_j)](\sum_{j=1}^{b} k_j) \geq (\sum_{j=1}^{b} 1)^2,$$

we can obtain

$$b \leq |n[\sum_{j=1}^{b}(1 / k_j)]|^{1/2} \tag{6}$$

which, obviously, improves considerably the bound on $b$ from [2]. Similarly, we can obtain

$$v \leq |n[\sum_{i=1}^{v}(1 / r_i)]|^{1/2}.$$

**Remark 3.** *In* (6)*, the equality holds if and only if* $k_1 = k_2 = \ldots = k_b$. *Additionally, it obviously follows that* (3) *and* (4) *are alternatively derived by using relations*

$$\sum_{j=1}^{b}(n_{ij} / k_j) \leq \sum_{j=1}^{b}(1 / k_j) \text{ and } \sum_{i=1}^{v}(n_{ij} / r_i) \leq \sum_{i=1}^{v}(1 / r_i),$$

*respectively.*

**Remark 4.** *The argument here is valid for general block designs. Therefore, the bounds derived here are applicable to a wide class of available block designs.*

**REFERENCES**

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