ANALYTIC TORSION FORMS AND TORSION-SIGNATURE

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Abstract. We define a torsion-signature invariant on the total space of a compact fiber bundle. The analytic torsion form of the fiber bundle is used to find an adiabatic limit formula of the torsion-signature. Mathematics Subject Classification (1991): 58G10, 53C07, and 57R20

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1. INTRODUCTION

The analytic torsion form of Bismut-Lott[BL] arises naturally from a transgression formula of the smooth version of the Riemann-Roch-Grothendieck theorem. This torsion form is further extended to certain non-compact fiber bundles in [GR]. It is natural to study the contribution of the analytic torsion form of degree $\geq 2$ to adiabatic limits of some invariants on the total space of the fiber bundle. Such a nontrivial contribution does not exist for the analytic torsion invariant simply because of the top dimensionality of the Euler form on the base space of the fiber bundle. Hence, we need to consider the adiabatic limits of other invariants.

The purpose of this paper is to introduce a new invariant, the torsion-signature on the total space, and to prove an adiabatic limit formula of this invariant, which involves both the analytic torsion form and a modified torsion form. This torsion-signature invariant is motivated by the following question: for a compact fiber bundle $Z \to M \xrightarrow{\pi} B$ and its analytic torsion form $T(T^HM, g^Z, g^E)$ associated with a flat complex vector bundle $E$ over $M$, what homotopy invariances does the pairing $\langle L(B)T(T^HM, g^Z, g^E), [B] \rangle$ have, provided $T(T^HM, g^Z, g^E)$ is closed? We refer to Section 2 for the detail of this question. We are led to consider an operator on the total space $M$ which combines the signature operator on the base space $B$ and the Euler-de Rham operator on the fiber. Instead of explicitly producing such an operator, we apply a new mixed super trace to the Euler-de Rham operator on $M$ and get our invariant. The idea is based on the fact that applying two super traces, which correspond to the two gradings given by the degree of forms and by the Hodge star operator on a closed Riemannian manifold, to the Euler-de Rham operator, we obtain the Euler characteristic number and the signature of the manifold (cf. [BGV], Chapter 4), respectively. The mixed super trace is just the tensor product of these two super traces on the exterior cotangent bundle $\wedge(T^*M) \simeq \pi^*(\wedge(T^*B)) \otimes (T^*Z)$. The advantage of this trace approach is that we can directly use many existing formulas. For instance, we can quickly get a local mixed index formula. Because of these new features, we have raised more questions than what we have solved in this paper.

This paper is arranged as follows. In Section 2 we recall the analytic torsion form of Bismut-Lott. Section 3 is devoted to a modified torsion form that we need in the adiabatic
limit formula. We introduce in Section 4 the mixed super trace and prove a local mixed index formula. We define the torsion-signature invariant in Section 5. This invariant reduces to the analytic torsion invariant when $B = \{pt\}$. Finally, we use in Section 6 the analytic and modified torsion forms to prove an adiabatic limit formula of the torsion-signature invariant. I am grateful to Mel Rothenberg for helpful comments.

2. ANALYTIC TORSION FORMS

Let $B$ and $Z$ be closed Riemannian manifolds and $Z \to M \xrightarrow{\pi} B$ a smooth compact fiber bundle over $B$ with the fiber $Z$. Suppose $E$ is a flat complex vector bundle over $M$ with a flat connection $\nabla^E$ and a Hermitian metric $g^E$. Denote by $TZ$ the vertical tangent bundle of $M$ consisting of tangent vectors of $M$ to the fibers. We choose a connection on the fiber bundle $M$. Then $TM = TZ \oplus T^HM$, where $T^HM$ is the bundle of horizontal vectors, $T^HM \simeq \pi^*(TB)$. From this we can write the exterior cotangent bundle as $\wedge^*(T^*M) \simeq \pi^*(\wedge^*(T^*B)) \otimes \wedge^*(T^*Z)$.

Let $E$ be the infinite dimensional $\mathbb{Z}$-graded vector bundle over $B$ whose fiber over $x \in B$ is isomorphic to the space of $E_x$, valued forms $\Omega(M, E_x)$ on $M_x$, where $\pi_x = \pi^{-1}(x) \simeq Z$. There is a connection $\nabla^E$ on $E$ given by

$$\nabla^E_U \phi = L_{U^H} \phi,$$

where $\phi \in C^\infty(B, E)$, $U^H \in C^\infty(M, T^HM)$ is the horizontal lift of a vector field $U$ on $B$, $\pi^*(U^H) = U$, and $L_{U^H}$ is the Lie differentiation on $C^\infty(B, E)$. We have $\Omega(M, E) \simeq \Omega(B, E)$. Let $T$ be the curvature of the fiber bundle $M$ given by

$$T(U_1, U_2) = -P^E[U^H_1, U^H_2] \in C^\infty(M, TZ),$$

where $P^E : TM = TZ \oplus T^HM \to TZ$ is the projection. $T$ is a $TZ$-valued horizontal 2-form on $M$. Denote by $i_T$ the interior multiplication by $T$. Then the exterior differential $d_M$ on $M$ defines a flat super connection on $E$ of total degree 1,

$$d_M = d_Z + \nabla^E + i_T. \quad (1)$$

The terms of (1) are degree-0, 1, and 2 parts of $d_M$, respectively. Let $N$ be the number operator of $E$ given by multiplication by $j$ on $C^\infty(M, N(TZ) \otimes E)$. Let $g^B$ and $g^Z$ be the Riemannian metrics on $B$ and $Z$, respectively. Using these two metrics, we can define an $L^2$-inner product $g^E$ on $E$, and hence an adjoint $d_M^*$ of $d_M$ as a super connection on $E$. For $t > 0$, let

$$D_t = \frac{1}{2}(t^{-N/2}d_M^* d_M^{-N/2} - t^{N/2}d_M d_M^{-N/2}) = \frac{\sqrt{t}}{2}(d_Z^* - d_Z) + \frac{1}{2}w(E, g^E) - \frac{1}{2\sqrt{t}}(T \wedge + i_T), \quad (2)$$

where $w(E, g^E) = (\nabla^E)^* - \nabla^E = (g^E)^{-1}(\nabla^E g^E)$. Take an odd function $f(z) = z\exp(z^2), z \in \mathbb{C}$, and define $\phi(\phi) = (2\pi i)^{-1}\phi$ for $\phi \in \Omega^4(B)$. We can use the usual super trace $Tr^s$ on $E$ to define real odd and even differential forms on $B$, respectively,

$$f(D_t, g^E) = (2\pi i)^{1/2} \phi Tr^s(f(D_t)), \quad (3)$$
and

\[ f'(D_t, g^E) = \varphi \text{Tr}^\varepsilon \left( \frac{N}{2} f(D_t) \right). \]  

(4)

Let \( H(Z, E|z) \) be the \( \mathbb{Z} \)-graded complex vector bundle over \( B \) whose fiber over \( x \in B \) is isomorphic to the cohomology \( H^*(M_x, E|z) \). By Hodge theory, the metric \( g^E \) on \( Z \) induces a metric \( g^{H(Z, E|z)} \) on \( H(Z, E|z) \). Furthermore, \( d_M \) induces a canonical flat connection \( \nabla^{H(Z, E|z)} \) on \( H(Z, E|z) \). Let \( O(TZ) \) be the orientation line bundle of \( TZ \), and \( e(TZ) \) be the Euler class of \( Z \) represented by an \( o(TZ) \)-valued closed form \( e(TZ, \nabla^Z) \) on \( M \),

\[ e(TZ, \nabla^Z) = \begin{cases} \text{Pf}(\frac{R^E}{2\pi}) \equiv T(\exp \frac{-R^E}{2\pi}), & \text{dim}(Z) \text{ even} \\ 0, & \text{dim}(Z) \text{ odd}, \end{cases} \]

where \( \nabla^Z = P^Z \nabla^M \) with \( \nabla^M \) the Levi-Civita connection on \( TM \) associated with metric \( g^M = \pi^*(g^B) \otimes g^E \). \( T \) is the Berezin integral from \( \wedge T^* Z \) to \( \mathbb{R} \) [BGV], and \( R^E \) is the curvature of \( \nabla^Z \). Define

\[ f(\nabla^E, g^E) = (2\pi i)^{\frac{1}{2}} \varphi \text{Tr}^\varepsilon(f(\frac{1}{2}((\nabla^E)^* - \nabla^E))), \]

where \( w(E, g^E) = (\nabla^E)^* - \nabla^E = (g^E)^{-1}(\nabla^E g^E) \). Similarly, we can define \( f(\nabla^{H(Z, E|z)}, g^{H(Z, E|z)}) \). Set

\[ \chi'(Z, E) = \sum_{j=0}^{\text{dim}(Z)} (-1)^j \dim(H^j(Z, E|z)). \]

Let \( \chi(Z) \) and \( rk(E) \) be the Euler characteristic number of \( Z \) and the rank of \( E \), respectively. The properties of the two forms in (2) – (3) are summarized in the following lemma [BL].

**Lemma 1.**

1) \( \frac{\partial}{\partial t} f(D_t, g^E) = \frac{1}{t} df'(D_t, g^E), \quad t > 0. \)

2) As \( t \to 0, \)

\[ f(D_t, g^E) = \begin{cases} \int_Z e(TZ, \nabla^Z)f(\nabla^E, g^E) + O(t), & \text{dim}(Z) \text{ even}, \\ O(\sqrt{t}), & \text{dim}(Z) \text{ odd}, \end{cases} \]

\[ f'(D_t, g^E) = \begin{cases} \frac{1}{4} \dim(Z) rk(E) \chi(Z) + O(t), & \text{dim}(Z) \text{ even}, \\ O(\sqrt{t}), & \text{dim}(Z) \text{ odd}. \end{cases} \]

3) As \( t \to \infty, \)

\[ f(D_t, g^E) = f(\nabla^{H(Z, E|z)}, g^{H(Z, E|z)}) + O\left(\frac{1}{\sqrt{t}}\right), \]

\[ f'(D_t, g^E) = \frac{1}{2} \chi'(Z, E) + O\left(\frac{1}{\sqrt{t}}\right). \]

This lemma enables us to define an analytic torsion form \( T(T^HM, g^Z, g^E) \) of the fiber bundle \( M \) as

\[ T(T^HM, g^Z, g^E) = \int_0^\infty \left[ f'(D_t, g^E) - \frac{1}{2} \chi'(Z, E) - \left(\frac{1}{4} \dim(Z) rk(E) \chi(Z) - \frac{1}{2} \chi'(Z, E) f'\left(\frac{i\sqrt{t}}{2}\right)\right) \right] \frac{dt}{t}. \]  

(5)
Note that this torsion form differs from that of [BZ] by a negative sign. Since the closed form $f(D_t, g^E)$ is independent of $t$, we get from Lemma 1 the following transgression formula [BL].

**Lemma 2.**

\[-dT(T^HM, g^Z, g^E) = \int_Z e(TZ, \nabla^Z)f(\nabla^E, g^E) - f(\nabla^{H(Z,E)}Z, g^{H(Z,E)}Z).\] (6)

The Riemann-Roch-Grothendieck theorem asserts that the two closed forms on the right hand side of (6) represent the same cohomology class. Lemma 2 gives explicitly the difference of these two forms. Formula (6) is also important in determining when the analytic torsion form is a closed even form. Indeed, we get by (6) the first part of the following lemma [BL].

**Lemma 3.** 1) If dim(Z) is odd and $H(Z, E|z) = 0$, then the analytic torsion form is closed.

2) If $g^F$ is covariantly constant with respect to $\nabla^E$ and dim(Z) is even, then $T(T^HM, g^Z, g^E) = 0$.

The property that $T(T^HM, g^Z, g^E)$ is closed in certain cases is important in the application. So we want to discuss another case where the analytic torsion form is closed. To this aim, let $\{e_i\}$ and $\{f_\alpha\}$ be local orthonormal bases of the vertical tangent bundle $TZ$ and $TB$, respectively, with the corresponding dual bases $\{e^i\}$ and $\{f^\alpha\}$. We also use the same symbols $\{f_\alpha\}$ and $\{f^\alpha\}$ to denote the liftings of $\{f_\alpha\}$ and $\{f^\alpha\}$ to $\pi^*TB$ and $\pi^*(T^*B)$, respectively. Denote by $e^\alpha$ (resp. $f^\alpha$) the exterior multiplication by $e^\alpha$ (resp. $f^\alpha$). Let $c(X) = (X \wedge) - i_X$ and $c_\alpha(X) = (X \wedge) + i_X$ for $X$ a tangent vector or 1-form. In particular, let $c^\alpha = c(e_i), c^\alpha = c(f_\alpha), c_i^\alpha = c_i(e_i), c_\alpha^\alpha = c_\alpha(f_\alpha)$. Then

\[D_i = \sqrt{\frac{1}{2}(\nabla_{e_i}^\alpha)^2 + \frac{1}{2}w(e_i, g^E) - \frac{1}{2\sqrt{2}}c_i(T),\]

where $\psi_j = (g^E)^{-1}(\nabla_{e_j}^\alpha g^E), \psi_\alpha = (g^E)^{-1}(\nabla_{f_\alpha}^\alpha g^E)$, and $w(e_i, g^E) = m^\alpha(w_{e_i e_j}c_i^\alpha + \psi_\alpha)$ with $w_{\alpha \beta \gamma}$ given by

\[w_{abc} = X^a(\nabla_{X_b}X_c)\]

for $X_a = e_j$ or $f_\alpha$.

Let $\Gamma = \pi_1(B)$ be the fundamental group of $B$, $\rho_1 : \Gamma \to Diff(Z)$ be a homomorphism, and $\tilde{B}$ be the universal classifying space of $B$, where $Diff(Z)$ is the group of diffeomorphisms on $Z$. Let $M = \tilde{B} \times_F Z \to B$ be a fiber bundle over $B$ with the fiber $Z$. Suppose $\rho_2 : \Gamma \to GL(C^k)$ is a representation of $\Gamma$ on $\mathbb{C}^k$. Then $E = \tilde{B} \times_F (Z \times \mathbb{C}^k)$ is a flat vector bundle over $M$ and its restriction to each fiber is a trivial bundle $\tilde{E} = Z \times \mathbb{C}^k$ over $Z$. Let $g^E$ be a Hermitian metric on $E$ such that $g^E$ is locally independent of $Z$. One can construct such a metric as follows. Choose a finite open covering $\{U_i\}^l_{i=1}$ of $B$ together with smooth cross-sections $\beta_i : U_i \to \tilde{B}$ for the projection $\tilde{B} \to B$. For $x \in \beta_i(U_i)$ we define a Hermitian metric $g_i$ on $\mathbb{C}^k \to \mathbb{C}^k$. The $g_i$’s may be different for $i \neq j$. Then for $\forall x \in \mathbb{V} \beta_i(U_i), \forall \in \Gamma$, we define a metric $\rho_2(\psi^{-1})g_i$ on $\mathbb{C}^k_{x^\alpha}$. We thus have a $\Gamma$-invariant Hermitian metric $\tilde{g}_i$ on $\beta_i(U_i) \times \mathbb{C}^k$. Let $\{\sigma_i\}$ be a partition of unity subordinate to $\{U_i\}$ and $\{\bar{\sigma}_i\}$ be the pull back of $\{\sigma_i\}$ to $\tilde{B}$. Then $\tilde{g} = \sum_i \bar{\sigma}_i \tilde{g}_i$ is a $\Gamma$-invariant metric on the trivial vector bundle $\tilde{B} \times \mathbb{C}^k$. We consider $\tilde{g}$ as a $\Gamma$-invariant
metric on the vector bundle $\tilde{B} \times Z \times \mathbb{C}^k$ which induces a required metric $g$ on $E$. Let the flat connection $\nabla^E$ on $E$ be locally given by the usual exterior differentiation. Since the fiber bundle $M$ is locally a product, the curvature tensor $T$ of $M$ is zero. The operator $D_i$ in (7) is now given by

$$D_i = \frac{\sqrt{f}}{2}(-c_i^j \nabla^E_{e_j} + \frac{1}{2} w(E, g^E)).$$

(8)

Since $\psi_j = (g^E)^{-1}(\nabla^E_{e_j}g^E) = 0$, $w(E, g^E) = m^\alpha \psi_\alpha$ and $w(E, g^E) = w(E, g^E) + m^{\alpha \omega \nu \rho} \langle \tilde{c}_\omega \tilde{c}_\nu \rangle$.

**Lemma 4.** If $w_{\alpha \beta} = 0$ and $g^E$ is independent of $Z$, then the analytic torsion form $T(M, g^Z, g^E)$ is closed on $B$.

**Proof.** By (6), it suffices to prove that

$$\int_Z e(TZ, \nabla^Z)f(\nabla^E, g^E) - f(\nabla^{H(Z, E|Z)}, g^{H(Z, E|Z)}) = 0.$$  

Recall that $f(\nabla^E, g^E) = (2\pi i)^{\frac{1}{2}} \varphi Tr^f(f(\frac{1}{2} w(E, g^E)))$ and

$$f(\nabla^{H(Z, E|Z)}, g^{H(Z, E|Z)}) = (2\pi i)^{\frac{1}{2}} \varphi Tr^f(f(\frac{1}{2} w(H, g^H)))$$

$$= (2\pi i)^{\frac{1}{2}} \varphi Tr^f(f(\frac{1}{2} P_Z w(E, g^E) P_Z)),$$

where $P_Z$ is the orthogonal projection of $E$ onto the kernel of $(d_z^* - d_z)^2$. Since $g^E$ is independent of $Z$, $w(E, g^E) = m^{\alpha}(g^E)^{-1}(\nabla^E_{e_\alpha}g^E)$ is a form on $B$ which is independent of $Z$. Note that $E|Z$ is trivial. We get

$$(2\pi i)^{\frac{1}{2}} \varphi Tr^f(f(\frac{1}{2} P_Z w(E, g^E) P_Z)) = \chi(Z)(2\pi i)^{\frac{1}{2}} \varphi Tr(f(\frac{1}{2} w(E, g^E))),$$

and

$$\int_Z e(TZ, \nabla^Z)f(\nabla^E, g^E) = \chi(Z)(2\pi i)^{\frac{1}{2}} \varphi Tr(f(\frac{1}{2} w(E, g^E))).$$

These two identities prove the result.

The condition that $w_{\alpha \beta} = 0$ can be satisfied in the following case. Let $q : \tilde{B} \times Z \to M = \tilde{B} \times \Gamma \times Z$ be the natural projection. We have that $q_*(T\tilde{B} \oplus T(Z)) = TM$. Here $T(Z)$ denotes the tangent bundle of $Z$ in order to distinguish the vertical bundle $TZ$. In particular, the vertical bundle $TZ$ is equal to $q_*(T(Z))$. Assume that there is a $\Gamma$-equivariant Riemannian metric on $T(Z)$. Let the Riemannian metric $g^Z$ on $TZ$ be the image under $q_*$ of the $\Gamma$-invariant metric on $T(Z)$. Thus $g^Z$ is locally independent of $B$. Let $T^H M = q_*(T\tilde{B})$. It follows that a horizontal vector $\tilde{U} \in T^H M$ is locally independent of fibers. This implies that $L_{\tilde{U}} g^Z = 0$. Hence $w_{\alpha \beta} = <S(e_\beta) \tilde{c}_\alpha, \tilde{c}_\beta> = \frac{1}{2} (L_{\tilde{U}} g^Z)(e_\beta, e_\alpha) = 0$ (cf. [BC]). Thus $w(E, g^E) = m^\alpha \psi_\alpha = w(E, g^E)$.

We now return to the general case. Let $Z \to M \to B$ be a fiber bundle over $B$ with a flat vector bundle $E$. When the analytic torsion form is closed, we define

$$T s(B, M, E) = <L(B)T(T^H M, g^Z, g^E), [B] >,$$

(9)
with the Hirzebruch L-class $L(B)$ of $B$ and the fundamental class $[B]$ of $B$.

$Ts(B, M, E)$ is a secondary invariant in the sense that the analytic torsion form appears in the Riemann-Roch-Grothendieck theorem as a secondary class form. Since $T(T^HM, g^w, g^E)$ is an even form, $Ts(B, M, E)$ is zero for the odd dimensional $B$. Obviously, if $M = B \times Z$, $E = B \times (\mathbb{Z} \times \rho \mathbb{C}^k)$ with $E|Z = \mathbb{Z} \times \rho \mathbb{C}^k$ and $\rho : \pi_1(Z) \to GL(\mathbb{C}^k)$. Then

$$Ts(B, M, E) = \text{sig}(B)T(Z, E|Z),$$

where $\text{sig}(B)$ is the signature of $B$, and $T(Z, E|Z)$ is the analytic torsion of $Z$ associated with the bundle $E|Z$. For convention, for $Z = \{pt\}, M = B$, we understand $Ts(B, M, E) = \text{sig}(B)$. But for $B = \{pt\}, M$ is equal to $Z$. We interpret $Ts(B, M, E) = T(Z, E|Z)$.

One open question is whether $Ts(B, M, E)$ has certain homotopy property. Namely, if $h : B_1 \to B$ is certain homotopy equivalence, does the following equality hold

$$<L(B_1)h^*(T(T^HM, g^Z, g^E)), [B_1]> = <L(B)T(T^HM, g^Z, g^E), [B]>?$$

If $M$ is the trivial fiber bundle $B \times Z$ over $B$, and $E|Z$ is independent of $B$, then $Ts(B, M, Z)$ is clearly a homotopy invariant. In view of the Kahn theorem [Ka], $Ts(B, M, Z)$ is unlikely a homotopy invariant in general. Now we list some elementary properties of the invariant $Ts(B, M, E)$.

**Proposition 1.** (i) $Ts(B, M, E)$ is a diffeomorphism invariant.

(ii) Let $Z_i \to M_i \to \pi_1 B_i$ be a fiber bundle and $E_i$ a flat bundle over $M_i$ such that the analytic torsion form $T(T^HM_i, g^{Z_i}, g^{E_i})$ is closed, $i = 1, 2$. Let $Z_1 \times Z_2 \to M_1 \times M_2 \to B_1 \times B_2$ be the product of the fiber bundles $M_1$ and $M_2$, and $P_i : M_1 \times M_2 \to M_i$ be the projection. Then

$$Ts(B_1 \times B_2, M_1 \times M_2, P_1^*(E_1) \otimes P_2^*(E_2)) = \text{rk}(E_1)\chi(Z_1)\text{sig}(B_1)Ts(B_2, M_2, E_2) + \text{rk}(E_2)\chi(Z_2)\text{sig}(B_2)Ts(B_1, M_1, E_1).$$

**Proof.** Since $L(B)$ and $[B]$ are diffeomorphism invariants, it suffices to prove that the analytic torsion form $T(T^HM, g^Z, g^E)$ is independent of the metric $g^B$. But this is obvious from the definition of the analytic torsion form.

Part (ii) follows from the product formula of the analytic torsion form (see Theorem 3.28 [BL] for the case $B_1 = B_2$).

3. MODIFIED ANALYTIC TORSION FORMS

In this section we introduce a modified analytic torsion form of the compact fiber bundle $M$. This torsion form will be used in the adiabatic limit in Section 6.

We use the notation of Section 2. Let for $t > 0$

$$C_t \equiv \frac{1}{2}(r^{-\frac{1}{2}}(d_m)^*l_{\frac{1}{2}}^* + l_{\frac{1}{2}}d_m^{-\frac{1}{2}})$$

$$= \frac{\sqrt{t}}{2}(d_Z^* + d_Z) + \nabla^E + \frac{1}{2}w(E, g^E) - \frac{1}{2\sqrt{t}}c(T).$$

(10)
Let \( \delta_t \) be the automorphism of \( \Omega(B, E) \) given by \( \delta_t(\xi) = t^{-\frac{1}{2}} \xi, \xi \in \Omega(B, E) \). Then \( C_t = t\frac{1}{2} \delta_t((dM)^* + dM)\delta_t^{-1} \) and \( D_t = t\frac{1}{2} \delta_t((dM)^* - dM)\delta_t^{-1} \). Hence, \( C_t^2 = -D_t^2 \). As shown in [BGV], \( Tr^*(e^{-C_t^2}) \) is a closed form on \( B \).

**Lemma 5.** 1) As \( t \to \infty \),

\[
e^{-C_t^2} = e^{\frac{1}{2} \frac{N}{2}(H(Z,E)_{(N)z}, g_{(n,\xi,\nu)|\nu}^{(n)z})^2} + O\left(\frac{1}{\sqrt{t}}\right),
\]

in the sense of all \( C^\infty \)-norms.

2) As \( t \to 0 \), the heat kernel \( e^{-C_t^2} (x,x) \) satisfies

\[
e^{-C_t^2}(x,x) \sim (4\pi t)^{-\frac{\dim Z}{2}} \sum_{j=0}^{\infty} j! K_j(x),
\]

where \( K_j(x) \in \sum_{2j \leq 2j} \Omega^{2j}(M, \text{End}_{C(T^{-Z})}(\wedge(T^* Z) \otimes E|_Z)). \)

**Proof.** See Theorems 9.19 and 10.21 in [BGV].

By Lemma 5, we can define the modified analytic torsion form of \( M \) as follows.

**Definition 1.** The modified analytic torsion form \( T_1(T^H M, g^Z, g^E) \) of \( M \) associated with the flat complex vector bundle \( E \) is

\[
T_1(T^H M, g^Z, g^E) = \int_\tau^\infty \phi Tr^s(\frac{N}{2}(e^{-C_t^2} - e^{\frac{1}{2} \frac{N}{2}(H(Z,E)_{(N)z}, g_{(n,\xi,\nu)|\nu}^{(n)z})^2}) dt
\]

\[
+ \frac{d}{dr}(\frac{1}{\Gamma(r)} \int_0^\tau t^{-1} \phi Tr^s(\frac{N}{2}(e^{-C_t^2} - e^{\frac{1}{2} \frac{N}{2}(H(Z,E)_{(N)z}, g_{(n,\xi,\nu)|\nu}^{(n)z})^2}) dt)_{r=0},
\]

where \( \tau \) is any small positive number.

\( T_1(T^H M, g^Z, g^E) \) is an even form on \( B \). It shares some properties of the analytic torsion form \( T(T^H M, g^Z, g^E) \). We list two properties in the following.

**Proposition 2.** Let \( B \) and \( Z \) be closed Riemannian manifolds. Suppose \( Z \to M \) is a compact fiber bundle, and \( E_i \) is a flat complex vector bundle over \( M \) with Hermitian metric \( g^{E_i}, i = 1, 2 \). Let \( Z = Z_1 \times Z_2 \) with the product metric \( g^Z = g^{Z_1} \times g^{Z_2} \), and \( Z \to M \to B \) be the product fiber bundle of \( M_1 \) and \( M_2 \). Form \( E = P_1^*(E_1) \otimes P_2^*(E_2) \) and \( g^E = P_1^*(g^{E_1}) \otimes P_2^*(g^{E_2}) \), where \( P_i : M \to M_1 \) is the natural projection. Then

\[
T_1(T^H M, g^Z, g^E) = rk(E_1)\chi(Z_1)T_1(T^H M_2, g^{Z_2}, g^{E_2}) + rk(E_2)\chi(Z_2)T_1(T^H M_1, g^{Z_1}, g^{E_1}).
\]

**Proof.** Note that

\[
\Omega(M, E) \cong \Omega(B) \hat{\otimes} C^\infty(M, E)
\]

\[
\cong \Omega(B) \hat{\otimes} C^\infty(B, (\wedge(T^* Z_1) \otimes E_1|_Z) \hat{\otimes} (\wedge(T^* Z_2) \otimes E_2|_Z)),
\]

\[
N = N_1 \hat{\otimes} I + I \hat{\otimes} N_2,
\]
and
\[ D_i^2 = D_i^2 \otimes I + I \otimes D_i^2, \]
Hence,
\[ Tr^*(N^2\frac{d^2}{2}) = Tr^*(\frac{N^2}{2} e^{D^2_1} e^{D^2_2}) = Tr^*(\frac{N_1}{2} e^{D^2_1}) Tr^*(e^{D^2_2}) + Tr^*(\frac{N_2}{2} e^{D^2_2}) Tr^*(e^{D^2_1}). \]
By local index theory techniques, one can check (see [BL]) that
\[ Tr^*(e^{D^2}) = rk(E) \chi(Z). \]
Now (13) follows clearly from (12).

Recall that the analytic torsion invariant \( T(M_x, E_x) \) of each fiber \( M_x \) associated with vector bundle \( E_x = E|_{M_x}, x \in B \), is defined to be
\[ T(M_x, E_x) = e^{\frac{1}{2} \int_{M_x} P r(N(-p_{V_x}^t p_{V_x}^t)^{t=0})}, \]
where \( V_x = (dZ - dZ)|_{M_x}, P_{V_x}^t = 1 - P_{V_x} \), and \( P_{V_x} \) is the orthogonal projection onto the kernel of \( V_x \). The following lemma asserts that the zero-form component \( T_1(T^H M, g^Z, g^E)_{0} \) of \( T_1(T^H M, g^Z, g^E) \) at \( x \in B \) is equal to \( \log T(M_x, E_x) \).

**Proposition 3.** \( T_1(T^H M, g^Z, g^E)_{0} = \log T(M_x, E_x) \).

**Proof.** This is obvious, since \( \omega(H(Z, E|Z), g^{H(Z, E|Z)}) \) is a one-form on \( B \), by (12) and (14),
\[ T_1(T^H M, g^Z, g^E)_{0}(x) = \int_{\tau}^\infty \varphi Tr^*(\frac{N}{2}(e^{V_x^2} - P_{V_x})) \frac{dt}{t} + \frac{d}{dr} \left( \frac{1}{r} \right) \int_{0}^{\tau} \varphi Tr^*(\frac{N}{2}(e^{V_x^2} - P_{V_x})) dt |_{t=0} = \log T(M_x, E_x). \]

The modified torsion form \( T_1(T^H M, g^Z, g^E) \) is still quite mystery. For instance, we do not know how this form depends on its arguments and when it is a closed form on \( B \). The difficulty is that we do not have at moment (2) and (3) of Lemma 1 for the complementary error function \( erfc(x) \). From this point of view, the analytic torsion form of Bismut-Lott has advantage over the modified torsion form. However, it is much more convenient to use the modified torsion form in the adiabatic limit. This is the main reason that we discuss the modified torsion form here.

4. LOCAL MIXED INDEX THEOREM

The main purpose of this section is to introduce a mixed super trace on \( \wedge(T^*M) \sim \pi^*(\wedge(T^*B)) \otimes \wedge (T^*Z) \), which combines the super trace on \( \pi^*(\wedge(T^*B)) \) corresponding to the grading given by the Hodge star operator and the super trace on \( \wedge(T^*Z) \) associated with
the grading given by the degree of forms. We apply this mixed super trace to the Euler-de Rham operator on the total space \( M \) and get a local mixed index theorem.

Let \( Z \to M \xrightarrow{\pi} B \) be the compact fiber bundle as in Section 2. Recall that the Hodge operator \( *_M \) on \( M \) is given by the chirality operator on \( \wedge^*(T^*M) \), namely, \( *_M(\xi) = i^p c(x^1) \cdots c(x^{\dim(M)}) \xi \), where \( \{ x^i \} \) is an orthonormal basis of \( T^*M \), \( c(x) = (x \wedge - i \xi) \), \( p = \frac{\dim(M)}{2} \) for \( \dim(M) \) even, and \( p = \frac{\dim(M)+1}{2} \) for \( \dim(M) \) odd. The definition of \( *_M \) is slightly different from the usual one since \( *_M^2 = 1 \). Let \( C(T^*M) \) be the bundle of Clifford algebras over \( M \) whose fiber over \( x \in M \) is the Clifford algebra \( C(T^*_xM) \) of the Euclidean space \( T^*_xM \). Then \( \wedge^*(T^*M) \) is a \( Z_2 \)-graded Clifford-bundle over \( M \) with a \( C(T^*M) \)-action given by \( \nu \xi = c(\nu) \xi \), \( \nu \in T^*_xM \) and \( \xi \in \wedge^*(T^*_xM) \), and with a grading given by the degree of forms. We have a natural \( Z_2 \)-grading on \( \text{End}(\wedge^*(T^*M)) \) and a super trace \( tr^s \) on \( C^\infty(M, \text{End}(\wedge^*(T^*M))) \) given by

\[
tr^s(a) = tr(a_{11}) - tr(a_{22})
\]

for \( a = (a_{ij})^{i,j=1}_{i,j=1} \) on \( \wedge^*(T^*M) = \wedge^{ev}(T^*M) \oplus \wedge^{odd}(T^*M) \). There is a super trace \( Tr^s \) on \( L^2(M, \wedge^*(T^*M)) \) which is given for an operator \( A \) with a smooth kernel \( K_A(x,y) \) by

\[
Tr^s(A) = \int_M tr^s(K_A(x,x)).
\]

We assume for the moment that \( M \) is even dimensional. Then \( \text{End}(\wedge^*(T^*M)) = C(T^*M) \otimes \text{End}_{C(T^*M)}(\wedge^*(T^*M)) \), and for \( a(x) \in C(T^*M), b \in \text{End}_{C(T^*M)}(\wedge^*(T^*M)) \),

\[
tr^s(a(x) \otimes b(x)) = (-2i)^{\frac{\dim(M)}{2}} \sigma_{\dim(M)}(a(x))tr^s(b(x)),
\]

where \( \sigma_{\dim(M)} : C(T^*M) \to \wedge^{\dim(M)}(T^*M) \) is the top dimensional piece of the symbol map, \((-2i)^{\frac{\dim(M)}{2}} \sigma_{\dim(M)} \) is the super trace on \( C(T^*M) \) and \( tr^s \) is the relative trace given by

\[
tr^s(b) = 2^{-\frac{\dim(M)}{2}} tr^s(*_M b).
\]

\( tr^s \) extends to a linear map from \( \Omega(M, \text{End}_{C(T^*M)}(\wedge^*(T^*M))) \) to \( \Omega(M) \). Thus,

\[
tr^s(K_A(x,x)) = (-2i)^{\frac{\dim(M)}{2}} tr^s(\sigma_{\dim(M)}(K_A(x,x))). \tag{15}
\]

On the other hand, we can also use the Hodge star operator \( *_M \) to grade \( \wedge^*(T^*M) \) and define a usual trace \( tr_* = 2^{-\frac{\dim(M)}{2}} tr_{\wedge^*(T^*M)} \) on \( \text{End}_{C(T^*M)}(\wedge^*(T^*M)) \), which can be extended to a linear map from \( \Omega(M, \text{End}_{C(T^*M)}(\wedge^*(T^*M))) \) to \( \Omega(M) \). We have thus another pointwise super trace

\[
tr_*^s(K_A(x,x)) = (-2i)^{\frac{\dim(M)}{2}} tr_*(\sigma_{\dim(M)}(K_A(x,x))) \tag{16}
\]

and then another super trace

\[
Tr_*^s(A) = \int_M tr_*^s(K_A(x,x)). \tag{17}
\]
The difference between the two traces $Tr^\alpha$ and $Tr^\alpha_\star$ is that we use a relative super trace in (15) and a usual trace in (16). Let $D_M = \tilde{d}_M + d_M$, where $\tilde{d}_M$ is the adjoint of $d_M$ with respect to the Hermitian structure on $M$. Applying these two traces to the operator $A = e^{-it\tilde{d}_M}$ and letting $t \to 0$, we obtain the index formulas for the Euler characteristic number and the signature of $M$, respectively. See ([BGV] Chapter 4) for details.

Now for the total space of the fiber bundle $M$, we have $T^*M = (T^HM)^* \oplus T^*Z$ and $\wedge^\ast(T^*M) = \wedge^\ast((T^HM)^*) \otimes \wedge^\ast(T^*Z)$. We may use a tensor product of the above two traces $tr^\alpha$ and $tr^\alpha_\star$ to get a new trace. More specifically, we assume throughout this section that the dimension of $B$ is even. Let $tr^\alpha_M$ be the tensor product of the trace $tr^\alpha_\star$ on $\wedge^\ast((T^HM)^*)$ and the super trace $tr^\alpha$ on $\wedge^\ast(T^*Z)$. For an even dimensional fiber $Z$, $tr^\alpha_M$ acts on the smooth kernel $K_A(x,y)$ of an operator $A$ as $tr^\alpha_M(K_A(x,x)) = (-2i)^{\text{dim}(Z)/2} tr^\alpha_\star(\sigma_{\text{dim}(M)}(K_A(x,x)))$, where $tr^\alpha_\star$ is the tensor product of the pointwise traces $tr^\alpha_\ast$ and $tr^\alpha$. The trace $tr^\alpha_M$ together with an integral on $M$ produces a super trace $Tr^\alpha_M$ on $L^2(M, \wedge^\ast(T^*M))$. If we tensor $\wedge^\ast(T^*M)$ with the flat vector bundle $E$ over $M$ and use the usual trace on $E$, we get a trace as above which is still denoted by $Tr^\alpha_M$. Thus if $A$ is of $Tr^\alpha_M$-trace class operator on $L^2(M, \wedge^\ast(T^*M) \otimes E)$ with a smooth kernel $K_A(x,y)$, then

$$Tr^\alpha_M(A) = \int_M tr^\alpha_M(K_A(x,x)).$$

We first apply the trace $Tr^\alpha_M$ to the operator $D_M = \tilde{d}_M + d_M$ and consider the following invariant for an even dimensional $Z$,

$$\text{ind}_\ast(D_M) = \lim_{t \to 0} Tr^\alpha_M(e^{-it\tilde{d}_M}).$$ (18)

$\text{ind}_\ast(D_M)$ is well defined by Theorem 4.1 [BGV]. The geometric meaning of $\text{ind}_\ast(D_M)$ is that it represents a mixture of the Euler characteristic number and the signature of $M$.

**Theorem 1.** Let $Z \to M \to B$ be a fiber bundle with $Z$ and $B$ closed Riemannian manifolds. Let $B$ be even dimensional and $E$ be a flat vector bundle over $M$ with a flat connection $\nabla^E$. Let $\hat{N}$ be the number operator of the total space $M$. As $t \to 0$,

$$\text{ind}_\ast(D_M) = (2\pi i)^{-\text{dim}(M)/2} rk(E) \int_B \hat{A}(B) \int_Z \hat{A}(Z) tr^\alpha_\star(e^{-t\hat{R}}),$$ (19)

where $\hat{A}(B)$ is the $\hat{A}$-class of $B$, $\hat{R}^M = \frac{1}{4} R^M_{jk} c^j c^k$, $R^M_{jk} = \langle e_j, R^M e_k \rangle$ with the curvature $R^M$ of the Levi-Civita connection $\nabla^M$. Let $\hat{N}$ be the number operator of the total space $M$. As $t \to 0$,

(ii) $Tr^\alpha_M(e^{-it\tilde{d}_M}) = \begin{cases} \text{ind}_\ast(D_M) + O(t), & \text{dim}(Z) \text{ even}, \\ O(\sqrt{t}), & \text{dim}(Z) \text{ odd}. \end{cases}$

(iii) $Tr^\alpha_M(\hat{N} e^{-it\tilde{d}_M}) = \begin{cases} \frac{\text{dim}(M)}{2} \text{ind}_\ast(D_M) + O(t), & \text{dim}(Z) \text{ even}, \\ \frac{1}{\sqrt{t}} rk(E) \int_M a_M + O(\sqrt{t}), & \text{dim}(Z) \text{ odd}. \end{cases}$

where $a_M$ is an odd form on $M$. 


(iv) Let $V_M = \tilde{d}_M - d_M$.

\[
Tr^s_M(\sqrt{t}V_me^{-tu}_u) = \begin{cases} 
O(t), & \text{dim}(Z) \text{ even}, \\
O(\sqrt{t}), & \text{dim}(Z) \text{ odd}.
\end{cases}
\]

(v) \quad \begin{align*}
Tr^s_M((1 - 2tD^2_M)e^{-u\tilde{d}_u}) = \begin{cases} 
\text{ind}_w(D_M) + O(t), & \text{dim}(Z) \text{ even}, \\
O(\sqrt{t}), & \text{dim}(Z) \text{ odd}.
\end{cases}
\end{align*}

(vi) \quad \begin{align*}
Tr^s_M(\frac{\tilde{N}}{2}(1 - 2tD^2_M)e^{-u\tilde{d}_u}) = \begin{cases} 
\frac{\dim(M)}{4} \text{ind}_w(D_M) + O(t), & \text{dim}(Z) \text{ even}, \\
O(\sqrt{t}), & \text{dim}(Z) \text{ odd}.
\end{cases}
\end{align*}

**Proof.** (i) Since (19) is local, we can assume $B$ and $Z$ are oriented and spin manifolds. By Theorem 4.1 [BGV], the heat kernel $K_t(x,y)$ of $D_M$ has the following expansion,

\[
K_t(x,x) \sim (4\pi t)^{-\frac{\dim(M)}{2}} \sum_{i=0}^{\infty} \frac{i^i}{i!} K_i(x),
\]

(20)

with $K_i(x) \in C^\infty(M, C_2(T^*M) \otimes \text{End}_{C(T^*M)}(\wedge^*(T^*M)))$. Here $C_2(T^*M)$ is the subset of elements in the Clifford bundle $C(T^*M)$ with degree $\leq 2i$. The symbol of sum of the $K_i$'s is

\[
\sigma(K) = \sum_{i=0}^{\infty} \sigma_2(K_i) = \hat{A}(M) \exp(-\hat{R}^{\otimes E,u}_M) \in \Omega(M, \text{End}_{C(T^*M)}(\wedge^*(T^*M))).
\]

Here $\hat{R}^{\otimes E,u} = \hat{R}^M \otimes I + I \otimes R^{E,u}$, and $R^{E,u}$ is the curvature of $\nabla^{E,u} = \nabla^E + \frac{1}{2} w(\nabla^E, g^E)$, $R^{E,u} = -\frac{1}{4} w(\nabla^E, g^E)^2$. For $a \in C_2(T^*M)$, $i < \frac{\dim(M)}{2}$, we know that its super trace is zero. Thus,

\[
\text{ind}_w(D_M) = \lim_{t \to 0} \frac{(4\pi t)^{-\frac{\dim(M)}{2}}}{i} \sum_{i \geq \frac{\dim(M)}{2}} \int_M Tr^s_M(K_i(x))
\]

\[
= (4\pi)^{-\frac{\dim(M)}{2}} \int_M Tr^s_M(K_{\frac{\dim(M)}{2}}(x))
\]

\[
= (4\pi)^{-\frac{\dim(M)}{2}} \int_M \hat{A}(M) Tr^s_M(e^{-\hat{R}^{\otimes E,u}})
\]

\[
= (2\pi)^{-\frac{\dim(M)}{2}} \int_M \hat{A}(M) tr^s_a(e^{-R^{E,u}})tr(e^{-R^{E,u}}).
\]

We obtain from ([BL], (3.77)) that $tr^s(e^{-R^{E,u}}) = rk(E)$. This implies (19).

(ii) If $Z$ is even dimensional, the result follows from Part (i) and (20). For odd dimensional $Z$, we still have Formula (20). However, the heat kernel in this case contains only the half-integer powers of $i$. Also $Tr^s_M(a) = 0$ for $a \neq c_1 \cdots c_{\frac{\dim(M)}{2}} \cdots c_{\frac{\dim(M)}{2}}$, since both $tr^s_a$ and $tr^i$ have such a property. Thus, by local index theory techniques, we get that as $t \to 0$,

\[
\lim_{t \to 0} Tr^s_M(e^{-\frac{1}{2}D^2_M}) = \lim_{t \to 0} \sum_{i \geq \frac{\dim(M)}{2}} \frac{t^{-\frac{\dim(M)}{2}}}{i} \int_M Tr^s_M(K_i(x)) = 0,
\]
since \( tr^\delta_M(K_i(x)) \) is an even form and \( \dim(M) \) is odd.

(iii) Let \( \{e_\alpha\} \) and \( \{e^\alpha\} \) be local orthonormal bases of \( TM \) and \( T^*M \), respectively. The operator \( \tilde{N} \) can be written as

\[
\tilde{N} = \frac{1}{2} \sum_{i=1}^{\dim(M)} c(e_{\alpha_i})c_\alpha(e_{\alpha_i}) + \frac{\dim(M)}{2},
\]

where \( c(e_{\alpha_i}) \) and \( c_\alpha(e_{\alpha_i}) \) are the Clifford variables associated with \( e_{\alpha_i} \) [BZ]. Using the Getzler rescaling \( G_{\sqrt{t}} \), we get that as \( t \to 0 \),

\[
G_{\sqrt{t}}(\sqrt{t} \sum_{i=1}^{\dim(M)} c(e_{\alpha_i})c_\alpha(e_{\alpha_i})) \to \frac{1}{2} \sum_{i=1}^{\dim(M)} e^{\alpha_i} \wedge c_\alpha(e_{\alpha_i})
\]

and

\[
G_{\sqrt{t}}(tD_M^2) \to (\partial_\alpha - \frac{1}{4} R_{\alpha\beta}^M \gamma_\beta)^2 + \hat{R}^M \otimes E^\alpha.
\]

Therefore, for the even dimensional \( Z \),

\[
\lim_{t \to 0} Tr_M(\frac{\sqrt{t}}{2} \sum_{i=1}^{\dim(M)} c(e_{\alpha_i})c_\alpha(e_{\alpha_i}) e^{-tD_M^2}) \to \frac{(2\pi i)}{2} \int_M \hat{A}(M)tr^\delta_M((\sum_{i=1}^{\dim(M)} e^{\alpha_i} c_\alpha(e_{\alpha_i}) e^{-\hat{R}^M tr^\delta_M e^{-\hat{R}^M tr^\delta_M}})).
\]

As shown in ([BZ], XI) with the usual trace replaced by the mixed one, we obtain that the above limit is zero. Hence by Part (ii), we have

\[
\lim_{t \to 0} Tr_M(\tilde{N} e^{-tD_M^2}) = \lim_{t \to 0} Tr_M(\frac{\dim(M)}{2} e^{-tD_M^2}) = \frac{\dim(M)}{2} in\mu(D_M).
\]

The approximate rate is clear. Similarly, one can prove the odd dimensional case.

(iv) Let \( \dim(Z) \) be even. We may apply the proof of Theorem 3.16 [BL] to the situation here. Indeed, let \( z \) be an odd Grassmann variable which anticommutes with all other Grassmann variables, \( z^2 = 0 \). For \( \xi \in \Omega(B) \otimes \mathbb{C}[z] \), \( \xi = \xi_0 + z\xi_1 \), \( \xi_i \in \Omega(B) \), we define \( \xi^z = \xi_1 \). Then

\[
\lim_{t \to 0} Tr_M(\sqrt{t} V_M e^{-tD_M^2}) = \lim_{t \to 0} Tr_M(\frac{1}{\sqrt{t}} e^{-t(D_M^2 - zV_M)}z = (2\pi i)^{-\frac{\dim(M)}{2}} \int_M \hat{A}(M)tr^\delta_M((e^{-\hat{R}^M tr^\delta_M e^{-\hat{R}^M tr^\delta_M}})z = (2\pi i)^{-\frac{\dim(M)}{2}} \int_M \hat{A}(M)tr^\delta_M((e^{-\hat{R}^M tr^\delta_M} tr^\delta_M(\frac{1}{2} \psi e^{-(\frac{1}{2} \psi)}) = 0,
\]

since \( tr(\frac{1}{2} \psi e^{-(\frac{1}{2} \psi)}) \) is an odd form. Similarly, by the argument of Part (ii), we get the result for the odd dimensional case.
(v) By local index theory techniques, we have that for the even dimensional $Z$,

$$
\lim_{t \to 0} Tr_M^s(-2tD_M^2e^{-itD_M^2}) = \lim_{t \to 0} Tr_M^s(e^{-t(1+2z)D_M^2})^\frac{1}{2} $$

$$\approx \{ \int_M (2\pi i(1+2z))^{-\min(M)} \frac{1}{\sinh(1+2z)\frac{\rho_M}{2}} \det(\frac{(1+2z)\rho_M}{2})tr_{*,r}^s(e^{-(1+2z)\bar{R}^{M\otimes E_u}}) \}^\frac{1}{2} $$

$$= (2\pi i)^{-\min(M)} \frac{1}{2} \int_M \hat{A}(M)tr_{*,r}^s(\bar{R}^M e^{-\bar{R}^M})tr(R^{E,u}e^{-R^{E,u}}) = 0.$$ 

The last step follows from the fact that $tr(R^{E,u}e^{-R^{E,u}}) = 0$ by Proposition 1.3 in [BL]. Hence by Part (ii), we obtain the result for the even dimensional $Z$. If $Z$ is odd dimensional, we use the above argument together with the proof of Part (ii) to get the assertion.

(vi) This part essentially follows from Part (v) and the proof of Theorem 3.21 [BL]. Indeed, consider the following trivial fiber bundle $Z' \to M' = M \times \mathbb{R}^+; \pi' \to \mathbb{R}^+$ over $\mathbb{R}^+$, where $\pi'(x,s) = s$ and $Z' \simeq M$. Let $\pi'' : M' \to M$ be the natural projection. Then $TZ' = (\pi''^*)^*(TM)$ and $(\pi''^*)^*E$ is a flat vector bundle over $M'$. We define a metric $g^{Z'}$ on $TZ'$ such that $g_{Z'}^{Z'}|_{M \times \{s\}}$ is equal to $(\pi''^*)^*g^M$. Let the horizontal distribution $T^HM'$ of $M'$ be given by $T^HM' = (\pi')^*TR^H_+$. We have

$$d_{M'} = d_M + ds \partial_s,$$

and the fiberwise adjoint $d_{M'}^* \in (d_{M'}^*)^*$ of $d_{M'}$ is

$$d_{M'}^* = \frac{1}{s}(N - \frac{\dim(M)}{2}) - ds \partial_s,$$

and $d_{M'}^*$ is the adjoint of $d_M$. Using the notation of Section 2, we get

$$2D' = t^{-\frac{\rho}{2}}d_{M'}^*t^{-\frac{3}{2}} - t^{-\frac{3}{2}}d_{M'}t^{-\frac{3}{2}}$$

$$= s^{-\frac{\rho}{2}}2D_Ms^{\frac{\rho}{2}} + \frac{ds}{s}(N - \frac{\dim(M)}{2}).$$

By the Taylor expansion,

$$Tr_M^s(f(2D'(t))(s) = Tr_M^s(f(s^{-\frac{\rho}{2}}f(2D_Ms^{\frac{\rho}{2}})) + \frac{2ds}{s}Tr_M^s(s^{-\frac{\rho}{2}}t^{N/2}f'(2D_Ms^{\frac{\rho}{2}}))$$

$$= \frac{\dim(M)}{2s}dsTr_M^s(s^{-\frac{\rho}{2}}f'(2D_Ms^{\frac{\rho}{2}})).$$

Using the proof of Part (iv) and the fact that $R^Z_v(\frac{\partial}{\partial s}, \cdot) = 0$, we obtain that the limits of $Tr_M^s(f(2D'(t))(s)$ and $Tr_M^s(s^{-\frac{\rho}{2}}f(2D_Ms^{\frac{\rho}{2}})(1)$ as $t \to 0$ do not contain a $ds$-term. Hence the $ds$ term of the above identity together with Part (v) proves the assertion.

Zhang [Zh] has recently obtained a similar index formula for the general case where $T^HM$ could be any sub-bundle $E_1$ of $TM$. In fact, for the even rank $E_1$, our method works in this
general case as well, since $\wedge^\ast(T^*M) \simeq \wedge^\ast(E^*_1) \otimes \wedge^\ast((E^*_1)^*)$ and a super trace could similarly be defined. The point of our method is that we can directly use the usual formulas since we only change the super trace. See also [Go] for the Euler-signature type operator.

For the use of the next section, we list the following proposition whose proof is obvious.

**Proposition 4.** With the assumption of Theorem 1, there is a constant $\lambda_0 > 0$ such that for $t \to \infty$,

1) $\Tr_M^*: e^{-tD^2_M} = \Tr_M^*(P_{Ker D_M}) + O(e^{-\lambda_0 t}),$

2) $\Tr_M^*(\tilde{N} e^{-tD^2_M}) = \Tr_M^*(\tilde{N} P_{Ker D_M}) + O(e^{-\lambda_0 t}),$

3) $\Tr_M^*(\tilde{N}(1 - 2tD^2_M)e^{-tD^2_M}) = \Tr_M^*(\tilde{N} P_{Ker D_M}) + O(e^{-\lambda_0 t}).$

5. TORSION-SIGNATURE INVARIANTS

In this section we use the mixed super trace to define a torsion type invariant for the operator $D_M$. Denote by $D^2_M$ the restriction of $D_M$ to the orthogonal complement of $Ker D_M$. As usual, we have that $\Tr_M^*(\frac{N}{2} e^{-t(D^2_M)^\perp}) \sim O(e^{-\lambda_0 t})$ as $t \to \infty$ and for $t \to 0$, $\Tr_M^*(\frac{N}{2} e^{-t(D^2_M)^\perp})$ has an asymptotic expansion in $t$ which is the integral powers of $t$ for dim$(M)$ even and the half-integer powers of $t$ for dim$(M)$ odd. Thus

$$\zeta(r) = \frac{1}{\Gamma(r)} \int_0^\infty \Tr_M^*(\tilde{N} e^{-t(D^2_M)^\perp})t^{r-1} dt, \quad \Re r > \frac{\dim(M)}{2},$$

(21)

extends to a meromorphic function of $r \in \mathbb{C}$ such that $\zeta(r)$ is holomorphic near the zero.

**Definition 2.** The torsion-signature $\mathcal{T}_w(M, E)$ of $M$ associated with the flat vector bundle $E$ is defined as

$$\log \mathcal{T}_w(M, E) = \frac{d\zeta(r)}{dr} \bigg|_{r=0}.$$

$\mathcal{T}_w(M, E)$ is very special for $M = B \times Z$ and $E = \pi_1^*(E_B) \otimes \pi_2^*(E_Z)$, where $\pi_2 : B \times Z \to Z$ is the natural projection and $E_B$ and $E_Z$ are two flat vector bundles over $B$ and $Z$, respectively. In this case, $D^2_M = D^2_B \otimes 1 + 1 \otimes D^2_Z$ and

$$\Tr_M^*(\frac{N'}{2} e^{-t(D^2_M)^\perp}) = \Tr^*(P_{Ker D_B}) \Tr^*(\frac{N}{2} e^{-t(D^2_Z)^\perp}) + \Tr^*(P_{Ker D_Z}) \Tr^*(\frac{N'}{2} e^{-t(D^2_B)^\perp}),$$

where $N'$ is the number operator of $B$. By the McKean-Singer index formula,

$$\Tr^*(\frac{N'}{2} e^{-t(D^2_Z)^\perp}) = const. \Tr(*_B \frac{N'}{2} e^{-t(D^2_Z)^\perp}) = const. \frac{\dim(B)}{4} \Tr(*_B e^{-t(D^2_Z)^\perp}) = 0,$$

where we used $*_B \frac{N'}{2} + \frac{N'}{2} *_B = \frac{\dim(B)}{2} *_B$. We get

$$\log \mathcal{T}_w(M, E) = \text{sig}(B, E_B) \log \mathcal{T}(Z, E_Z).$$
Hence $T_w(M, E)$ is not trivial in general. Because of this, we call $T_w(M, E)$ the torsion-signature invariant of $M$ associated with $E$. In particular, we consider it for the fiber bundle $M = \tilde{B} \times_f Z$ and the flat vector bundle $E$ associated with two representations $\rho_i$ of $\Gamma = \pi_1(B)$ as we discussed in Section 2. In this case, we denote $T_w(M, E)$ by $T_w(B, Z, \rho_1, \rho_2)$.

**Proposition 5.** (i) For $B = \{pt\}$, $T_w(\{pt\}, Z, id, id)$ is equal to the analytic torsion $T(Z, E)$ of $Z$ associated with the trivial vector bundle $E$. (ii) For $Z = \{pt\}$, $\log T_w(\{pt\}, id, id) = 0$. (iii) $\log T_w(B, Z, id, id) = \text{sig}(B, E_B) \log T(Z, EZ)$, where $id$ means the trivial representation of $\Gamma$.

**Proof.** (i) For $B = \{pt\}$, $Tr^*_M$ is the usual super trace on $L^2(Z, \Lambda^*(T^*Z) \otimes E)$. The result is obviously true.

(ii) Since $Tr^*_M = Tr^*_\Sigma$ on $L^2(B, \Lambda^*(T^*B) \otimes E)$ and $Tr^*_\Sigma(\frac{N}{2} e^{-\kappa D^2_{g,1}}) = 0$ by the above argument, the assertion clearly holds.

(iii) This is the trivial case that we just discussed above. Q.E.D.

Part (iii) shows that $T_w(M, E)$ might be independent of the metric in general only when the cohomology $H^*(Z, E|_Z) = 0$. An unsolved problem is to find a formula for the dependence of $T_w(M, E)$ on the metric.

The following version of $T_w(M, E)$ will be used in the next section. Let $u(x) = (1 - 2x)e^{-x}$.

**Proposition 6.**

\[
\log T_w(M, E) = \int_0^{\infty} \left[ Tr^*_M(\frac{N}{2} u(tD^2_M)) - Tr^*_M(\frac{N}{2} P_{\text{Ker}D_M}) \right] dt - \left( \frac{\dim(M)}{4} \text{ind}_w(D_M) - Tr^*_M(\frac{N}{2} P_{\text{Ker}D_M}) u(t) \right) \frac{dt}{t}. \tag{22}
\]

**Proof.** By Theorem 1 and Proposition 4, the right hand side of (22) is well defined. For an even dimensional $Z$, we have

\[
\text{RHS of (22)} = \int_0^{\infty} \left[ Tr^*_M(\frac{N}{2} u(tD^2_M)) - (\frac{\dim(M)}{4} \text{ind}_w(D_M) - Tr^*_M(\frac{N}{2} P_{\text{Ker}D_M}) u(t) \right] \frac{dt}{t} \tag{22}
\]

\[
= \frac{d}{dt} \left\{ \frac{1}{\Gamma(r)} \int_0^{\infty} \left[ Tr^*_M(\frac{N}{2} e^{-tD^2_M}) \right] dt \right\} _{t=0} + \int_0^{\infty} 2 \frac{d}{dt} Tr^*_M(\frac{N}{2} e^{-tD^2_M}) dt
\]

\[
= \log T_w(M, E) + Tr^*_M(\tilde{N} P_{\text{Ker}D_M}) - \frac{\dim(M)}{2} \text{ind}_w(D_M)
\]

\[
- \frac{d}{dt} \left\{ \frac{1}{\Gamma(r)} \int_0^{\infty} \left( \frac{\dim(M)}{4} \text{ind}_w(D_M) - Tr^*_M(\frac{N}{2} P_{\text{Ker}D_M}) u(t) \right) \frac{dt}{t} \right\} _{t=0}
\]

\[
= \log T_w(M, E).
\]

Here we used Theorem 1 (iii). The above proof is also valid for the odd dimensional case.
except that we have to consider
\[ \frac{d}{dr} \left\{ \frac{1}{r} \int_0^\infty T_M^r \left( \frac{\tilde{N}}{2} (-2tD_M^2) e^{-itD_M^2} \right) \frac{dt}{r} \right\}_{r=0} \]
\[ = \int_\varepsilon^\infty \frac{d}{dr} T_M^r (\tilde{N} e^{-itD_M^2}) dt + \frac{d}{dr} \left\{ \frac{1}{r} \int_0^\varepsilon \frac{d}{dr} T_M^r (\tilde{N} e^{-itD_M^2}) dt \right\}_{r=0} \]
\[ = T_M^\varepsilon (\tilde{N} \mathcal{P}_{KerD_M}). \]

We will link the \( T_M^\varepsilon (M, E) \) to the analytic torsion form and the invariant \( TS(B, M, E) \) by an adiabatic limit formula in the next section. We refer to [GP] for the corresponding invariant in the operator situation.

6. ADIABATIC LIMITS OF TORSION-SIGNATURE INVARIANTS

Our purpose of this section is to find an adiabatic limit formula of the invariant \( T_M^\varepsilon (M, E) \) which is related to the analytic torsion form and the invariant \( TS(B, M, E) \) defined in Section 2. Unless specifically stated, we assume throughout this section that \( B \) is an even dimensional closed Riemannian manifold, \( M \) is a compact fiber bundle over \( B \) with a closed Riemannian manifold \( Z \) as typical fiber, and that \( E \) is a flat vector bundle over \( M \).

Recall the exterior differential \( d_M \) of \( M \) with coefficients in \( E \) is equal to
\[ d_M = d_Z + \nabla^E + i_H. \]

Let \( \tilde{d}_M, \tilde{d}_Z, \tilde{\nabla}^E \) and \( i_H \) be the adjoints of these operators with respect to the Hermitian structure on the total space \( M \). We know \( \Omega(M, E) \cong \Omega(B) \otimes \mathcal{C}^\infty(B, \wedge^k(T^*Z, E|_Z)) \). As before, denote by \( \tilde{N} \) and \( N' \) the number operators on \( \Omega(M, E) \) given by \( \tilde{N} \xi = j \xi \) and \( N' \xi = k \xi \) for \( \xi \in \Omega^k(B) \otimes \mathcal{C}^\infty(B, \wedge^l(T^*Z, E|_Z)) \), \( k + l = j \). Now the adiabatic limit amounts to blowing up the metric of the base space \( B \) by considering the metric \( g_{\varepsilon}^M = \frac{\pi^*(g_{\varepsilon})}{\varepsilon} \oplus g_{\varepsilon}^Z \) on \( M \) and then taking the limit as \( \varepsilon \to 0 \). With respect to this new metric, the adjoint of the exterior differential \( d_M, \varepsilon \) of \( M \) is
\[ \tilde{d}_M, \varepsilon = \varepsilon^{-N'} \tilde{d}_M \varepsilon^{N'} = \tilde{d}_Z + \varepsilon^{-1} \tilde{\nabla}^E \varepsilon + \varepsilon^{-2} i_H \varepsilon. \]

Consider
\[ C_\varepsilon = \frac{1}{2} (\varepsilon^{-N'} \tilde{d}_M \varepsilon^{N'} + d_M) \]
and
\[ D_\varepsilon = \frac{1}{2} (\varepsilon^{-N'} \tilde{d}_M \varepsilon^{N'} - d_M). \]

We have that with a local orthonormal frame \( \{ e_i, e_\alpha \} \) of \( TM \) as in Section 2 where we used \( f_{ij} \) for \( e_{ij} \),
\[ C_\varepsilon = \frac{1}{2} \left( \{ m^l \nabla^M \otimes e_j, \varepsilon e_\alpha \} + m^l \nabla^M \otimes e_j e_\alpha \right) + \left( m^l \nabla^M \otimes e_j + m^l \sqrt{\varepsilon} \nabla^M \otimes e_j e_\alpha \right) \]
\[ = \frac{1}{2} \left( c^l \nabla^M \otimes e_j + \sqrt{\varepsilon} \nabla e_\alpha \right) \]
\[ = \frac{1}{4} c^l \psi_j - \frac{\sqrt{\varepsilon}}{4} c^l \psi_\alpha. \]  \hfill (23)
where $\nabla_{x^\varepsilon}^{M \otimes E} = \nabla_{x^\varepsilon}^M \otimes 1 + 1 \otimes \nabla_{x^\varepsilon}^E$, $\nabla_{x^\varepsilon}^{M \otimes E, u} = \nabla_{x^\varepsilon}^{M \otimes E} + \frac{1}{2} \psi$, and $\nabla_{x^\varepsilon}^M$ is the Levi-Civita connection of $M$ with respect to the metric $g_{x^\varepsilon}^M$. Using the usual Lichnerowicz formula ([BGV], P.126), we get

$$C_{x^\varepsilon}^2 = \frac{1}{4} \left\{ c^I (\nabla_{\alpha, x^\varepsilon}^{M \otimes E, u} + \sqrt{\varepsilon} c^\alpha \nabla_{\alpha, x^\varepsilon}^{M \otimes E, u}) - \frac{1}{2} c^I (\nabla_{\alpha, x^\varepsilon}^{M \otimes E, u} + \sqrt{\varepsilon} c^\alpha \nabla_{\alpha, x^\varepsilon}^{M \otimes E, u}, c^I \psi) + \sqrt{\varepsilon} c^\alpha \psi \alpha \} + \frac{1}{4} (c^I (\nabla_{\alpha, x^\varepsilon}^{M \otimes E, u} + \sqrt{\varepsilon} c^\alpha \nabla_{\alpha, x^\varepsilon}^{M \otimes E, u}, c^I \psi) = \frac{1}{4} \left\{ -\Delta_x^{M \otimes E, u} + \frac{K_x}{4} + \frac{1}{2} c^I (\nabla_{\alpha, x^\varepsilon}^{M \otimes E, u} c^I) \psi + \sqrt{\varepsilon} c^I (\nabla_{\alpha, x^\varepsilon}^{M \otimes E, u} c^I) \psi \right\} + \frac{1}{2} c^I (\nabla_{\alpha, x^\varepsilon}^{M \otimes E, u} c^I) \psi + \sqrt{\varepsilon} c^I (\nabla_{\alpha, x^\varepsilon}^{M \otimes E, u} c^I) \psi \right\} + \sqrt{\varepsilon} c^I (\nabla_{\alpha, x^\varepsilon}^{M \otimes E, u} c^I) \psi \right\}_{(24)}$$

where $\Delta_x^{M \otimes E, u} = \sum_{\alpha} \left( (\nabla_{\alpha, x^\varepsilon}^{M \otimes E, u})^2 - \nabla_{\alpha, x^\varepsilon}^{M \otimes E, u} \right)$, $K_x$ is the scalar curvature of $(M, g_{x^\varepsilon}^M)$, and $R_x = R_x^M \otimes 1 + 1 \otimes R_x^E$. To further compute the terms in (24), let $\nabla = \pi^* (\nabla^B \oplus \nabla^Z)$ and $S = \nabla^M - \nabla^\varepsilon$. The torsion $S$ has the following useful properties.

(i) $2 < S(X_1)X_2, X_3 > + < T(X_1, X_2), X_3 > + < T(X_3, X_1), X_2 > - < T(X_2, X_3), X_1 > = 0,$

where $T(X_1, X_2) = \nabla^\varepsilon_{X_1} X_2 - \nabla^\varepsilon_{X_2} X_1 - [X_1, X_2]$ is the torsion of $\nabla^\varepsilon$.

(ii) $S$ is a one-form with values in antisymmetric elements of End($TM$). If $X \in TM$ , then $S(X)$ maps $TZ$ to $T^H M$. If $X \in T^H M$, then $S(X)$ maps $T^H M$ to $TZ$.

(iii) $S(\bar{U}) = 0$, for $\bar{U} \in T^H M$.

(iv) For $\bar{U} \in T^H M$ and $X_i \in TZ$,

$$< S(X_1)X_2, \bar{U}_1 > = - < S(X)\bar{U}_1, X_2 >, \quad < S(\bar{U}_1)X_1, \bar{U}_2 > = - < S(\bar{U}_1)\bar{U}_2, X_1 > = < S(X_1)\bar{U}_1, \bar{U}_2 >.$$

All other components $< S(\cdot), \cdot >$ vanish. These numbers are related to $w_{abc}$ in Section 2.

(i) $< S(\cdot), \cdot >$ is independent of the metric on $B$.

We refer to [Bij][BC] for more details about these facts. Note that $T$ takes values in $TZ$ and $T(X_1, X_2) = 0$ for $X_i \in TZ$ and $T(\bar{U}, \bar{U}) = 0$ for $\bar{U} \in T^H M$. In view of these facts, we have the following.

$$\nabla_{\varepsilon, x^\varepsilon}^{M \otimes E, u} = \nabla_{\varepsilon, x^\varepsilon}^{Z \otimes E, u} + \frac{1}{2} < S(e_i) e_j, e_\alpha > c^I \sqrt{\varepsilon} c^\alpha + \frac{1}{4} < S(e_i) e_\alpha, e_\beta > \varepsilon c^\alpha c^\beta,$$

$$\nabla_{\varepsilon, x^\varepsilon}^{M \otimes E, u} = \sqrt{\varepsilon} (\nabla_{\varepsilon, x^\varepsilon}^{Z \otimes E, u} + \frac{1}{2} < S(e_i) e_j, e_\beta > c^I \sqrt{\varepsilon} c^\beta),$$

$$\nabla_{\varepsilon, x^\varepsilon}^{M \otimes E, u} = (\nabla_{\varepsilon, x^\varepsilon}^{Z \otimes E, u} + S^\varepsilon) \nabla_{\varepsilon, x^\varepsilon}^{Z \otimes E, u} + (\nabla_{\varepsilon, x^\varepsilon}^{Z \otimes E, u} + S^\varepsilon) \nabla_{\varepsilon, x^\varepsilon}^{Z \otimes E, u}.$$
and
\[ \nabla_{\varepsilon}^{M \otimes E, u} = \sqrt{\varepsilon}(\nabla_{\varepsilon}^{M \otimes E, u} + S_{\varepsilon}^{\varepsilon}) \nabla_{\varepsilon}^{u, e_{\alpha}}. \]

Here \( S_{\varepsilon}^{\varepsilon} = \nabla_{\varepsilon}^{M} - \nabla_{\varepsilon}^{E}. \) We use the convention that \((\nabla_{\varepsilon}^{e_{\alpha}} + A(e_{\alpha}))^2 = \sum_{\alpha} (\nabla_{\varepsilon}^{e_{\alpha}} + A(e_{\alpha}))_r\) = \((\nabla + A)\sum_{\alpha} \nabla_{\varepsilon}^{e_{\alpha}} e_{\alpha}, \) where \((.)_r\) means an honest square. Then

\[
\begin{align*}
\Delta_{\varepsilon}^{M \otimes E, u} &= (\nabla_{e_{j}}^{Z \otimes E, u} + \frac{1}{2} < S(e_{j}) e_{j}, e_{\alpha} > c^{j} \sqrt{\varepsilon} c^{\alpha} + \frac{1}{4} < S(e_{j}) e_{\alpha}, e_{\beta} > \varepsilon c^{\alpha} c^{\beta} )^2 \\
&- \nabla_{\varepsilon}^{Z \otimes E, u} \sum_{\alpha} S(e_{\alpha}) e_{\alpha} - S^{\alpha} \left( \sum_{i} \varepsilon S(e_{i}) e_{i} \right) \\
&+ (\sqrt{\varepsilon}(\nabla_{\varepsilon}^{\otimes E, u} + \frac{1}{2} < S(e_{\alpha}) e_{j}, e_{\beta} > c^{j} \sqrt{\varepsilon} c^{\beta} )^2. \quad (25)
\end{align*}
\]

By (24) and (25), we obtain the following important Lichnerowicz type formula.

**Theorem 2.**

\[
\begin{align*}
C_{\varepsilon}^{2} &= \frac{1}{4} (-\nabla_{e_{j}}^{Z \otimes E, u} + \frac{1}{2} < S(e_{j}) e_{j}, e_{\alpha} > c^{j} \sqrt{\varepsilon} c^{\alpha} + \frac{1}{4} < S(e_{j}) e_{\alpha}, e_{\beta} > \varepsilon c^{\alpha} c^{\beta} )^2 \\
&+ \nabla_{\varepsilon}^{Z \otimes E, u} \frac{\varepsilon}{2} < S(S(e_{i}) e_{i}, e_{\alpha} > c^{i} \sqrt{\varepsilon} c^{\alpha} \\
&- (\sqrt{\varepsilon}(\nabla_{\varepsilon}^{\otimes E, u} + \frac{1}{2} < S(e_{\alpha}) e_{i}, e_{\beta} > c^{i} \sqrt{\varepsilon} c^{\beta} )^2 \\
&+ \frac{K_{\varepsilon}}{4} + \frac{1}{2} c^{i} c^{j} R_{\varepsilon}(e_{i}, e_{j}) + \sqrt{\varepsilon} c^{i} c^{\alpha} R_{\varepsilon}(e_{i}, e_{\alpha}) + \frac{1}{2} \varepsilon c^{\alpha} c^{i} R_{\varepsilon}(e_{\alpha}, e_{\beta}) \\
&- \frac{1}{2} \left[ c^{i} c^{j} c^{\alpha} (\nabla_{\varepsilon}^{M \otimes E, u} e_{\alpha}) \psi_{1} + \sqrt{\varepsilon} c^{i} c^{\alpha} (\nabla_{\varepsilon}^{M \otimes E, u} e_{\alpha}) \psi_{\alpha} + c^{j} c^{\alpha} (\nabla_{\varepsilon}^{M \otimes E, u} e_{\alpha}) \psi_{1} + \varepsilon c^{\alpha} c^{i} (\nabla_{\varepsilon}^{M \otimes E, u} e_{\alpha}) \psi_{1} \\
&+ \sqrt{\varepsilon} c^{i} c^{\alpha} (\nabla_{\varepsilon}^{M \otimes E, u} e_{\alpha}) \psi_{1} + \varepsilon c^{\alpha} c^{i} (\nabla_{\varepsilon}^{M \otimes E, u} e_{\alpha}) \psi_{1} \right] \\
&+ \frac{1}{4} \left( c^{i} c^{j} c^{\alpha} \psi_{1} \psi_{1} + \varepsilon c^{\alpha} c^{i} c^{j} \psi_{1} \psi_{1} \right) \}. \quad (26)
\end{align*}
\]

More generally, let \( z \) be an auxiliary odd Grassmann variable which anticommutes with all other odd Grassmann variables and \( z^{2} = 0. \) We want to compute \( 4iz_{\varepsilon}^{2} = 2z_{\frac{1}{2}} D_{\varepsilon}. \) We have

\[
\begin{align*}
D_{\varepsilon} &= \frac{1}{2} \left[ (m^{i} \nabla_{e_{j}}^{M \otimes E} + m^{\alpha} \sqrt{\varepsilon} \nabla_{e_{\alpha}}^{M \otimes E})^{*} - (m^{i} \nabla_{e_{j}}^{M \otimes E} + m^{\alpha} \sqrt{\varepsilon} \nabla_{e_{\alpha}}^{M \otimes E}) \right] \\
&= \frac{1}{2} \left[ -c^{i} \nabla_{e_{j}}^{M \otimes E, u} - \sqrt{\varepsilon} c^{\alpha} \nabla_{e_{\alpha}}^{M \otimes E, u} + \frac{1}{2} c^{i} \psi_{j} + \frac{\sqrt{\varepsilon}}{2} c^{\alpha} \psi_{\alpha} \right]. \quad (27)
\end{align*}
\]

Let

\[
D_{i, \varepsilon} = \nabla_{e_{i}}^{Z \otimes E, u} + \frac{1}{2} < S(e_{i}) e_{j}, e_{\alpha} > c^{j} \sqrt{\varepsilon} c^{\alpha} + \frac{1}{4} < S(e_{i}) e_{\alpha}, e_{\beta} > \varepsilon c^{\alpha} c^{\beta},
\]

\[
D_{\varepsilon} = \nabla_{\varepsilon}^{\otimes E, u} + \frac{1}{2} < S(e_{\alpha}) e_{j}, e_{\beta} > c^{j} \sqrt{\varepsilon} c^{\beta}.
\]
and

\[ D_{\alpha, \varepsilon} = \sqrt{\varepsilon} (\nabla_{e_{\alpha}}^{\otimes E, \mu} + \frac{1}{2} < S(e_{\alpha}) e_{i}, e_{\beta} > c^{i} \sqrt{\varepsilon} c^{\beta}). \]

\[ D_{j, \varepsilon} \frac{z}{2 \sqrt{t}} c^{i} + \frac{z}{2 \sqrt{t}} c^{i} D_{j, \varepsilon} - \frac{z}{2 \sqrt{t}} c^{i} (\nabla_{e_{i}}^{\varepsilon} e_{j}) \]

\[ = \frac{z}{2 \sqrt{t}} [c_{i} (\nabla_{e_{i}}^{\varepsilon} e_{j}) + 2c^{i}_{\alpha} \nabla_{e_{\alpha}}^{\otimes E, \mu} + \frac{1}{2} < S(e_{i}) e_{i}, e_{\alpha} > \sqrt{\varepsilon} (c^{i}_{\alpha} c^{j} - c^{j} c^{i}_{\alpha}) c^{\alpha} \]

\[ + \frac{1}{4} < S(e_{j}) e_{\alpha}, e_{\beta} > \varepsilon 2c^{\alpha} c^{j} c^{\alpha} - c_{i} (\nabla_{e_{i}}^{\varepsilon} e_{j})]. \]

\[ D_{\alpha, \varepsilon} \frac{z}{2 \sqrt{t}} c^{\alpha} + \frac{z}{2 \sqrt{t}} c^{\alpha} D_{\alpha, \varepsilon} - \frac{z}{2 \sqrt{t}} c^{\alpha} (\sqrt{\varepsilon} \nabla_{e_{\alpha}}^{\otimes E, \mu} e_{\alpha}) \]

\[ = \frac{z}{2 \sqrt{t}} [c_{i} (\nabla_{e_{i}}^{\otimes E, \mu} e_{\alpha}) + 2c^{i}_{\alpha} \nabla_{e_{\alpha}}^{\otimes E, \mu} + \frac{1}{2} < S(e_{\alpha}) e_{i}, e_{\beta} > \sqrt{\varepsilon} 2c^{\alpha} c^{\beta} c^{\alpha} \]

\[ - c_{i} (\sqrt{\varepsilon} \nabla_{e_{i}}^{\otimes E, \mu} e_{\alpha})]. \]

Hence by (27) – (29), we obtain

\[ t (D_{j, \varepsilon} \frac{z}{2 \sqrt{t}} c^{i} + \frac{z}{2 \sqrt{t}} c^{i} D_{j, \varepsilon} - \frac{z}{2 \sqrt{t}} c^{i} (\nabla_{e_{i}}^{\varepsilon} e_{j}) + D_{\alpha, \varepsilon} \frac{z}{2 \sqrt{t}} c^{\alpha} + \frac{z}{2 \sqrt{t}} c^{\alpha} D_{\alpha, \varepsilon} \]

\[ - \frac{z}{2 \sqrt{t}} c_{i} (\sqrt{\varepsilon} \nabla_{e_{i}}^{\otimes E, \mu} e_{\alpha})] = -2 \sqrt{t} D_{\varepsilon} + \sqrt{t} (\frac{1}{2} c^{i} \psi_{j} + \frac{\varepsilon}{2} c^{\alpha} \psi_{\alpha}). \]

Therefore, we have proved by (26) and (30) the following Lichnerowicz type formula, which is useful in many situations.

**Theorem 3.**

\[ 4t c^{2}_{\varepsilon} - 2 \varepsilon \sqrt{t} D_{\varepsilon} = t (- (\nabla_{e_{i}}^{\otimes E, \mu} + \frac{1}{2} < S(e_{i}) e_{j}, e_{\beta} > c^{j} \sqrt{\varepsilon} c^{\alpha} + \frac{1}{4} < S(e_{i}) e_{\alpha}, e_{\beta} > \varepsilon c^{\alpha} c^{\beta} \]

\[ - \frac{z}{2 \sqrt{t}} c^{i} + \sqrt{\varepsilon} \epsilon c^{i} R_{\varepsilon}(e_{i}, e_{j}) + \sqrt{\varepsilon} c^{i} c^{\alpha} R_{\varepsilon}(e_{i}, e_{\alpha}) + \frac{1}{2} \varepsilon c^{\alpha} c^{\beta} R_{\varepsilon}(e_{\alpha}, e_{\beta}) \]

\[ - \frac{1}{2} (c^{i} c_{i} (\nabla_{e_{i}}^{\otimes E, \mu} e_{\alpha}) \psi_{\alpha} + \sqrt{\varepsilon} c^{i} c_{i} (\nabla_{e_{i}}^{\otimes E, \mu} e_{\alpha}) \psi_{\alpha} + c^{j} (\nabla_{e_{j}}^{\otimes E, \mu} e_{\alpha}) \psi_{\alpha}) \]

\[ + \sqrt{\varepsilon} c^{i} c_{i} (\nabla_{e_{i}}^{\otimes E, \mu} e_{\alpha}) \psi_{\alpha} + \sqrt{\varepsilon} c^{i} c_{i} (\nabla_{e_{i}}^{\otimes E, \mu} e_{\alpha}) \psi_{\alpha} + \varepsilon c^{\alpha} c_{i} (\nabla_{e_{i}}^{\otimes E, \mu} e_{\alpha}) \psi_{\alpha} \psi_{\beta} \psi_{\alpha} \psi_{\beta} \]

\[ + \frac{z}{2 \sqrt{t}} (c^{i} \psi_{j} + \sqrt{\varepsilon} c^{\alpha} \psi_{\alpha}). \]

(31)
We now apply the Getzler rescaling \( G_{(\varepsilon t)^{1/2}} \) to \( 4tC^2_{\varepsilon} \). Namely, we make the following changes: \( \nabla e_{\alpha} \rightarrow (\varepsilon t)^{-1/2} \nabla e_{\alpha}, c^\alpha \rightarrow ((\varepsilon t)^{-1/2} e_{\alpha} \wedge) - (\varepsilon t)^{1/2} e_{\alpha} \) and \( c^\alpha_t \rightarrow c^\alpha_t \). Then by (26),

\[
\begin{align*}
&\lim_{\varepsilon \to 0} G_{(\varepsilon t)^{1/2}} (4tC^2_{\varepsilon}) = t \left( - (\nabla Z_{\alpha} e_{\alpha} + \frac{1}{2} < S(e_i) e_j, e_\alpha > c^\alpha \frac{e_\alpha}{\sqrt{t}} + \frac{1}{4t} < S(e_i) e_\alpha, e_\beta > e_\alpha e_\beta \right)^2 \\
&- \frac{1}{t} (\delta - \frac{1}{4} R^B_{\alpha \beta} x_{\beta})^2 + \frac{K}{4} + \frac{1}{c^\alpha \sqrt{t}} R(e_i, e_j) + \frac{c^\alpha m^\alpha}{\sqrt{t}} R(e_i, e_\alpha) + \frac{1}{2t} m^\alpha m^\beta R_0 (e_\alpha, e_\beta) \\
&- \frac{1}{2} [c^\alpha c^\alpha \left( \nabla_{e_\alpha} e_i \right) \psi_i + \frac{m^\alpha}{\sqrt{t}} c^\alpha \left( \nabla_{e_\alpha} \eta e_i \right) \psi_i + \frac{m^\alpha}{\sqrt{t}} c^\alpha \left( \nabla_{e_\alpha} \eta e_i \psi_i \right) + \frac{1}{4} c^\alpha \psi_i \psi_i].
\end{align*}
\]

(32)

Here we used the facts that with appropriate local orthonormal bases of \( TM \) and \( T^* M \), \( \nabla_{e_\alpha} e_i = \varepsilon (\nabla_{e_\alpha} e_i)^H \) (horizontal component) and \( \nabla_{e_\alpha} e_\alpha = (\nabla_{e_\alpha} e_\alpha)^H + \varepsilon (\nabla_{e_\alpha} e_\alpha)^H \), \((-)^H \) is the vertical component. Note that \( \nabla_{e_\alpha} e_i = \lim_{\varepsilon \to 0} \nabla_{e_\alpha} e_i \nabla Z_{\alpha} e_i = \nabla M_{\alpha} e_i \), \( \nabla_{e_\alpha} e_\alpha = \nabla M_{\alpha} e_\alpha \).\( p^H : T = T^H \) \( TM \to T^* M \) is the projection. \( R_0 (e_\alpha, e_\beta) = R(e_\alpha, e_\beta) + \hat{R}^B (e_\alpha, e_\beta). \)

Let \( C_t = \frac{1}{2} (r^N/2 d^N/2 + t^N/2 d^M t^N/2) \). Here \( d^M \) is the fiberwise adjoint of \( d_M \). Theorem 3.11 \([BL]\) together with (32) proves

\[
\begin{align*}
\lim_{\varepsilon \to 0} G_{(\varepsilon t)^{1/2}} (4tC^2_{\varepsilon}) &= C^2_{A_i} - \frac{\sqrt{t} m^\alpha}{2} c^\alpha \left( \nabla_{e_\alpha} e_i \right) \\
&- (\delta - \frac{1}{4} R^B_{\alpha \beta} x_{\beta})^2 + \frac{1}{2} m^\alpha m^\beta R^B (e_\alpha, e_\beta) \\
&= C^2_{A_i} - \sqrt{t} C' - (\delta - \frac{1}{4} R^B_{\alpha \beta} x_{\beta})^2 + \frac{1}{2} m^\alpha m^\beta R^B (e_\alpha, e_\beta) \\
&\equiv \text{def} B_{A_i},
\end{align*}
\]

(33)

where

\[
C' = \frac{m^\alpha}{2} c^\alpha \left( \nabla_{e_\alpha} e_i \right).
\]

(34)

\( C' \) comes from the term \( m^\alpha c^\alpha \left( \nabla_{e_\alpha} e\right) \psi_i \) in (32). Note that if the metric \( g_E \) on \( E \) is either locally independent of \( B \) or covariantly-constant with respect to \( \nabla_E \), then \( C' = 0 \). In particular, if \( \nabla_E \) is fiberwise unitary, then \( C' = 0 \). This is the case where the adiabatic limit of \( \eta \)-invariants is discussed \([BC]\).

We now come to one of our main results. Let \( M_{\varepsilon} \) be the manifold \( M \) with the metric \( g^M_{\varepsilon} \).

**Theorem 4.** Let \( \dim(B) \) be even, \( C' = 0 \) and \( H^*(Z, E) = 0 \). If \( Z \) is even dimensional, assume further \( \text{ind}_u(D_M) = 0 \). Then

\[
\begin{align*}
&\lim_{\varepsilon \to 0} \log \mathcal{T}_r(M_{\varepsilon}, E) = 2^{\dim(B)} \frac{1}{2} \int_B \varphi L(B) \log \left( - \frac{R^B}{2} \coth \frac{R^B}{2} \right) \mathcal{T}_r(T^H M, g^Z, g_E) \\
&+ 2^{\dim(B)} \frac{1}{2} \int_B \varphi L(B) \mathcal{T}(T^H M, g^Z, g_E) + \int_B \varphi \hat{A}(B) \text{tr}_a (\tilde{R}^B e^{-h^B} \tilde{F}) \mathcal{T}_r(T^H M, g^Z, g_E),
\end{align*}
\]

(35)

where \( L(B) = \text{det}^{\frac{1}{2}} \left( \frac{R^B}{\text{unh} \frac{R^B}{2}} \right) \) is the stable Hirzebruch \( L \)-class of \( B \).
Proof. Since \( H^*(Z, E|Z) = 0 \), the operator \( d^*_Z + d_Z \) is invertible. We can apply Proposition 4.41 [BC] to conclude that \( D_M \) is invertible for \( \varepsilon \) very close to zero. Thus, \( P_{KerD_M} = 0 \). Hence \( Tr_M^*(\frac{\tilde{N}}{2} P_{KerD_M}) = 0 \). By Proposition 6, this implies

\[
\lim_{\varepsilon \to 0} \log T(M, E) = \lim_{\varepsilon \to 0} \int_0^{\infty} Tr_M^*(\frac{\tilde{N} f'(i\sqrt{t}D_M)}{2t}) \, dt.
\] (36)

Recall that \( N' \) is the number operator of \( B \). \( N' = \frac{1}{2} \sum_{i=0}^{\dim(B)} c(e_{i+})c(e_{i+}) + \frac{\dim(B)}{2} \). Then

\[
\tilde{N} = (N + \frac{\dim(B)}{2}) + \frac{1}{2} \sum_{i=0}^{\dim(B)} c(e_{i+})c(e_{i+}).
\]

We thus break up (36) into two terms which correspond to the two terms of \( \tilde{N} \). Let us first consider the term associated to \( (N + \frac{\dim(B)}{2}) \). Let \( g(x) = (1 + 2x)e^x \). By the fact that \( V_{M,\varepsilon}^2 = -C_{\varepsilon}^2 \) and (33), we have

\[
\lim_{\varepsilon \to 0} Tr_M^*(\frac{2N + \dim(B)}{4} g(4tV_{M,\varepsilon}^2)) = \lim_{\varepsilon \to 0} Tr_M^*(\frac{2N + \dim(B)}{4} g(G_{\varepsilon}\frac{1}{2} (-tC_{\varepsilon}^2)))
\]

\[
= Tr_M^*(\frac{2N + \dim(B)}{4} g(-B_{4t}^2))
\] (37)

uniformly for \( t \in [\delta, T] \) and arbitrary \( \delta, T > 0 \). (37) also holds uniformly for \( t \in [0, \delta] \).

Indeed by (33), \( \lim_{\varepsilon \to 0} G_{\varepsilon}\frac{1}{2} (4C_{\varepsilon}^2) = B_4 \). The asymptotic expansion of \( Tr_M^*(\frac{2N + \dim(B)}{4} g(-tG_{\varepsilon}\frac{1}{2} (4C_{\varepsilon}^2))) \) at the origin depends on the local symbol of the operator \( G_{\varepsilon}\frac{1}{2} (4C_{\varepsilon}^2) \), and its coefficients approximate to those in the expansion of \( Tr_M^*(\frac{2N + \dim(B)}{4} g(-B_{4t})) \) with remainder bounded by a number independent of \( \varepsilon \). Hence we obtain

\[
\int_0^T Tr_M^*(\frac{2N + \dim(B)}{4} g(4tV_{M,\varepsilon}^2)) \, dt \to \int_0^T Tr_M^*(\frac{2N + \dim(B)}{4} g(-B_{4t})) \, dt, \quad \varepsilon \to 0.
\]

To estimate the large time contribution of \( Tr_M^*(\frac{2N + \dim(B)}{4} g(4tV_{M,\varepsilon}^2)) \), we need the assumption that \( H^*(Z, E|Z) = 0 \). Thus there exists a \( \lambda_0 > 0 \) such that the eigenvalues of \( -V_{M,\varepsilon}^2 \) are greater than \( \lambda_0 \) (cf. [BC], Prop. 4.41).

\[
Tr_M^*(\frac{2N + \dim(B)}{2} e^{4tV_{M,\varepsilon}^2} ) \leq const. e^{-(4t-1)\lambda_0} Tr_M^*(e^{V_{M,\varepsilon}^2}) \leq const. e^{-(4t-1)\lambda_0} \varepsilon^{-\frac{\dim(B)}{2}}, \quad t \to \infty.
\] (38)

Here we used the fact that the metrics \( g_M^\varepsilon \) have uniformly bounded geometry with volume growth as \( \varepsilon^{-\frac{\dim(B)}{2}} \). Therefore,

\[
\left| \int_0^\infty Tr_M^*(\frac{2N + \dim(B)}{4} e^{4tV_{M,\varepsilon}^2}) \, dt \right| \leq const. \varepsilon^{-\frac{\dim(B)}{2}} \frac{1}{4T\lambda_0} e^{-(4T-1)\lambda_0}.
\]
By choosing $T$ such that $(4T - 1)\lambda_0 = \frac{\dim(b)}{2} |\ln \varepsilon|$, we get
\[ \int_T^\infty T_r^s(M) \left( \frac{2N + \dim(B)}{2} \right) e^{4TV_{M,\varepsilon}} dt \to 0, \quad \varepsilon \to 0. \tag{39} \]
Similarly, one can prove that as $\varepsilon \to 0$,
\[ \int_T^\infty T_r^s(M) \left( \frac{2N + \dim(B)}{2} \right) 2(4tV_{M,\varepsilon}^2) e^{4TV_{M,\varepsilon}} dt \to 0. \]
Note that $\int_T^\infty T_r^s(M) \left( \frac{2N + \dim(B)}{4} \right) g(-B_{4t}) dt \to 0$ as $T \to \infty$ (see the proof of Lemma 3.2 [GR]). Hence, we obtain
\[ \lim_{\varepsilon \to 0} \int_0^\infty T_r^s(M) \left( \frac{2N + \dim(B)}{4} \right) g(4tV_{M,\varepsilon}^2) dt = \int_0^\infty T_r^s(M) \left( \frac{2N + \dim(B)}{4} \right) g(-B_{4t}^2) dt. \tag{40} \]
We now use local index theory techniques (cf. [BGV], Chapter 4) and get
\[ \lim_{\varepsilon \to 0} \int_0^\infty T_r^s(M) \left( \frac{2N + \dim(B)}{4} \right) g(4tV_{M,\varepsilon}^2) dt = \int_0^\infty T_r^s(M) \left( \frac{2N + \dim(B)}{4} \right) g(-B_{4t}^2) dt. \]
and
\[ \lim_{\varepsilon \to 0} \int_0^\infty T_r^s(M) \left( \frac{2N + \dim(B)}{4} \right) g(4tV_{M,\varepsilon}^2) dt = \left\{ \det \left( \frac{(1 + 2z)^{\frac{R_0^b}{2}}}{\sinh(1 + 2z)^{\frac{R_0^b}{2}}} (2\pi i)^{\frac{\dim(b)}{2}} \right) \cdot \frac{(1 + 2z)^{\frac{R_0^b}{2}}}{\sinh(1 + 2z)^{\frac{R_0^b}{2}}} \right\} \cdot tr^s(e^{-(1+2z)\hat{R}_b^1}) \cdot \left( e^{-(1+2z)\hat{C}_{b+1}^1} + \sqrt{c'} \right). \]

On the other hand, by Theorem 3.15 [BL] and the assumption $H^s(Z, E|Z) = 0$, we have that $\varphi T^s((1 - 2C_{\text{b+1}}^1) e^{-C_{b+1}^1})$ is also exact. Hence, by (40) and $C' = 0$, we get
\[ \lim_{\varepsilon \to 0} \int_0^\infty T_r^s(M) \left( \frac{2N + \dim(B)}{4} \right) g(4tV_{M,\varepsilon}^2) dt = \int_B (2\pi i)^{\frac{\dim(b)}{2}} \left\{ \det \left( \frac{(1 + 2z)^{\frac{R_0^b}{2}}}{\sinh(1 + 2z)^{\frac{R_0^b}{2}}} (2\pi i)^{\frac{\dim(b)}{2}} \right) \cdot \frac{(1 + 2z)^{\frac{R_0^b}{2}}}{\sinh(1 + 2z)^{\frac{R_0^b}{2}}} \right\} \cdot \left( e^{-(1+2z)\hat{R}_b^1} \right). \tag{41} \]
Now, since $z^2 = 0, (1 + 2z)^{\frac{\dim(b)}{2}} = 1 - \frac{\dim(b)}{2} 2z$. Let $h(x) = \frac{1}{2} \log \frac{1}{\sinh \frac{1}{2} x}$. Then $h'(x) = \frac{1}{2} \log \frac{1}{\sinh \frac{1}{2} x} = 1 - \frac{1}{\sinh \frac{1}{2} x}$. Therefore,
\[ \frac{1}{2} \log \frac{(1 + 2z)^{\frac{R_0^b}{2}}}{\sinh(1 + 2z)^{\frac{R_0^b}{2}}} = \exp tr^s(h((1 + 2z)\hat{R}_b^1)) = \exp tr^s(h(\hat{R}_b^1) + z(1 - \frac{R_0^b}{\tanh \frac{R_0^b}{2}})) = \exp tr^s(h(\hat{R}_b^1) + z(1 - \frac{R_0^b}{2} \coth \frac{R_0^b}{2})). \]
Since $\hat{A}(B)tr_*(e^{-\hat{h}}) = 2^{\dim(B)/2} \mathcal{L}(B)$, a computation proves that the second term in the right hand side of (41) is equal to

$$
\left\{ \int_B \varphi \mathcal{L}(B)(-\dim(B)) - 1 + tr((1 - \frac{R^B}{2} \coth \frac{R^B}{2})) T_I(T^H M, g^Z, g^E) \\
+ \int_B \varphi \mathcal{L}(B) \int_0^{\infty} \left( \frac{1}{2} \varphi Tr^z(N(1 + 2(-C^2_{\Delta t}))e^{-C_{\Delta t}^z}) \frac{dt}{t} \right) 2^{\frac{\dim(B)}{2}} \\
- 2 \int_B \varphi \hat{A}(B)tr_*(\hat{R}^B e^{-\hat{h}}) T_I(T^H M, g^Z, g^E).
$$

From this we get that the left hand side of (41) is equal to the right hand side of (35). Therefore, by (36) it remains to check

$$
\lim_{\varepsilon \to 0} \int_0^{\infty} tr_{B}(\frac{1}{2} \sum_{i=0}^{\dim(B)} c(e_{i})c(e_{i})g(4tV_{d,\varepsilon}^2)) \frac{dt}{t} = 0.
$$

But this clearly follows from the proofs of Theorem 1 (iii) and of (41). \hfill \Box

By the above proof, we get the following corollary.

**Corollary 1.** With the condition of Theorem 4,

$$
\lim_{\varepsilon \to 0} \log T_w(M_{\varepsilon}, E) = 2^{\frac{\dim(B)}{2}} \int_B \varphi \mathcal{L}(B) T_I(T^H M, g^Z, g^E).
$$

**Proof.** By assumption, we have that for $\tau > 0$ very small and $\varepsilon \to 0$,

$$
\log T_w(M_{\varepsilon}, E) = \frac{d}{dr}(\frac{1}{\Gamma(r)} \int_0^{\tau} t^r Tr^z_M(\tilde{N} \frac{2}{2} e^{-iD_{\Delta t}^z}) \frac{dt}{t})_{r=0} + \int_{\tau}^{\infty} Tr^z_M(\tilde{N} \frac{2}{2} e^{-iD_{\Delta t}^z}) \frac{dt}{t}.
$$

The result follows from the proof of Theorem 4. \hfill \Box

**Remark 1.** (i) Formula (35) is much more complicated than the formula in Corollary 1. This is due to the fact that compared to $T_I(T^H M, g^Z, g^E)$, the analytic torsion form has an extra term. Up to a constant, the second term of (35) is the invariant $\mathcal{T}(B, M, E)$ in Section 2.

(ii) The argument above also works for the adiabatic limit of the analytic torsion invariant of the total space. But for this case one can only see the contribution of the degree zero component of the analytic torsion form, since the Euler form $e(TB, \nabla^B)$ is in the top dimension $\dim(B)$. The formula under the assumption of Theorem 4 is

$$
\lim_{\varepsilon \to 0} \log T(M_{\varepsilon}, E) = (2\pi i)^{\frac{\dim(B)}{2}} \int_B \varphi e(TB, \nabla^B) T_I(T^H M, g^Z, g^E) \\
+ (2\pi i)^{\frac{\dim(B)}{2}} \int_B \varphi e(TB, \nabla^B) tr(\frac{R^B}{2} \coth \frac{R^B}{2}) T_I(T^H M, g^Z, g^E) \\
+ \int_B \varphi \hat{A}(B)tr_*(-2\hat{R}^B e^{-\hat{h}}) T_I(T^H M, g^Z, g^E). \tag{42}
$$
Indeed, to apply the proof of Theorem 4 to the situation here, we need only to change the trace $\text{Tr}_M^a$ to $\text{Tr}^a$ and $\text{ind}_w(D_M)$ to $\text{ind}(D_M)$ and note that

$$\hat{A}(B)\text{tr}_r^a(e^{-R^0}) = (2\pi i)^{-\frac{d+1}{2}} \text{Pf}(-\frac{R^0}{2\pi}).$$

Similar formula to that in Corollary 1 also holds.

(iii) For $C' \neq 0$, one needs other forms to get an adiabatic limit formula. The case where $H^*(\mathbb{Z}, E|_{\mathbb{Z}}) = 0$ is much more involved, and is considered by Dai-Melrose [DM] and Forman [Fo] for the analytic torsion invariant. Theorem 4 can be extended to the $L^2$-case.
REFERENCES