Note di Matematica Vol. 16 - n. 2, 267-285 (1996)

ON A CLASS OF ALGEBRAS

LORENA GIACOBAZZI

Abstract. *We introduce the notion of projective gruppal algebra that connects to every subgroup of order n of PGL (n, K) a n-dimensional algebra over K.*

In particular we classify the projective gruppal algebras 4-dimensional over a perfect field K studying the conjugacy classes of the groups $Z_2 \times Z_2$ and Z_4 in PGL $(4, K)$.

1. INTRODUCTION

Denote by $M_n(K)$ the algebra of the $n \times n$ matrices over K and by $(E_{ij}|\ i,j = 0, 1, \ldots, n-1)$ its canonical basis.

$A = (K^n, f)$ is the algebra n-dimensional over K defined by a bilinear map (multiplication)

$$f : K^n \times K^n \to K^n, f(x,y) = xy.$$

We will write the elements $x \in K^n$ as colum matrices $(x_0\ x_1 \ldots x_{n-1})^T$.

In K^n we fix the canonical basis $\mathbf{B} = (e_i | i = 0, 1, \ldots, n-1)$.

The matrices, $\mathbf{L}(x)$ and $\mathbf{R}(y)$, of the endomorphisms of K^n

$$L_x : A \to A, L_x(y) = xy$$

and

$$R_x : A \to A, R_x(y) = yx$$

are called *left* and *right multiplication matrices* respectively (briefly l.m.m. and r.m.m.).

Obviously

$$xy = \mathbf{L}(x)y = \mathbf{R}(y)x, \quad \forall x, y \in A. \tag{1}$$

Then the multiplicative structures of A is determinated giving $\mathbf{L}(x)$ or $\mathbf{R}(x)$. We write $A = (K^n, \mathbf{L}(x))$ or $A = (K^n, \mathbf{R}(y))$.

We say that $A' = (K^n, \mathbf{L}'(x)) = (K^n, \mathbf{R}'(y))$ is d-*isotopic to A* if there exsists an isotophism $d = (D_1, D_2, D_3) \in GL^3\ (n, K)$ so that

$$\mathbf{L}'(x) = D_1^{-1}\mathbf{L}(D_2(x))D_3 \tag{2}$$

or equivalently

$$\mathbf{R}'(y) = D_1^{-1}\mathbf{R}(D_3(y))D_2. \tag{3}$$

Let D be a subgroup of $GL(n, K)^3$. We say that A' is *D-isotopic to A* if there exists $d \in D$ so that (2) or (3) is satisfied.

Every subgroup D defines in a natural way an equivalence relation in a given set $\mathbf{A}\ (n, K)$ of n-dimensional K-algebras. The corrispondent partition is said *D-classification of* $\mathbf{A}\ (n, K)$.

$x = (x_0\ x_1 \dots x_{n-1})^T \in A$ is a left zero divisor (respectively a right zero divisor) if and only if $P(x_0, x_1, \dots, x_{n-1})$ belongs to the hypersurface Φ : det $(\mathbf{L}(x)) = 0$ (respectively Ψ : det $(\mathbf{R}(y)) = 0$) in the projective space $P_{n-1}(K)$, $(n-1)$-dimensional over K. From now on we say that P, instead of x, is a left zero divisor (respectively a right zero divisor).

Remark. *An algebra A' d-isotopic to A, possesses zero divisors if and only if A has some.*

Remark. *If A' is isotopic to A, then the surfaces Φ' : det $(\mathbf{L}'(x)) = 0$ and Ψ' : det $(\mathbf{R}'(y)) = 0$ are projectively equivalent to Φ and Ψ respectively* (cf. also [1]).

Let be

$$\mathbf{L}(x) = (L_0x, L_1x, \dots, L_{n-1}x) \tag{4}$$

and

$$\mathbf{R}(y) = (R_0y, R_1y, \dots, R_{n-1}y) \tag{5}$$

where L_ix and R_iy are the i-th columns. If L_i, $R_i \in GL(n, K)$ then, as usual, we identify the linear automorphisms

$$\lambda_i : P_{n-1}(K) \to P_{n-1}(K), kx' = L_ix, i = 0, 1, \dots, n-1, k \in K^* = K - 0$$

and

$$\rho_i : P_{n-1}(K) \to P_{n-1}(K), ky' = R_iy, i = 0, 1, \dots, n-1, k \in K^*,$$

with the images $[L_i], [R_i]$ of L_i, R_i in the canonical map

$$GL(n, K) \to PGL(n, K).$$

Definition 1.1. *Let $A = (K^n, \mathbf{L}(x)) = (K^n, \mathbf{R}(y))$ be a n-dimensional K-algebra whose l.m.m. and r.m.m. are given by (4) and (5) respectively. We say that A is a left projective gruppal algebra (l.p.g.a.) if $T(A) = \{[L_i] \mid i = 0, 1, \dots, n-1\}$ is a subgroup of $PGL(n, K)$.*

Analogously A is a right projective gruppal algebra (r.p.g.a.) if $T'(A) = \{[R_i] \mid i = 0, 1, \dots, n-1\}$ is a subgroup of $PGL(n, K)$.

A projective gruppal algebra (p.g.a) is a l.p.g.a. and a r.p.g.a.

Let $P \subseteq GL(n, K)$ be the subgroup of the matrices having only one element different from zero in every row and in every column. We put

$$G = \{(A, A, D) \in GL(n, K)^3 \mid D \in P\} \tag{6}$$

The map $A \mapsto T(A)$ from the set of n-dimensional l.p.g.a. A to the set of the subgroups $T(A) \subseteq PGL(n, K)$, card $(T(A)) = n$, is surjective. Moreover $T(A) = T(A')$ if and only if $\exists D \in P$ so that A' is (I_n, I_n, D)-isotopic to A.

We can easily prove the following

Proposition 1.2. *Let A, A' be l.p.g.a. n-dimensional over K. A' is G-isotopic to A if and only if $T(A')$ is conjugate to $T(A)$ in PGL (n, K).*

An analogous proposition can be enunciate for r.p.g.a. In this case, instead of G, we fix the group

$$G' = \{(A, D, A) \in GL(n, K)^3 | D \in P\} \tag{7}$$

isomorphic to G.

In Section 2 we give some general propositions concerning the projective gruppal algebras. Sections 3 and 4 are devoted to the G-classification of the l.p.g.a. 4-dimensional over a perfect field K.

2. PROJECTIVE GRUPPAL ALGEBRAS

Let A be a l.p.g.a. whose l.m.m. is given by (4).

The hypothesis that $[L_i]$ are elements belonging to the group $T(A)$ can be expressed substituting the set $\{0, 1, \dots, n-1\}$ for an additive group N isomorphic to $T(A)$ and assuming that $N \to T(A)$, $c \mapsto [L_c]$ is an isomorphism.

Consequently we set

$$\mathbf{L}(x) = (L_0x, L_ax, \dots, L_gx), 0, a, \dots, g \in N, \tag{8}$$

$B = (e_g | g \in N)$, etc.

Comparing

$$L_ax = \mathbf{L}(x)e_a, \forall a \in N, \tag{9}$$

with (1) we have

$$L_ax = xe_a, \forall a \in N.$$

In particular

$$L_ae_b = e_be_a, \forall a, b \in N. \tag{10}$$

and

$$x = xe_0, \forall x \in A.$$

Remark. *e_0 is the unity of A if and only if*

$$e_0e_a = e_a, \forall a \in N,$$

hence, from (10), if and only if

$$L_ae_0 = e_a, \forall a \in N. \tag{11}$$

Now if $[L_a]$, $[L_b] \in T(A)$, then

$$L_aL_b = k_{a,b}L_{a+b}, k_{a,b} \in K^*, \forall a, b \in N. \tag{12}$$

Furthermore if we suppose that e_0 is the unity of A, then multipling by e_0 both the sides of (12) and comparing with (10) and (11) we obtain

$$e_be_a = k_{a,b}e_{a+b}, \forall a, b \in N. \tag{13}$$

Proposition 2.1. *Let A be a l.p.g.a. with unity e_0. Then*
a) A is associative;
b) A is commutative if and only if

$$k_{a,b}e_{a+b} = k_{b,a}e_{b+a}, \forall a, b \in N.$$

Proof. a) We have

$$L_c(L_bL_a) = (L_cL_b)L_a, \forall a, b, c \in N,$$

and appling (12),

$$k_{b,a}k_{c,b+a} = k_{c,b}k_{c+b,a}, \forall a, b, c \in N.$$

From (13) we obtain

$$(e_ae_b)e_c = (k_{b,a}e_{b+a})e_c = k_{b,a}k_{c,b+a}e_{a+b+c}$$

and

$$e_a(e_be_c) = e_a(k_{c,b}e_{c+b}) = k_{c,b}k_{c+b,a}e_{c+b+a}.$$

Hence

$$(e_ae_b)e_c = e_a(e_be_c), \forall a, b, c \in N.$$

b) follows from (13) and from $e_ae_b = e_be_a$, $\forall a, b \in N$.

Proposition 2.2. *Let A be a l.p.g.a. with l.m.m. (8). If e_0 is the unity of A and if*

$$k_{a,b} = 1, \forall a, b \in N \qquad (cf. (12)),$$

then A is anti-isomorphic to the group algebra of N over K.

Proof. If $e_a, e_b \in B$, then from (13) we deduce $e_be_a = e_{a+b} \in B$, $\forall a, b \in N$. Therefore $N \to B$, $a \mapsto e_a$ is an anti-isomorphism.

Proposition 2.3. *Let A be a l.p.g.a. whose l.m.m. is given by (8). Then the group $\{\lambda_a | \, a \in N\}$ fixes the hypersurface*

$$\phi : \det(\mathbf{L}(x)) = 0.$$

Proof. $\det(\mathbf{L}(L_sx)) = \pm\det(\mathbf{L}(x))$, $\forall s \in N$. Then $P \in \phi$ implies $\lambda_s(P) \in \phi$.

It is easy to verify that r.p.g.a. satisfy the anologous propositions to 2.1, 2.2 and 2.3.

Proposition 2.4. *Let A be a l.p.g.a. with unity e_0 and suppose that (8) and*

$$\mathbf{R}(y) = (R_0y, R_ay, \dots, R_gy), 0, a, \dots, g \in N,$$

are its l.m.m. and r.m.m. respectively. Then
a) A is a r.p.g.a.;
b) group $T = \{[L_a] | a \in N\}$ and $T' = \{[R_a] | a \in N\}$ are anti-isomorphic.

Proof. a) From hypothesis and from Proposition 2.1. it follows that $(xy)z = x(yz), \forall x, y, z \in A$. Consequently

$$\mathbf{L}(xy)z = \mathbf{L}(x)\mathbf{L}(y)z$$

or

$$\mathbf{L}(\mathbf{L}(x)y) = \mathbf{L}(x)\mathbf{L}(y), \forall x, y \in A.$$

In particular

$$\mathbf{L}((\mathbf{L}(e_r)e_s) = \mathbf{L}(e_r)\mathbf{L}(e_s), \forall r, s \in N.$$

By virtue of (9) and (10)

$$\mathbf{L}(e_r)e_s = L_s e_r = e_r e_s, \forall r, s \in N.$$

Substituting and comparing with (1) we obtain

$$\mathbf{L}(e_r e_s) = R_r R_s, \forall r, s \in N.$$

From (1) and (13)

$$\mathbf{L}(e_r e_s) = k_{s,r}\mathbf{L}(e_{s+r}) = k_{s,r}R_{s+r}.$$

Hence

$$R_r R_s = k_{s,r}R_{s+r}, \forall r, s \in N,$$

or

$$[R_r][R_s] = [R_{s+r}], \forall r, s \in N.$$

b) From this it follows that

$$h : N \to T', s \mapsto [R_s]$$

is an anti-isomorphism. Consequently if

$$\delta : N \to T, c \mapsto [L_c],$$

then

$$\delta^{-1} oh : T \to T'$$

is also an anti-isomorphism.

3. SUBGROUPS OF ORDER 4 IN $PGL(4, K)$.

In this and next Sections we suppose K a perfect field and $n = 4$.

Let $F = K^* / K^{*2}$ be the quotient of the multiplicative group K^* of K, char $(K) \neq 2$, over the subgroup K^{*2} of the squares.

Putting

$$[k_1][k_2] = [-k_1 k_2], \forall [k_1][k_2] \in F$$

we define an abelian group F' isomorphic to F. In particular if -1 is a square, i.e. if $i \in K^*$ and $i^2 = -1$, then $F' = F$.

$G(k_1, k_2) \subseteq F$ and $G'(k_1, k_2) \subseteq F'$ denote the subgroups generated by $[k_1]$ and $[k_2]$.

Let H be a ring with unit element, u, char $(H) = 2$, and let H^* be the subgroup of the invertible elements. Denote by $\Delta(h) \subseteq H$ the subset that contains a given $h \in H$ and satisfying the following conditions:

(a) $\Delta(h)$ is invariant under the action of the maps $s : H \to H$, $s(x) = x + u$ and $q : H \to H$, $q(x) = \begin{cases} x^{-1}, x \in H^* \\ x, x \in H - H^* \end{cases}$;
(b) $\Delta(h)$ is minimal respect to the condition (a).

$Q(A, B) \subseteq PGL\ (4, K)$ denote the subgroup generated by $[A]$, $[B]$, and isomorphic to quadrinomial group. $C(R) \subseteq PGL\ (4, K)$ denote the cyclic subgroup of order 4 generated by $[R]$.

The G-classification of l.p.g.a. (or r.p.g.a.) 4-dimensional over K is subordinated to the determination of the conjugacy classes of the subgroups of order 4 in $PGL\ (4, K)$ (cf. Proposition 1.1.). For this reason we prove the following

Proposition 3.1. *Every subgroup of $PGL(4, K)$, char $(K) \neq 2$, isomorphic to the quadrinomial group is conjugated to a subgroup $Q(A, B)$ where A, B coincide with one of the following couples:*

$$A = diag(1, 1, 1, -1), B = diag(1, 1, -1, -1), \tag{14}$$

$$A = A_1(k_1) = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}, K = \begin{pmatrix} 0 & k_1 \\ 1 & 0 \end{pmatrix}, B = B(k_2) = \begin{pmatrix} 0 & k_2 I_2 \\ I_2 & 0 \end{pmatrix}, k_i \in K^*, i = 1, 2, \tag{15}$$

$$A = A_2(k_1) = \begin{pmatrix} K & 0 \\ 0 & -K \end{pmatrix}, B = B(k_2). \tag{16}$$

Couples A, B belonging to distinct classes (14), (15) and (16) define not conjugated subgroups.

$Q(A_1(k_1), B(k_2))$ and $Q(A_1(k_1'), B(k_2'))$ are conjugated if and only if $G(k_1, k_2) = G\ (k_1', k_2')$.

$Q(A_2(k_1), B(k_2))$ and $Q(A_2(k_1'), B(k_2'))$ are conjugated if and only if $G'\ (k_1, k_2) = G'\ (k_1', k_2')$.

Proposition 3.2. *Every subgroup of $PGL(4, K)$, char $(K) = 2$, isomorphic to the quadrinomial group is conjugated to a subgroup $Q(A, B)$ where A, B coincide with one of the following couples:*

$$A = \begin{pmatrix} K_1 & 0 \\ 0 & K_1 \end{pmatrix}, K_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \tag{17}$$

$$A = W(R_1) = \begin{pmatrix} I_2 & R_1 \\ 0 & I_2 \end{pmatrix}, B = W(R_2) = \begin{pmatrix} I_2 & R_2 \\ 0 & I_2 \end{pmatrix}, \tag{18}$$

with

$$R_2 = S_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, R_1 = \begin{cases} S' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\ S'' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\ S_k = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}, k \in K^* - \{1\}, \end{cases} \tag{19}$$

or

$$R_2 = I_2, R_1 = R \in M_2(K) - \{0, I_2\}. \tag{20}$$

$Q(W(S_k), W(S_1))$ and $Q(W(S_{k'}), W(S_1))$ are conjugated if and only if $\Delta(k) = \Delta(k')$.

$Q(W(R), W(I_2))$ and $Q(W(R'), W(I_2))$ are conjugated if and only if $\exists\ Y \in GL(2, K)$ such that $\Delta(Y^{-1}RY) = \Delta(R')$.

The other above-mentioned couples of groups are not conjugated.

Proposition 3.3. *Every subgroup of* $PGL(4, K)$, *char* $(K) \neq 2$, *cyclic of order 4 is conjugated to the subgroup* $C(R)$ *generated by one of the following matrices:*

$$R = R(k) = \begin{pmatrix} 0 & k \\ I_3 & 0 \end{pmatrix}, k \in K^*. \tag{21}$$

Furthermore:
if $i^2 = -1, i \in K^*$,

$$R = diag(1, 1, i, r), r = 1, -1, i, -1, \tag{22}$$

if $i^2 = -1, i \notin K^*$

$$R = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{23}$$

or

$$R = \begin{pmatrix} 0 & -2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{pmatrix}. \tag{24}$$

$C(R(k))$ *and* $C(R(k'))$ *are conjugated if and only if* $k' = c^4 k$ *or* $k' = c^4 k^3$, $c \in K^*$.
The other above-mentioned couples of groups are not conjugated.

Proposition 3.4. *Every subgroup of* $PGL(4, K)$, *char* $(K) = 2$, *cyclic of order 4 is conjugated to the subgroup* $C(R)$ *generated by one of the following matrices:*

$$R = R(1) = \begin{pmatrix} 0 & 1 \\ I_3 & 0 \end{pmatrix} \tag{25}$$

or

$$R = R' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{26}$$

The matrices (25) and (26) define not conjugated subgroups.

Proof of Propositions. 3.1, 3.2, 3.3 and 3.4 requires some Lemmas.

Let K' be the splitting field of the characteristic polynomial of $M \in GL(4, K)$. The Jordan canonical forms of M belong to the group $GL(4, K')$ and if $J(M)$ and $J'(M)$ are two of them, then there exists a permutation-matrix $E_\sigma \in S_4$, such that

$$J'(M) = E_\sigma^{-1}(J(M))E_\sigma$$

Briefly we will say that $J'(M)$ is equivalent to $J(M)$ and we will denote by $[J]$ the image of J in the natural homomorphism of $GL(4, K')$ over $PGL'(4, K') = GL(4, K') / kI_4, k \in K^*$.

Lemma 3.5. *Let $J = J(M), M \in GL(4, K)$. Card $\langle [J] \rangle = 2$ if and only if J is equivalent to one of the following matrices:*

$$J_1 = \rho_0 diag(1, 1, -1, -1), \rho_0^2 = k_2 \in K^*, [K' : K] \leq 2$$

or

$$J_2 = \rho_0 diag(1, 1, 1, -1), \rho_0 \in K^*, K' = K$$

if char $(K) \neq 2$; to one of the following:

$$J_3 = \rho_0 I_4 + E_{34}, \rho \in K^*, K' = K$$

or

$$J_4 = \rho_0 I_4 + E_{13} + E_{24}, \rho_0 \in K^*, K' = K$$

if char $(K) = 2$.

Card $\langle [J] \rangle = 4$ if and only if J is equivalent to one of the following matrices:

$$J_5 = diag(\rho_0, \rho_1, \rho_2, \rho_3), \rho_i^4 = k \in K^*, \rho_i \neq \rho_j \quad \forall i \neq j, [K' : K] \leq 8,$$

$$J_6 = diag(\rho_0, \rho_1, \rho_2, \rho_2), \rho_i^4 = k \in K^*, \rho_i \neq \rho_j \quad \forall i \neq j, [K' : K] \leq 2,$$

$$J_7 = diag(\rho_0, \rho_0, \rho_2, \rho_2), \rho_i^4 = k \in K^*, \rho_0 \neq \pm\rho_2, [K' : K] \leq 2,$$

or

$$J_8 = diag(\rho_0, \rho_1, \rho_1, \rho_1), \rho_i^4 = k \in K^*, \rho_0 \neq \pm\rho_1, K' = K$$

if char $(K) \neq 2$; to one of the following:

$$J_9 = \rho_0 I_4 + E_{12} + E_{23} + E_{34}, \rho_0 \in K^*, K' = K$$

or

$$J_{10} = \rho_0 I_4 + E_{23} + E_{34}, \rho_0 \in K^*, K' = K$$

if char $(K) = 2$.

Proof. If $B_r = \rho_r I_{h(r)} + N_{h(r)}$, $N_{h(r)}^{h(r)} = 0$, $\rho_r \in K'^*$, $1 \leq h(r) \leq 4$, $r = 0, 1$, are the Jordan blochs of J then

$$B_r^n = \sum_{s=0}^{n} \binom{n}{s} \rho_r^{n-2} N_{h(r)}^s, \quad \forall n \in \mathbb{N} \tag{$*$}$$

$Card\langle [J] \rangle = 2$ if and only if $B_r^2 = kI_{h(r)}$, $k \in K^*$, $r = 0, 1, ..$ and $[J] \neq [I_4]$. By virtue of $(*)$ these conditions are equivalent to $2N_{h(r)} = N_{h(r)}^2 = 0$, $\rho_r^2 = k$, $r = 0, 1, \ldots$, apart the case $h(r) = 1$, $\rho_r = \rho_0 \in K^*$, $r = 0, 1, ..$

If char $(K) \neq 2$ we deduce

$$J = diag(\rho_0, \rho_1, \rho_2, \rho_3), \rho_r^2 = k, r = 0, 1, ..$$

where the scalars ρ_r are not all coincident. Therefore J is equivalent to a matrix J_1 or J_2.

If char $(K) = 2$ the previous conditions are equivalent to $h(r) \leq 2$, $\rho_r = \rho_0$, $r = 0, 1, ..$, apart the case $h(r) = 1$, $r = 0, 1, ..$
Therefore J is equivalent to a matrix J_3 or J_4.

Analogously we prove the second part of the thesis.

Observe that if $J(M) = J_5$, then $K' = K[\rho_0, i]$, where ρ_0 is a root of the polynomial $\rho^4 = k$ and $i^2 = -1$.

Lemma 3.6. *Let be $M \in GL(4,K), J = J(M) \in GL(4,K')$, $K' \neq K$, char $(K) \neq 2$ and let $Q(A', J) \subseteq PGL'(4, K')$ be a subgroup isomorphic to the quadrinomial group. If Q_1, Q_2 are subgroups of $PGL(4, K)$ conjugated to $Q(A', J)$ then they are conjugated in $PGL(4, K)$.*

Proof. By hypothesis $Q_i = Q_i(A_i, B_i)$, $A_i, B_i \in GL(4, K)$, $i = 1, 2$. Moreover there exists $X_i \in GL(4, K')$ such that $X_i A' X_i^{-1} = A_i$ and

$$X_i J X_i^{-1} = B_i, i = 1, 2. \qquad (*)$$

$Card <[J]> = 2$ then (cf. Lemma 3.5) $J = diag(\rho_0, \rho_1, \rho_2, \rho_3)$, $\rho_r^2 = k \in K^*$, $\rho_r \in K'$, $r = 0, 1$.

From $(*)$ it follows that the r-th column of X_i is an eigenvector of B_i corresponding to the eigenvalue ρ_r. A K-automorphism, Γ, of K' induces a permutation σ on the eigenvalues ρ_r of A_i and the same permutation on the columns of X_i, $i = 1, 2$.

Moreover if E_σ is the σ-permutation matrix, we have $\Gamma\ (X_i) = X_i\ E_\sigma$, $i = 1, 2$. Hence $\Gamma\ = (X_1\ X_2^{-1}) = \Gamma\ (X_1)\ \Gamma(X_2)^{-1} = X_1 X_2^{-1}$, $\forall\ \Gamma$ and we deduce that $X_1 X_2^{-1} \in GL(4, K)$.

Analogously we prove the following

Lemma 3.7. *Let $B \in GL(4,K), J = J(B) \in GL(4,K')$, $K' \neq K$, char $(K) \neq 2$, and $Card\langle[J]\rangle = 4$. If $C(R)$, $C(R')$ are cyclic subgroups of $PGL(4, K)$ conjugated to $C(J)$, then they are conjugated in $PGL\ (4, K)$.*

We will prove Propositions 3.1 and 3.2 using the following scheme.

Fixed $J = J(M)$, $M \in GL(4, K)$, such that $Card\langle[J]\rangle = 2$, (cf. Lemma 3.5), we determine the matrices $A' \in GL\ (4, K')$ such that conditions

$$\langle[A']\rangle \times \langle[J]\rangle = Q(A', J) \qquad (27)$$

and

$$\exists X \in GL(4, K') : XA'X^{-1} = A, XJX^{-1} = B \in GL(4, K) \qquad (28)$$

are satisfied. Now we determine conditions of conjugacy among all the groups $Q_j(A, B)$.

From Lemma 3.6 every subgroup of $PGL(4, K)$ isomorphic to the quadrinomial group is conjugated to a group $Q_J(A, B)$.
Moreover observe that if J' is equivalent to J then the groups $Q_{J'}\ (A, B)$ are conjugated to the previous ones. Therefore it is not limitative to suppose J coincident with J_1, J_2 if char $(K) \neq 2$ or with J_3, J_4 if char $(K) = 2$.

It is worth observing that (27) is equivalent to condition:

$$A' \in C(J) = \{U \in GL(4, K') | [UJ] = [JU]\}, Card\langle A'\rangle = 2, [A'] \neq [J]. \qquad (29)$$

Proof of Proposition 3.1.

STEP 1. *Up to conjugation the groups $Q_{J_1}(A,B)$ are whose in which the matrices A,B coincide with (14), (15) or (16).*

Proof. $C(J_1) = C_1(J_1) \cup C_2(J_1)$,

$$C_1(J_1) = \{U_1 = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} | V_i \in GL(2,K')\},$$

$$C_2(J_1) = \{U_2 = \begin{pmatrix} 0 & W_1 \\ W_2 & 0 \end{pmatrix} | W_i \in GL(2,K')\}.$$

The matrices $A' = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \in C_1(J_1)$ such that $Card\langle [A'] \rangle = 2$, $[A'] \neq [J_1]$ are those in which

$$tr(A_i) = 0, \det(A_i) = -k_1 \in K^*, i = 1,2, \qquad (a)$$

or

$$A_j = h_0 I_2, h_0^2 \in K^*, \det(A_i) = -h_0^2, tr(A_i) = 0, i,j = 1,2, i \neq j. \qquad (b)$$

In case (a) A_1 and A_2 have the same eigenvalues $\pm\sqrt{k_1} \in K"$, $[K" : K'] \leq 2$. Then there exists $S_i \in GL\,(2,K")$ such that

$$S_i^{-1} A_i S_i = \sqrt{k_1} diag(1,-1), i = 1,2,$$

and if we put

$$X^{-1} = \begin{pmatrix} S_1 D & \rho_0 S_1 D \\ S_2 D & -\rho_0 S_2 D \end{pmatrix}, D = \begin{pmatrix} 1 & \sqrt{k_1} \\ 1 & -\sqrt{k_1} \end{pmatrix},$$

easily we can verify that

$$XA'X^{-1} = A_1(k_1), XJ_1X^{-1} = B(k_2).$$

We observe that $X^{-1} \in GL(4,K')$ also when $[K" : K'] = 2$. In fact if Γ is the not identic K'-automorphism of $K"$, then

$$S_i = \begin{pmatrix} S_{i1} & \Gamma(S_{i1}) \\ S_{i2} & \Gamma(S_{i2}) \end{pmatrix}$$

and $S_i D \in GL(2,K')$.

In case (b) A' is diagonalizable and has three eigenvalues coincident with h_0 that lies in K^* for satisfy condition (28). Moreover also $\rho_0 \in K^*$. In fact A' and J_1 commute and their product has three eigenvalues coincident with $\pm\rho_0 h_0$. Then there exists $S_i \in GL(2,K)$ so that

$$S_i^{-1} A_i S_i = h_0 diag(1,-1), h_0 \in K^*.$$

If we fix

$$X^{-1} = \begin{cases} diag(I_2, S_2), i = 2, \\ E_\sigma diag(I_2, S_2) E_\sigma, E_\sigma = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, i = 1, \end{cases}$$

we have $XA'X^{-1} = h_0 diag(1, 1, 1, -1)$ and $XJ_1X^{-1} = J_1$.

A matrix

$$A' = \begin{pmatrix} 0 & F_1 \\ F_2 & 0 \end{pmatrix} \in C_2(J_1)$$

satisfies condition $Card\langle [A'] \rangle = 2$ if and only if $F_2 = k_1 \, F_1^{-1}, k_1 \in K^*$.

If we put

$$X^{-1} = \begin{pmatrix} I_2 & \rho_0 I_2 \\ F_1^{-1}K & -\rho_0 F_1^{-1}K \end{pmatrix}$$

we have

$$XA'X^{-1} = A_2(k_1) \quad \text{and} \quad XJ_1X^{-1} = B(k_2).$$

STEP 2. *Up to conjugation the subgroups* $Q_{J_2}\ (A, B)$ *are those in which* A, B *coincide with (14).*

Proof. Proceeding as above we observe that the matrices A' satisfying (17) or (19) with $J = J_2$ are the ones of the type $A' = diag\ (M, a)$, $M \in GL\ (3, K)$, $a \in K^*$, $M \neq \pm aI_3$ and $M^2 = a^2 I_3$.

The last condition implies that $J(M)$ is diagonal, then there exists $S \in GL\ (3, K)$ such that $SMS^{-1} = \pm a diag(1, 1, -1)$.

If we put $X = diag(S, 1)$ we have $XJ_2X^{-1} = A$ and $XA'X^{-1} = aAB$ or $XA'X^{-1} = -aB$ where A and B indicate the matrices (14).

Let Q and S be the set of the subgroups $Q(A_1(k_1), B(k_2)) \subseteq PGL\ (4, K)$ and $G\ (k_1, k_2) \subseteq F$ respectively. Moreover let be $\chi : Q \to S$, $Q(A_1(k_1), B(k_2)) \mapsto G\ (k_1, k_2)$.

STEP 3. $Q(A_1(k_1'), B(k_2'))$ *is conjugated to* $Q(A_1(k_1), B(k_2))$ *if and only if belongs to the inverse image of* $G\ (k_1, k_2)$ *in* χ.

Proof. $(t^2 - \alpha^2 k_1)^2, (t^2 - \beta^2 k_2)^2\ (t^2 - \gamma^2 k_1 k_2)^2$, $\alpha, \beta, \gamma \in K^*$ are the characteristic polynomial of $\alpha A(k_1)$, $\beta B(k_2)$, $\gamma C(k_1, k_2) = \gamma A(k_1)\, B(k_2)$ respectively. If $Q(A_1(k_1'), B(k_2'))$ is conjugated to $Q(A_1(k_1), B(k_2))$ then characteristic polynomial of each matrix $A_1(k_1')$, $B(k_2')$, $C(k_1', k_2')$ coincides with one of the previous ones. Then $[k_1']$, $[k_2']$ are generators of $G(k_1, k_2)$.

Viceversa if $Q(A_1(k_1'), B(k_2'))$ belongs to the inverse image of $Q(A_1(k_1), B(k_2))$ in χ, then obviously $[k_1']$, $[k_2']$ generate $G(k_1, k_2)$.

Therefore thesis follows if we prove that the groups $Q\,(A_1\,(\alpha^2 k_1), B\,(\beta^2 k_2))$, $Q\ (A_1\ (k_2),\ B\ (k_1))$, $Q(A_1\ (k_1 k_2),\ (k_2))$ and $Q(A_1(k_1), B(k_1 k_2))$ are conjugated to $Q(A_1(k_1), B(k_2))$, $\forall k_1, k_2, \alpha, \beta \in K^*$.

Easily we prove that

$$S^{-1}Q(A_1(\alpha^2 k_1), B(\beta^2 k_2))S = Q(A_1(k_1), B(k_2), S = \begin{pmatrix} \beta D & 0 \\ 0 & D \end{pmatrix}, D = \begin{pmatrix} 0 & \alpha k_1 \\ 1 & 0 \end{pmatrix};$$

$$S^{-1}Q(A_1(k_2), B(k_1))S = Q(A_1(k_1), B(k_2)), S = E_{11} + E_{23} + E_{32} + E_{44};$$

$$S^{-1}Q(A_1(k_1 k_2), B(k_2))S = Q(A_1(k_1), B(k_2)), S = \begin{pmatrix} D & k_2 D' \\ D' & D \end{pmatrix};$$

$$D = diag(a, b), D' = diag(b, ak_2^{-1}), a^2 - k_2 b^2 \neq 0;$$

$$S^{-1}Q(A_1(k_1), B(k_1 k_2))S = Q(A_1(k_1), B(k_2)), S = \begin{pmatrix} k_1 I_2 & 0 \\ 0 & K \end{pmatrix}.$$

Let be $\chi' : Q(A_2(k_1), B(k_2)) \mapsto G'(k_1, k_2), Q(A_2(k_1), B(k_2)) \subseteq PGL(4, K)$.

STEP 4. *$Q(A_2(k_1'), B(k_2'))$ is conjugated to $Q(A_2(k_1), B(k_2))$ if and only if belongs to the inverse image of $G'(k_1, k_2)$ in χ'.*

Proof. It is analogous to the one of the Step 3. In particular it fan be verify that:

$$S^{-1}Q(A_2(k_2), B(k_1))S = Q(A_2(k_1), B(k_2)), S = E_{11} + E_{23} + E_{32} - E_{44};$$

$$S^{-1}Q(A_2(\alpha^2 k_1), B(\beta^2 k_2))S = Q(A_2(k_1), B(k_2)), S = \begin{pmatrix} \beta D & 0 \\ 0 & D \end{pmatrix}, D = \begin{pmatrix} 0 & \alpha k_1 \\ 1 & 0 \end{pmatrix};$$

$$S^{-1}Q(A_2(-k_1 k_2), B(k_2))S = Q(A_2(k_1), B(k_2)), S = \begin{pmatrix} D & k_2 D' \\ D' & D \end{pmatrix};$$

$$D = diag(a, b), D' = diag(-b, ak_2^{-1}), a^2 - k_2 b^2 \neq 0;$$

$$S^{-1}Q(A_2(k_1), B(-k_1 k_2))S = Q(A_2(k_1), B(k_2)), S = \begin{pmatrix} K & 0 \\ 0 & I_2 \end{pmatrix}.$$

The group $Q(A, B)$ with A, B given by (14), can not be conjugated neither to a group $Q(A_1(k_1), B(k_2))$ nor to a group $Q(A_2(k_1), B(k_2))$ because A possesses three eigenvalues 1 while each matrix $A_1(k)$, $B(k_2)$, $A_1 B(k_2)$, $A_2(k)$, $A_2 B(k_2)$ has at most two eigenvalues coincident.

Now we suppose by way of contradiction, that $Q(A_1(k_1), B(k_2))$ is conjugated to $Q(A_2(k_1'), B(k_2'))$.

From Steps 3 and 4 we deduce that there exists $S_1 \in GL(4, K)$ so that $[S^{-1} A_1(k_1) S_1] = [A_2(k_1')]$ and $[S_1^{-1} B(k_2) S_1] = [B(k_2')]$. Then $k_1' = \alpha^2 k_1$, $k_2' = \beta^2 k_2$, $\alpha, \beta \in K^*$. But $[A_2(\alpha^2 k_1)] = [S_2^{-1} A_2(k_1) S_2]$, $[B(\beta^2 k_2)] = [S_2^{-1} B(k_2) S_2]$, $S_2^{-1} = \begin{pmatrix} \beta D & 0 \\ 0 & D \end{pmatrix}$, $D = \begin{pmatrix} 0 & \alpha k_1 \\ 1 & 0 \end{pmatrix}$, implies $[S^{-1} A_1(k_1) S] = [A_2(k_1)]$, $[S^{-1} B(k_2) S] = [B(k_2)]$, $S \in GL(4, K)$, that is impossible.

Observe that in some cases it is easy to determine the number of the conjugation classes of the quadrinomial group in $PGL(4, K)$, char $(K) \neq 2$.

For example if $K = \overline{K}$ the groups S and S' coincide with the trivial group. Then we can fix $k_1 = k_2 = 1$ in both the cases (15) and (16).

If $K = \mathbf{R}$ or $K = F_q$, $q = p^h$, $p \neq 2$ then S and S' are isomorphic to $\{1, -1\}$. Both the groups generated by matrices (14) and (15) respectively, are classified in two conjugacy classes.

Proof of Proposition 3.2.

STEP 1. *Up to conjugation the groups $Q_{J_3}(A, B)$ are those in which*

$$A = W(R) = \begin{pmatrix} I_2 & R \\ 0 & I_2 \end{pmatrix}, B = W(S_1) = \begin{pmatrix} I_2 & S_1 \\ 0 & I_2 \end{pmatrix}, S_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, R \in M_2(K) - \{0, S_1\}.$$

Proof. Let

$$C(J_3) = \{(a_{ij}) \in GL(4, K) | a_{41} = a_{42} = a_{43} = a_{13} = a_{23} = 0, a_{44} = a_{33} \neq 0,$$
$$a_{11}a_{22} - a_{12}a_{21} \neq 0\}$$

A matrix $(a_{ij}) \in C(J_3)$ satisfies condition $[(a_{ij})^2] = [I_4]$ if and only if

$$a_{11} = a_{22}, rank \begin{pmatrix} a_{14} & a_{32} & a_{12} & a_{11}+a_{33} \\ a_{24} & a_{31} & a_{11}+a_{33} & a_{21} \end{pmatrix} < 2. \qquad (*)$$

The conjugation $M \to X_0^{-1}MX_0$, $X_0 = E_{12}+E_{21}+E_{33}+E_{44}$ maps J_3 into itself and $(a_{ij}) \in C(J_3)$ into $\begin{pmatrix} a_{22} & a_{21} & 0 & a_{24} \\ a_{12} & a_{11} & 0 & a_{14} \\ a_{32} & a_{31} & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{33} \end{pmatrix} \in C(J_3)$. Then it is not limitative to suppose $A' = \begin{pmatrix} a_{11} & c^2a_{21} & 0 & ca_{24} \\ a_{21} & a_{11} & 0 & a_{24} \\ a_{31} & ca_{31} & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{33} \end{pmatrix}$, $a_{33} \neq 0$, $a_{11}+a_{33} = ca_{21}$, $c \in K$ and $[A'] \neq [J_3]$.

If we put $X = \begin{pmatrix} 0 & \rho_0 a_{31} & \rho_0(h+a_{21}) & 0 \\ 0 & h & 0 & 0 \\ h & hc & 0 & a_{24} \\ 0 & 0 & 0 & h+a_{21} \end{pmatrix}$, $h \neq 0, a_{21}$, we have $XJ_3X^{-1} = \rho_0 \begin{pmatrix} I_2 & S_1 \\ 0 & I_2 \end{pmatrix}$, $XA'X^{-1} = a_{33}\begin{pmatrix} I_2 & R \\ 0 & I_2 \end{pmatrix}$, $R = a_{33}^{-1}\begin{pmatrix} \rho_0 a_{31} & \rho_0 a_{34} \\ a_{21} & a_{24} \end{pmatrix}$.

STEP 2. *Up to conjugation the groups $Q_{J_4}(A, B)$ are those in which A and B coincide with (17), (18) or (20).*

Proof. Let $C(J_4) = \left\{ \begin{pmatrix} D & C \\ 0 & D \end{pmatrix} \in M_4(K) | D \in GL(2,K) \right\}$.

A matrix $\begin{pmatrix} D & C \\ 0 & D \end{pmatrix} \in C(J_4)$ satisfies condition $\left[\begin{pmatrix} D & C \\ 0 & D \end{pmatrix}^2\right] = [I_4]$ if and only if

$$CD = DC, tr(D) = 0. \qquad (*)$$

If $D = dI_2, d \in K^*$, then we put $X = \begin{pmatrix} \rho_0 I_2 & 0 \\ 0 & I_2 \end{pmatrix}$ and we have $XJ_4X^{-1} = \rho_0 \begin{pmatrix} I_2 & I_2 \\ 0 & I_2 \end{pmatrix}$, $X\begin{pmatrix} dI_2 & C \\ 0 & dI_2 \end{pmatrix}X^{-1} = d\begin{pmatrix} I_2 & R \\ 0 & I_2 \end{pmatrix}$, $R = d^{-1}\rho_0 C$. If $D \in GL(2,K)$ is not diagonal, then from (*) it follows $tr(C) = 0$ and $\exists U \in GL(2,K) : UDU^{-1} = T = \begin{pmatrix} t_0 & 1 \\ 0 & t_0 \end{pmatrix}$, $t_0 \in K^*$. Consequently $tr(C') = 0$, $UCU^{-1} = C'$, $TC' = C'T$ and therefore $C' = \begin{pmatrix} c & e \\ 0 & c \end{pmatrix}$.

Put $X = \begin{pmatrix} \rho_0 t_0 & \rho_0 & t_0 & 1 \\ \rho_0 t_0 & 0 & c\rho_0 + t_0 & \rho_0(ct_0^{-1}+e) \\ \rho_0 t_0 & \rho_0 & 0 & 0 \\ \rho_0 t_0 & 0 & c\rho_0 & \rho_0(ct_0^{-1}+e) \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}$ we obtain $X\begin{pmatrix} D & C \\ 0 & D \end{pmatrix}X^{-1}$ $= t_0\begin{pmatrix} K_1 & 0 \\ 0 & K_1 \end{pmatrix}$, $XJ_4X^{-1} = \rho_0\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$.

It is useful observing that the additive subgroup $\langle R_1, R_2\rangle \subseteq M_2(K)$, char $(K) = 2$, generated by R_1, R_2 is isomorphic to $Q(W(R_1), W(R_2))$, $W(R_i) = \begin{pmatrix} I_2 & R_i \\ 0 & I_2 \end{pmatrix}$, $i = 1, 2$.

Easily we verify

STEP 3. *Let be $R_3 = R_1 + R_2, R'_3 = R'_1 + R'_2$. $Q(W(R_1), W(R_2))$ and $Q(W(R'_1), W(R'_2))$ are conjugated if and only if there exists a permutation σ and a matrix $X = (X_{ij}) \in GL(4, K)$, $X_{ij} \in M_2(K)$ such that $R_iX_{22} = X_{11} R'_{\sigma(i)}$, $R_iX_{21} = X_{21} R'_{\sigma(i)} = 0$, $i = 1, 2, 3$.*

In particular we deduce

$$R_{\sigma(i)} \text{ is singular if and only if } R_i \text{ is singular;} \tag{30}$$

and

$$\text{if at least one of the matrices } R_i, i = 1, 2, 3, \text{ is not singular, then}$$

$$X_{11} \text{ and } X_{22} \text{ are not singular and } X_{21} = 0. \tag{31}$$

STEP 4. *A group $Q = Q(W(R), W(S_1))$, (cf. Step 1) is conjugated to a group $Q(W(R_1), W(I_2))$ with R_1 defined in (20) or to a group $Q(W(R_1), W(S_1))$ with R_1 and S_1 defined in (19).*

Proof. We distinguish two cases.

i) Let us suppose that there exists $B = (b_{ij}) \in \{R, R + S_1\}$, $\det(B) \neq 0$.

If $b_{21} = 0$ and then $b_{11} \neq 0 \neq b_{22}$, we put $X_{11} = \begin{pmatrix} 1 & b_{12} \\ 0 & b_{22} \end{pmatrix}$, $X_{22} = \begin{pmatrix} b_{11}^{-1} & 0 \\ 0 & 1 \end{pmatrix}$, $X_{12} = X_{21} = 0$ and we have $BX_{22} = X_{11} I_2$, $S_1 X_{22} = X_{11}S_1$. From this and from Step 3 we deduce that Q is conjugated to $Q(W(I_2), W(S_1))$.

If $b_{21} \neq 0$, we put $X_{11}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & b_{11}b_{21}^{-1} \end{pmatrix}$, $X_{22} = \begin{pmatrix} b_{21}^{-1} & b_{22}\det^{-1}(B) \\ 0 & b_{21}\det^{-1}(B) \end{pmatrix}$, $X_{21} = X_{12} = 0$ and we verify that $X_{11}^{-1} BX_{22} = I_2$, $X_{11}^{-1} S_1X_{22} = \begin{pmatrix} 0 & 0 \\ 0 & b_{21}\det^{-1}(B) \end{pmatrix} = R_1$. Consequently Q is conjugated to $Q(W(I_2), W(R_1))$.

ii) Let us suppose $\det(R) = \det(R + S_1) = 0$. Hence we have $R = \begin{pmatrix} r_{11} & r_{12} \\ 0 & 0 \end{pmatrix}$ or $R = \begin{pmatrix} 0 & r_{12} \\ 0 & r_{22} \end{pmatrix}$.

If $R = \begin{pmatrix} r_{11} & r_{12} \\ 0 & 0 \end{pmatrix}$, $r_{11} \neq 0$, we put $X_{11} = \begin{pmatrix} r_{11} & 0 \\ 0 & 1 \end{pmatrix}$, $X_{22} = \begin{pmatrix} 1 & r_{12} \\ 0 & r_{11} \end{pmatrix}$, $X_{12} = X_{21} = 0$ and we observe that $RX_{22} = X_{11}S''$, $S_1X_{22} = X_{11}S_1$.

If $R = \begin{pmatrix} 0 & r_{12} \\ 0 & r_{22} \end{pmatrix}$, $r_{22} \neq 0$, then $RX_{22} = X_{11}S'$, $S_1X_{22} = X_{11} S_1$, $X_{11} = \begin{pmatrix} 1 & r_{11} \\ 0 & r_{22} \end{pmatrix}$, $X_{22} = I_2$, $X_{12} = X_{21} = 0$.

STEP 5. *$Q(W(S_1), W(S_k))$ and $Q(W(S_1), W(S_{k'}))$, $k, k' \in K^* - \{1\}$ are conjugated if and only if $\Delta(k) = \Delta(k')$. $Q(W(I_2), W(R))$ and $Q(W(I_2), W(R'))$, $R, R' \neq 0, I_2$ are conjugated if and only if there exists $Y \in GL(2, K)$ such that $\Delta(Y^{-1}RY) = \Delta(R')$.*

Proof. From Step 3 we deduce that $Q(W(S_1), W(S_k))$ and $Q(W(S_1), W(S_{k'}))$ are conjugated if and only if there exists $c \in K^*$ such that $\{1, k', k'+1\} = \{c, ck, c(k+1)\}$. Then the first part of thesis follows.

Let us put $I_2 = R_1 = R'_1, R = R_2, R' = R'_2, I_2 + R = R_3, I_2 + R' = R'_3$. If $Q(W(I_2), W(R))$ and $Q(W(I_2), W(R'))$ are conjugated then conditions in Step 3 are satisfied with X_{11}, X_{22} not singular and $X_{21} = 0$ (cf. (31)).

In particular if $\sigma(1) = 1$, then $X_{11} = X_{22}, X_{11}^{-1} RX_{11} = R'_{\sigma(2)} = \begin{cases} R' \\ R' + I_2 \end{cases}$ and if $\sigma(1) \neq 1$, then

$$X_{11}^{-1}X_{22} = R'_{\sigma(1)} = \begin{cases} R' \\ R' + I_2 \end{cases}, X_{11}^{-1}R_{\sigma^{-1}(1)}X_{22} = I_2, R_{\sigma^{-1}(1)} = \begin{cases} R \\ R + I_2 \end{cases},$$
$$X_{11}^{-1}R_{\sigma^{-1}(1)}X_{11} = X_{11}^{-1}R_{\sigma^{-1}(1)}X_{22}X_{22}^{-1}X_{11} = R'^{-1}_{\sigma(1)}$$

Viceversa let us suppose $R' \in \Delta(Y^{-1}RY)$. If $R' = Y^{-1}RY$ or $R' = Y^{-1}(R + I_2)Y$, then evidently conditions in Step 3 are satisfied. If $Y^{-1}RY \neq R' \neq Y^{-1}(R + I_2)Y$ then $Y^{-1}MY = M'^{-1}, M = \begin{cases} R \\ R + I_2 \end{cases}, M' = \begin{cases} R' \\ R' + I_2 \end{cases}$. Consequently, if we put $M' = ZY$, we have $ZMY = I_2$.

Easily we verify that the group generated by matrices (17) is not conjugated to a group generated by a couple of matrices (18).

From (30) it follows that every group defined by a couple of matrices (18), (19) is not conjugated to a group defined by a couple (18), (20).

By Step 3 we deduce other conditions of conjugation.

We can express condition $\Delta(Y^{-1}RY) = \Delta(R'), Y \in GL(2, K)$ through the eigenvalues x_1, x_2 and x'_1, x'_2 of R and R' respectively. In fact it is equivalent to require that $\Delta((x_1, x_2)) = \Delta((x'_1, x'_2))$ for a suitable ordering of the couples.

To proof Propositions 3.3 and 3.4 the following observation is useful.

Let be $J = J_5, J_6, J_7$ or J_8, if char $(K) \neq 2$ and let be $J = J_9$ or J_{10} if char $(K) = 2$. If we determine $X \in GL(4, K')$ such that $X^{-1} JX = R \in GL(4, K)$, then every cyclic subgroup of $PGL(4, K)$ is conjugated to a group $C_J(R) = C(X^{-1}JX)$ (cf. Lemma 3.7).

Proof of Proposition 3.3.

(a) Suppose $X = (x_{ij}), x_{ij} = \rho_i^j, i, j = 0, 1, 2, 3$, and hence $X^{-1} J_5 X = R_k$ coincident with (21).

If $C(R_{k'})$ is conjugated to $C(R_k)$ then the characteristic polynomial of $[R_{k'}]$ coincides with the one of a generator of $C(R_k)$. Hence $k' = c^4 k$ or $k' = c^4 k^3, c \in K^*$.

Viceversa if $k' = c^4 k$ or $k' = c^4 k^3, c \in K^*$, then $H^{-1}R_kH = C^{-1}R_{c^4k}, H =$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c^2 \\ c^{-1}k^{-1} & 0 & 0 & 0 \end{pmatrix} \text{ or } H'^{-1}R_k^3H' = C^{-1}R_{c^4k^3}, H' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & (ck)^{-1} & 0 \\ 0 & (ck)^{-2} & 0 & 0 \\ (ck)^{-3} & 0 & 0 & 0 \end{pmatrix}$$

respectively.

(b) Let us suppose $J = J_6$.

Necessarily $\rho_2 = \rho_3 = \rho \in K^*$ and at least one of the eigenvalues ρ_0, ρ_1 coincides with $\pm\rho i$, $i^2 = -1$. Therefore we distinguish two cases:

(b′) $i \notin K$. Then ρ_0 and ρ_1 are not conjugated. Consequently $J_6 = \rho\, diag(\pm i, i, 1, 1)$, but being $J_6^3 = \rho^3\, diag\,(i, \pm i, 1, 1)$, we can suppose $J_6 = \rho\, diag\,(i, -i, 1, 1)$, $\rho \in K^*$. Put $X = \begin{pmatrix} X_{11} & 0 \\ 0 & I_2 \end{pmatrix}$, $X_{11} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$ we have $X^{-1}\, diag\,(i, -i, 1, 1)\, X = R = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

(b") $i \in K$. Then up to conjugation, $R = diag(1, 1, i, r)$, $r = -i, -1$.

(c) Characteristic polynomial of matrix J_7 belongs to $K[t]$. Hence $\rho_2 = \pm i\rho_0$ and $\rho_0 \in K$ if and only if $i \in K$.

If $i \in K$, then $R = diag\,(1, 1, i, i)$. If $i \notin K$, then $\rho_0 = a + ib$, $a, b \in K$ and $\pm i\rho_0$ are conjugated in $K(i)$. Consequently $\rho_0 = a(1 \pm i)$, $J_7 = a diag\,(1 + i,\ 1 + i,\ 1 - i,\ 1 - i)$. Lastly, put $X = \begin{pmatrix} 1 & 1+i & 0 & 0 \\ 0 & 0 & 1 & 1+i \\ 1 & 1-i & 0 & 0 \\ 0 & 0 & 1 & 1-i \end{pmatrix}$, we obtain $X^{-1}\, diag\,(1 + i, 1 + i, 1 - i, 1 - i)\, X =$
$= \begin{pmatrix} 0 & -2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{pmatrix}$.

(d) In J_8 we have $\rho_1 \in K^*$ and $\rho_0 = \pm i\rho_i \in K^*$. Then up to conjugation, $A = (1, 1, i, 1)$.

The groups generated by matrices R correspondent to J_5, J_6, J_7, J_8 respectively, are not conjugated because they have a different number of distinct eigenvalues.

Proof of Proposition 3.4.

(a) If we put $X^{-1} = \begin{pmatrix} 1 & \rho_0^{-1} & \rho_0^{-2} & \rho_0^{-3} \\ 1 & 0 & \rho_0^{-2} & 0 \\ 1 & \rho_0^{-1} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, then $X^{-1}J_9X = \rho_0 R(1)$.

(b) If we put $X^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \rho_0^2 & 0 & 0 \\ 0 & 0 & \rho_0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, then we have $X^{-1}J_{10}X = \rho_0 R'$.

$C(R(1))$ and $C(R')$ are not conjugated because $rank(R(1) + I_4) = rank(R^3(1) + I_4) = 3$, $rank(R' + I_4) = rank\,(R'^3 + I_4) = 2$.

G-CLASSIFICATION OF THE 4-DIMENSIONAL LEFT PROJECTIVE GRUPPAL ALGEBRAS

In Section 3 we have determined the conjugacy classes of the groups $Z_2 \times Z_2$ and Z_4 in $PGL\,(4, K)$, fixing an element $Q(A, B)$ or $C(R)$ in every class. In this Section we examine the main properties of l.p.g.a. which are defined by each of the above-mentioned groups (cf. Section 1).

If $\mathcal{A}$ is the l.p.g.a. defined by $Q(A_1(k_1), B(k_2))$ (cf. (15)), then $\mathbf{L}(x) = (I_4x, A_1(k_1)x, B(k_2)x, A_1\,(k_1)B(k_2)\,x) = x_0\, I_4 + x_1 A_1\,(k_1) + x_2\, B(k_2) + x_3 A_1\,(k_1)B(k_2) = \mathbf{R}(y)$. It follows that $\mathcal{A}$ is commutative. $\mathcal{A}$ is also associative and gruppal because $e_0e_i = e_ie_0 = e_i$, $i = 1, 2, 3$ (cf. Propositions 2.1 and 2.4).

$L(e_0, e_1) = \{y = x_0e_0 + x_1e_1 | x_0, x_1 \in K\}$ is a subalgebra in which

$$e_1^2 = k_1e_0, \tag{32}$$

$$e_1e_2 = e_3 \tag{33}$$

and then we can write every $x \in \mathcal{A}$ as it follows:

$$x = \sum_{i=0}^{3} x_ie_i = y_0e_0 + y_1e_2, y_0 = x_0e_0 + x_1e_1, y_1 = x_2e_0 + x_3e_1 \in L(e_0, e_1).$$

We deduce that $\mathcal{A}$ is an algebra 2-dimensional over $L(e_0, e_1)$ in which

$$e_2^2 = k_2e_0. \tag{34}$$

Put $X^{-1} = \begin{pmatrix} D & \sqrt{k_2}D \\ D & -\sqrt{k_2}D \end{pmatrix}$, $D = \begin{pmatrix} 1 & \sqrt{k_1} \\ 1 & -\sqrt{k_1} \end{pmatrix}$, we have $\det(\mathbf{L}(x)) = \det\ (X^{-1}\mathbf{L}(x)X)$ $= \det(diag\ (f_0, f_1, f_2, f_3))$, where

$$f_0 = x_0 + \sqrt{k_1}x_1 + (x_2 + \sqrt{k_1}x_3)\sqrt{k_2}, f_1 = x_0 - \sqrt{k_1}x_1 + (x_2 - \sqrt{k_1}x_3)\sqrt{k_2},$$
$$f_2 = x_0 + \sqrt{k_1}x_1 - (x_2 + \sqrt{k_1}x_3)\sqrt{k_2}, f_3 = x_0 - \sqrt{k_1}x_1 - (x_2 - \sqrt{k_1}x_3)\sqrt{k_2}.$$

Therefore the surfaces Φ and Ψ are union of four linearly indipendent planes and their rational points over K are the zero divisors of $\mathcal{A}$. In particular we deduce that $\mathcal{A}$ is a division algebra if and only if $\xi^2 - k_1$ and $\xi^2 - k_2$ are irreducible in $K[\xi]$ and in $K_1[\xi]$, $K_1 = K(\sqrt{k_1})$ respectively. In this case $\mathcal{A} = K(\sqrt{k_1}, \sqrt{k_2})$.

If $k_1 = k_2 = 1$, $\mathcal{A}$ is the group algebra of $Z_2 \times Z_2$ over K (cf. (32), (33), (34)).

Comparing (15) with (17) we observe that l.p.g.a. defined by $Q(A, B)$, whit A and B given in (17), is the group algebra of $Z_2 \times Z_2$ over K, char $(K) = 2$.

$\mathbf{L}(x) = (I_4x, A_2(k_1)x, B(k_2)x, A_2(k_1)B(k_2)x) = \begin{pmatrix} x_0 & k_1x_1 & k_2x_2 & -k_1k_2x_3 \\ x_1 & x_0 & k_2x_3 & -k_2x_2 \\ x_2 & -k_1x_3 & x_0 & k_1x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{pmatrix}$ is the l.m.m. of l.p.g.a. defined by the group $Q(A_2(k_1), B(k_2))$ (cfr. (16)) and then it is the generalized quaternion algebra $\left(\frac{k_1, k_2}{K}\right)$ over K (cf. [2]).

$\det(\mathbf{L}(x)) = \det(\mathbf{R}(x)) = [(x_0^2 - k_1x_1^2) - k_2(x_2^2 - -k_1x_3^2)]^2$. Hence $\left(\frac{k_1, k_2}{K}\right)$ is a division algebra if and only if quadric surface $\Phi : x_0^2 - k_1x_1^2 - k_2x_2^2 + k_1k_2x_3^2 = 0$ has not rational points over K.

Now suppose $\mathcal{A}$ the l.p.g.a. defined by cyclic group $C(R(k))$, $R(k)$ as in (21). Then $\mathbf{L}(x) = (I_4x, R(k)x, R(k)^2x, R(k)^3x) = x_0I_4 + x_1\ R(k) + x_2\ R(k)^2 + x_3R(k)^3 = \mathbf{R}(x)$ and therefore $\mathcal{A}$ is a gruppal algebra commutative with unity e_0.

Put $X = (x_{ij}), x_{ij} = \rho_i^j, \rho_i^4 = k, i, j = 0, \ldots, 3$, we have $\det(\mathbf{L}(x)) = \det(X\mathbf{L}(x)X^{-1})$ $= \det(diag\ (f_0, f_1, f_2, f_3)), f_i = \sum_{j=0}^{3} x_j\rho_i^j$, $i = 0, \ldots, 3$. Therefore surfaces Φ and Ψ are union of four linearly indipendent planes.

Such planes have not rational points over K if and only if $\xi^4 - k$ is irreducible in $K[\xi]$. In this case $\mathcal{A} = K(\rho)$, $\rho^4 = k$.

If $k = 1$ we obtain the group algebra of Z_4 over K. When char $(K) = 2$ we obtain such group algebra from $C(R(1))$, $R(1)$ as in (25).

L.p.g.a. defined by groups $Q(A, B)$ with A, B as in (14), or by $C(R)$, with R as in (22), are particular examples of l.p.g.a. $\mathcal{A}$ whose l.m.m. is given by $\mathbf{L}(x) = (L_0x, L_1x, L_2x, L_3x)$, $L_i = diag\ (l_{i0}, l_{i1}, l_{i2}, l_{i3}) \in GL\,(4, K)$. Then $e_ie_J = l_{ji}\ e_i$, $i,j = 0, \ldots, 3$.

We deduce that $\mathcal{A}$ is not commutative, not associative and every $x \in \mathcal{A}$ is a left zero divisor. Moreover we observe that every subspace $L(e_i) = ke_1$ is a left ideal of $\mathcal{A}$ isomorphic to K.

Easily we verify that l.p.g.a. $\mathcal{A}$ defined by remaining groups in Propositions 3.2, 3.3 and 3.4, are not commutative and not associative. Moreover every $x \in \mathcal{A}$ is a left zero divisor.

Also l.p.g.a. $\mathcal{A}$ defined by groups $Q(A, B), A, B$ as in (18), (19) or in (18), (20) are not commutative, not associative and every element $x \in A$ is a left zero divisor.

$L(e_0, e_1, e_2)$ and $L(e_0, e_1, e_3)$ are left ideals of A and their intersection is an associative ideal. Moreover $L(e_0, e_1, e_2)$ is associative when $R_2 = S_1$ and $R_1 = S'$ or $R_1 = S_k$ (cf. (19)).

L.p.g.a. $\mathcal{A}$ defined by $C(R), R$ as in (23) or in (24) and by $C(R'), R'$ as in (26) are not commutative, not associative and every element $x \in \mathcal{A}$ is a left zero divisor.

If R is (23) then $L(e_0, e_1)$ is a left ideal isomorphic to $K(i)$ and $L(e_2)$, $L(e_3)$ are left ideals isomorphic to K.

If R is (24) then $L(e_0, e_1)$ and $L(e_2, e_3)$ are left ideals isomorphic to $K(1 + i)$. In fact we observe that $K(1 + i) \rightarrow L(e_2, e_3)$, $x + (1 + i)\ y \rightarrow x\ (e_2 - e_3)\ - e_3y$ is an isomorphic of algebras.

If R' is (26) then $L(e_1, e_2)$ and $L(e_1, e_2, e_3)$ are left ideals and $L(e_0)$ is a left ideal isomorphic to K.

REFERENCES

[1] G. MENICHETTI, *n-dimensional algebras over a field with a cyclic extension of degree n*, Geom. Dedicata 63 (1996), 69-94.
[2] R.S. PIERCE, *Associative Algebras*, Springer, New York, 1982.

Received July 1, 1997 and in revised form November 14, 1997
Lorena Giacobazzi
Via della Repubblica, 41
40046 Porretta Terme (Bo)
ITALY