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BAIRE PROPERTIES OF LOCALLY CONVEX SPACES

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INTRODUCTION

The property of topologica1 spaces to be a Baire space is of importane in many parts in mathematics. So it is a fundamental tool in the theorems of Banach Steinhaus, the closed graph theorem or the open mapping theorem.

Unfortunately it has bad permanence properties. For example Oxtoby [28] showed that the product of a Baire space with itself need not be Baire, which was extended by Arias de Reyna [3], cf. also [42], to products of two normed Baire spaces. Moreover dense hyperplanes of Baire spaces need not be Baire, which was proved by Arias de Reyna in [2], cf. also [33]. So it hns become a classica1 tradition in Pure Mathematics to consider properties of locally convex spaces which are weaker than the property of being a Baire space and which preserve the applications of the Baire property. In fact, there is a whole hierarchy of such properties, e. g. unordered Bairelike, totally barrelled, db, Bairelike, quasi-Baire and more. They were introduced e. g. by S. Saxon, A. R. Todd, M. Valdivia, W. J. Robertson, I. Tweddle and F. E. Yeomans in [34, 31, 41, 39].

Spaces with Baire properties are certainly interesting. For instance, Saxon proved that Grothendieck's factorization theorem for closed linear maps from a locally convex Baire space into an LF-space remains true for closed linear maps from a Bairelike space into an LB-space. Moreover an interesting classification of LF-spaces is given by P. P. Narayanaswami and S. Saxon in [27] and a connection to the classica1 separable quotient problem is given by S. Saxon and A. Wilansky in [37]. In contrast to the Baire property, these weaker properties have good permanence properties. For instance they are stable under arbitrary products, quotients, countably codimensional linear subspaces, the 3-space problem and are inherited from dense linear subspaces by the whole space.

The aim of this thesis is to continue the study of Baire properties mentioned above. More specifically, we investigate the behaviour of Baire properties with respect to projective limits, to the formation of vector valued sequence spaces h(X) and more generally to inductive and projective limits of Moscatelli type. Finally we give a contribution to a problem posed by Valdivia: Are complete Bairelike spaces Baire? In fact, we construct a quasicomplete Bairelike space which is not a Baire space.

In chapter 0 we first of all give the definitions of the spaces of type h(X) and inductive- and projective limits of Moscatelli type and summarize their basic properties. Afterwards we define the Baire properties mentioned above and recall some known connections and results. Moreover, at the end of chapter 0 we present a result about Baire properties of LF-spaces of Moscatelli type.

Chapter 1 is devoted to the investigation of the property quasi-Baire. These are spaces which are barrelled and which are not the union of an increasing sequence of nowheredense subspaces. This last property, called "without S_σ" is treated independently. We give a characterization of this property and study its permanence properties. The independence of barrelledness and "without S_σ" is shown by examples and we discuss its stability with respect t" the formation of the bidual. In the main part of chapter 1 we characterize the quasi-B&-property of a space X in terms of the existence of a continuous norm on the dual $(X', \beta(X', X))$ – at least for a large class of locally convex spaces.

In chapter 2 we examine the stability of these Baire properties with respect t" projective limits. In general they are not inherited even by countable projective limits of Baire spaces. Next we study projective limits with open linking maps, called strict. For uncountable strict projective limits we present an example of a strict projective limit of Banach spaces which is not barrelled. In the countable case we get positive results for the properties barrelled, "without S_σ", quasi-Baire and Bairelike. Fora special class of strict projective limits we can also present positive results for the properties db, unordered Bairelike and Baire. Moreover we give an example of a projective limit of discrete abelian topologica1 groups which is not Baire. We close this chapter with a positive result on a projective sequence of df-spaces (the strong dual is a Fréchet space) for the property "without S_σ".

In chapter 3 we first investigate spaces of type h(X). We start with an example of a quasi-Baire space E, such that $\ell_\infty(E)$ is not quasi-Baire. For X which satisfies the countable boundedness condition we can present a positive result for the property "without S_σ". Next we turn t" the class of Bairelike spaces, which contains the following two large subclasses: the metrizable barrelled spaces and the weakly barrelled spaces. The fact that h(X) is Bairelike, if X is metrizable and barrelled, is already contained in L. Frerick's dissertation. We are able t" prove thar this fact remains true for weakly barrelled X – at least for $\lambda = \ell_\infty$ and λ having the property of sectional convergence. After that we prove that for infinite dimensiona1 Hausdorff spaces X, such that the weak dual $(X', \sigma(X', X))$ is barrelled, the space h(X) is not Baire. These results are then used to give an example of a quasicomplete Bairelike space which is not Baire.

In the second part of this capter we treat projective limits of Moscatelli type. We obtain positive results for metrizable barrelled spaces and for weakly barrelled spaces, if the linking map is open. We finally show that in the general case there exist two weakly barrelled Hausdorff spaces, such that the corresponding projective limit of Moscatelli type is not barrelled.

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0. **DEFINITIONS,** NOTATIONS, TERMINOLOGY

In this section we introduce basic well known definitions and recall some properties of the corresponding notion. These are first of all spaces of type h(X) and inductive and projective limits of Moscatelli type. Then we present a hierachy of Baire properties we will deal with in the future and recollect some important known results about Baire properties of LF-spaces. At the end of this section we give a first partial description of the Baire properties of LF-spaces of Moscatelli type.

Now we start with the definition of normal Banach sequence spaces, spaces of type h(X) and the inductive and projective limits of Moscatelli type. A comprehensive discussion of the spaces h(X) can be found in [16] and the inductive- and projective limits of Moscatelli type are studied in detail in [6] and [7].

0.1 Definition:

Let $(\lambda, \|\,\|)$ be a Banach space with the following properties:

i) $\varphi \subset \lambda \subset \omega$,

ii) the inclusion $(\lambda, \|\,\|) \hookrightarrow \omega$ is continuous.

iii) if $\alpha \in \lambda$ and $\beta \in \omega$ with $|\alpha_n| \geq |\beta_n|$ for all $n \in \mathbb{N}$, then $\beta \in \lambda$ and $\|\alpha\| \geq \|\beta\|$.

Then we call $(\lambda, \|\,\|)$ a normal Banach sequence space (nBss for abbreviation). Examples for such spaces are $(\ell_p, \|\,\|_p)$ with $1 \leq p \leq \infty$ or $(c_0, \|\,\|_\infty)$.

A nBss $(\lambda, \|\,\|)$ has the property of sectional convergence if for all $x = (x_n)_{n\in\mathbb{N}} \in \lambda$ it is true that $((x_k)_{k\leq n}, (0)_{k>n}) \xrightarrow{n\to\infty} x$ in $(\lambda, \|\,\|)$. Spaces with sectional convergence are for example $(\ell_p, \|\,\|_p)$ with $1 \leq p < \infty$ or $(c_0, \|\,\|_\infty)$ but not $(\ell_\infty, \|\,\|_\infty)$.

Now let in the following $(\lambda, \|\,\|)$ be a nBss and let $(X, \Im)$ be a locally convex space (lcs). Then $cs(X)$ denotes the set of all continuous seminorms on X, and for $p \in cs(X)$ let

$$\lambda(X,p) := \lambda((X,p)_{n\in\mathbb{N}}) := \{(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}} : (p(x_n))_{n\in\mathbb{N}} \in \lambda\}$$

be provided with the seminorm $\hat{p} : (x_n)_{n\in\mathbb{N}} \mapsto \|(p(x_n))_{n\in\mathbb{N}}\|$. Furthermore we define

$$h(X) := \bigcap_{p\in cs(X)} \lambda(X,p)$$

and h(X) carries the initial topology with respect to the inclusion $(h(X) \hookrightarrow \lambda(X,p))_{p\in cs(X)}$. We put

$$\lambda((X)_{k\geq n}) := \{(x_k)_{k\geq n} \in \prod_{k\geq n} X : ((0)_{k<n}, (x_k)_{k\geq n}) \in \lambda(X)\}$$

provided with the induced topology of h(X). Now let X, Y be lcs with continuous inclusion $Y \hookrightarrow X$. Then we put

$$E_n := E_n(Y \hookrightarrow X, \lambda) := \prod_{k<n} X \times \lambda((Y)_{k\geq n})$$

provided with the product topology. This defines an inductive sequence of lcs with continuous inclusions $E_n \hookrightarrow E_{n+1}$ for all $n \in \mathbb{N}$ and we put

$$E := E(Y \hookrightarrow X, \lambda) := \operatorname{ind}_{n\in\mathbb{N}} E_n(Y \hookrightarrow X, \lambda)$$

Then E is called the inductive limit of Moscatelli type with respect to Y $\hookrightarrow$ X and λ. If X and Y are Bunach or Fréchet spaces, the E is an LB- or an LF-space. respectively and is also called an LB- or LF-space of Moscatelli type.

Now let X and Y be lcs and let f : Y $\to$ X be a continous linear map. We define

$$F_n := \prod_{k<n} Y \times \lambda((X)_{k\geq n})$$

provided with the product topology. Moreover let

$$g_{n,n+1} : F_{n+1} \longrightarrow F_n \ (x_k)_{k\in\mathbb{N}} \mapsto ((x_k)_{k<n}, f(x_n), (x_k)_{k>n})$$

This is a welldefined, linear, continuous map and $((F_n)_{n\in\mathbb{N}}, (g_{n,n+1})_{n\in\mathbb{N}})$ is a projective sequence of Ics, such that we can define $F := \mathrm{proj}_{n\in\mathbb{N}}(F_n, g_{n,n+1})$. F is called the projective limit of Moscatelli type with respect to $(\lambda, \|\ \|)$, Y, X and f. We write for this $F(Y \xrightarrow{f} X, \lambda)$. If X and Y are Banach spaces, the also F_n is a Banach space (see the remark after the definition), such that F is a Fréchet space. Then F is a also called a Fréchet space of Moscatelli type.

Now we recall some basic properties of these spaces, see [6], [7] and [Ih] for proofs.

0.2 Remark.

For every nBss $(\lambda, \|\ \|)$ and Ics X the inclusions $\bigoplus_{n\in N} X \hookrightarrow$ h(X) $\hookrightarrow \prod_{n\in N} X$ are continuous. A basis of the zeroneighbourhood (0-nbhd) filter of h(X) is given by

$$\lambda(U) := \{(x_k)_{k\in N} \in \lambda(X) : \|(p_U(x_n))_{n\in N}\| \leq 1\}$$

for $U = \Gamma U \in \mathcal{U}_0(X)$, where $\mathcal{U}_0(X)$ denotes the set of all 0-nbhds in X. h(X) is Hausdorff, (semi-)metrizable, (semi-)normable or complete if and only if X has the same property. For Ics X and Y with a continuous linear map g : Y $\to$ X the continuous map $\tilde{g} : (x_n)_{n\in N} \mapsto (g(x_n))_{n\in N}$ maps h(Y) continuously into h(X). If X is a reflexive Fréchct space and λ is reflexive then also h(X) is reflexive.

For the inductive limit of Moscatelli type. we have the continuous inclusions $\lambda(Y) \hookrightarrow E(Y \hookrightarrow X, \lambda) \hookrightarrow$ h(X) and the continuous linear mapping

$$\Phi : \bigoplus_{k\in N} X \times \mathrm{h(Y)} \longrightarrow E(Y \hookrightarrow X, \lambda)$$

$$((x_k)_{k\in N}, (y_k)_{k\in N}) \mapsto (x_k + y_k)_{k\in N}$$

which is also surjective and open [24, page 176], such that we will write

$$E(Y \hookrightarrow X, \lambda) = \bigoplus_{k\in N} X + \lambda(Y)$$

with the meaning above behind and get a description of the basis of the 0-nbhd filter by

$$\{\bigoplus_{n\in N} V_n + V : V_n \in \mathcal{U}_0(X), V \in \mathcal{U}_0(\lambda(Y))\}$$

For projective limits of Moscatelli type $F := F(Y \xrightarrow{f} X, \lambda)$ we will also use the following representation that F is topologically isomorphic to

$$H := \{(y_n)_{n\in N} \in Y^N : (f(y_n))_{n\in N} \in \lambda(X)\}$$

provided with the initial topology with respect to the inclusion $H \hookrightarrow \prod_{n\in N} Y$ and $H \to \lambda(X)$, $(x_n)_{n\in N} \mapsto (f(x_n))_{n\in N}$, see [[]Proposition 3.1 .1.]Kh. A basis of the 0-nbhd filter is given by

$$\prod_{n<n} V \times \{(y_k)_{k\geq n} \in \lambda((X)_{k\geq n}) : \|((0)_{k<n}, (p_U(f(y_k)))_{k\geq n})\| \leq 1\}$$

with $U = \Gamma U \in \mathcal{U}_0(X)$, $V \in \mathcal{U}_0(Y)$ and $n \in$ N. The spaces F_n are Hausdorff, metrizable, normable, complete if und only if X and Y have the same property. $F(Y \xrightarrow{f} X, \lambda)$ is Hausdorff, metrizable or complete if X and Y have the same property. If we replace X by $f(Y)$ we get the same space F. such that we can suppose f to have a dense range orto be surjective.

Now we introduce some Baire properties. These are well known and a complete description of them can be found in [29,9].

0.3 Definition.

Let $(E, \Im)$ be a lcs. Then $(E, \Im)$ is called

i) barrelled, if every barrel U in $(E, 3)$ is a 0-nbhd in $(E, \Im)$.

ii) quasi-Baire, if $(E, \Im)$ is barrelled and E is not the union of an increasing sequence of nowhere dense linear subspaces of $(E, \Im)$.

iii) Bairelike, if one of the following equivalent conditions holds:

a) If $(A_n)_{n\in N}$ is an increasing sequence of closed absolutely convex subsets of E, such thnt it's union spani E, than there exists an $n \in$ N such that $A_n \in \mathcal{U}_0(E, \Im)$.

b) E cannot be covered by an increasing sequence of rare absolutely convex subsets of E.

c) E is not the union of an increasing sequence of nowhere dense absolutely convex subsets.

iv) (db)-space, if the following is true:

If E is the union of an increasing sequence of subspaces $(E_n)_{n\in N}$, then there is an $n \in$ N such that E_n is dense in $(E, 3)$ and $(E_n, \Im \cap E_n)$ is barrelled.

v) unordered Bairelike, if E is not the union of a sequence of nowhere dense absolutely convex subsets

vi) Baire space, if the countable intersection of open subsets which al-e dense in $(E, \Im)$ is dense in $(E, \Im)$.

The properties i) and vi) are classical, ii) and iii) are due to S. Saxon in [34], iv) also known as suprabarrelled, were independently introduced by W. J. Robertson. I. Tweddle, F. E. Yeomans in [31] and M. Valdivia in [41] and v) is due to A. R. Todd and S. Saxon in [39]. For further comments and historical notes see [29, Chapter 9.4]. The importance of the properties ii-v) arises in their applications e.g. in closed graph theorems [29, Chapter 9.1] or in measure theory, see [15] for details. Moreover they have good permanence properties. In the following remark we summarize some important properties of them. Pl-oofs and an intensive discussion can be found in [29]

0.4 Remark.

The properties i)-vi) are stable under quotients; moreover they are inherited by spaces from a dense linear subspace. The properties i)-v) are stable with respect to countable codimensional linear subspaces, the 3-space problem and arbitrary products. Furthermore the following implications are true:

Baire ⇒ unordered Bairelike ⇒ (db)-space

⇒ Bairelike ⇒ quasi-Baire ⇒ barrelled

[29, Chapters 9.1, 9.3] establishes all the implications and provides counterexamples, which show that none of these arrows can be reversed. For metrizable spaces I. Amemiya and Y. Kōmura showed in [4, Satz 1] the equivalence of countably barrelled and Bairelike. This theorem was extended by M. Valdivia in [40] to the statement, that every ℓ_∞-barrelled space whose completion is Bairelike is Bairelike, where a lcs E is called ℓ_∞-barrelled if every weakly bounded sequence in E' is E-equicontinuous [29, Definition 8.2.13]. Clearly "countably barrelled" implies "ℓ_∞-barrelled".

For LF-spaces one has the important result of P. P. Narayanaswami and S. Saxon [27, Th. 3] that an LF-space is Bairelike if and only if it is metrizable. On the other hand one has the following classification which can be found in [23, Corollary 7.2.10].

0.5 Proposition.

For an LF-space $(E, \Im) = \text{ind}_{n\in N}(E_n, \Im_n)$ the following are equivalent:

i) $(E, \Im)$ is a Fréchet space.

ii) $(E, \Im)$ is a Baire space.

iii) $(E, \Im)$ is a (db)-space.

iv) There is $n \in$ N such that $E_k = E_{k+1}$ for all $k \geq n$.

Altogether it is true that every metrizable incomplete LF-space is Bairelike but not a (db)-space. An example of an incomplete LF-space can be found in [11, page 285]. Now we prove an extension of Proposition 0.5 for the class of LF-spaces of Moscatelli type.

0.6 Proposition.

Let X and Y be Fréchet spaces with continuous inclusion $Y \hookrightarrow X$ and let $(\lambda, \|\ \|)$ be a nBss. Then for $E = E(Y \hookrightarrow X, \lambda)$ the following are equivalent:

i) E is a Fréchet space.

ii) E is a Baire space.

iii) Y = X

iv) E is Bairelike.

v) E is metrizable.

Proof:

The equivalence of i), ii) and iii) follows immediately by Proposition 0.5 and iv) ⇔ v) by the theorem of P. P. Narayanaswami and S. Saxon mentioned in Remark 0.4. As ii) ⇒ iv) holds in general, it suffices to show iv) ⇒ iii).

We suppose X $\neq$ Y, hence the inclusion Y $\hookrightarrow$ X is not nearly open. Then there exists a 0-nbhd. $U = \overline{\Gamma U}^Y$ in Y such that the closure $\overline{U}^X$ of U in X is not a 0-nbhd in X and hence $\overline{U}^\circ = \emptyset$. We put $\rho_n := \|(\delta_{kn})_{k\in N}\|$ and

$$A_n := \left(\prod_{k<n} X \times \prod_{k\geq n} \frac{k}{\rho_k} \overline{U}^X \right) \cap E$$

for all $n \in$ N. Then $(A_n)_{n\in N}$ is an increasing sequence of closed, absolutely convex and nowhere dense subsets of E. To get a contradiction there is only to show that $E = \bigcup_{n\in N} A_n$ holds. So let $x = (x_n)_{n\in N} \in E$ be given. Then there exists an $n \in$ N, such that $(x_k)_{k\geq n} \in \lambda((Y)_{k\geq n})$ and consequentely there is $m \in$ N with $\|((0)_{k<n}, (p_U(x_k))_{k\geq n})\| \leq m$. Let now $j \geq r := \max\{n, m\}$ be given. Then

$$p_U(x_j)\rho_j = \|(\delta_{jk} p_U(x_j))_{k\in N}\| \leq$$

$$\|((0)_{k<n}, (p_U(x_k))_{k\geq n})\| \leq m \leq r \leq j$$

such that $x_j \in \frac{j}{\rho_j} U \subset \frac{j}{\rho_j} \overline{U}^X$ and thus $x \in A_r$. Since x was arbitrary, we are done.

1. QUASI-BAIRE SPACES

Our next aim in this section is to investigate quasi-Baire spaces. For this purpose it is quite useful to treat the following property "not S," separately, which is a part of the quasi-Baire property and which is in fact due to [37] where a connection to the separable quotient problem is given.

1.1 Definition.

A lcs E has S_σ, if there is a strictly increasing sequence of closed linear subspaces $(E_n)_{n\in N}$ in E, such that $E = \bigcup_{n\in N} E_n$.

We say that E contains φ (complemented), if there is a (complemented) subspace $L \subset E$, which is topologically isomorphic to φ.

From the definition of S_σ and quasi-Baire the following equivalence follows directly

i) E is quasi-Baire.

ii) E is barrelled and has not S_σ.

An important characterization of quasi-Baire spaces is the following proposition.

1.2 Proposition.

For a barrelled lcs E the following are equivalent:

i) E is not quasi-Baire.

ii) E contains φ complemented

iii) E contains a closed subspace of countably infinite codimension.

The equivalence i $\Leftrightarrow$ ii) is due to J. Bonet and P. Pérez Carreras in [9, Lemma 1] and the whole equivalence can be found in [27, Theorem 1]. For general lcs the following result holds.

1.3 Proposition.

Let $(E, \Im)$ be a lcs. Then the following are equivalent:

i) $(E, \Im)$ has S,.

ii) $(E, 3)$ contains a closed subspace of countably infinite codimension.

Proof:

"ii) $\Rightarrow$ i)"

Suppose E contains a closed subspace L of countably infinite codimension. Then E / L is Hausdorff and contains a strictly increasing sequence $(M_n)_{n\in N}$ of subspaces with $\dim M_n = n$ and $E / L = \bigcup_{n\in N} M_n$. Since E / L is Hausdorff the subspaces M_n are closed such that also $L_n := q^{-1}(M_n)$ is closed, where $q : E \longrightarrow E / L$ is the canonical quotient map. Furthermore the subspaces L,, are increasing and $E = \bigcup_{n\in N} L_n$ holds. Thus $(E, \Im)$ has S_σ.

"i) $\Rightarrow$ ii)"

Suppose that $(E, \Im)$ has S,. Then there is a strictly increasing sequence of closed subspaces $(L_n)_{n\in N}$ in $(E, \Im)$ with $E = \bigcup_{n\in N} L$,,. Thus we can choose a sequence $(x_n)_{n\in N}$ in E with x,, $\in L_{n+}$, $\setminus L_n$ for all $n \in$ N. With the theorem of Hahn-Banach we get that for all $n \in$ N there exists $f_n \in E'$ such that f_n L,, $= 0$ and $f_n(x_n) = 1$. Consequently $f : (E, \Im) \longrightarrow \omega, x \mapsto (f_n(x))_{n\in N}$ is welldefined, linear and continuous. Since $(L_n)_{n\in N}$ is increasing with $E = \bigcup_{n\in N} L_n$ it follows that $f(E)$ c φ and because of $f(x_n) = (f_1(x_n), \ldots, f_{n-1}(x_n), 1, 0, \ldots)$ for all $n \in$ N we have $f(E) = \varphi$. Thus ker$f$ is closed and of countably infinite codimension in $(E, \Im)$, which establishes ii). ❑

Putting Proposition 1.3 and [29, Proposition 8.2.16] together we obtain the following extension of Proposition 1.2.

1.4 Proposition.

Let $(E, \Im)$ be an ℓ_∞-barrelled Ics. Then the following are equivalent:

i) $(E, 3)$ has S,.

ii) $(E, 3)$ contains a closed subspace of countably infinite codimension.

iii) $(E, \Im)$ contains $(\varphi, \tau(\varphi, \omega))$ complemented.

In the following proposition we remark some permanence properties of S_σ-spaces

1.5 Proposition.

a) Arbitrary products of spaces without S_σ are without S_σ. The proof is exactly the same as the one for quasi-Baire given in [29, Theorem 9.2.6].

b) Let $(E, \Im)$, $(F, \Re)$ be lcs. $(E, \Im)$ without S_σ and let $f : (E, \Im) \longrightarrow$ (F, $\Re$) be continuous, linear and surjective. Then also $(F, \Re)$ is without S_σ.

Proof:

Let $(F_n)_{n\in N}$ be an increasing sequence of closed subspaces of $(F, \Re)$ with $F = \bigcup_{n\in N} F$,,. Then $E = \bigcup_{n\in N} f^{-1}(F_n)$ and since $(E, \Im)$ has not S_σ, there is $n \in$ N. such that $E = f^{-1}(F_n)$. Asf is surjective $F = F_n$ and hence $(F, \Re)$ is without S,. □

Consequently quotients of spaces without S_σ are without S_σ.

c) Let (E, 3) be a Ics and $L \subset E$ a dense linear subspace. If $(L, \Im \cap L)$ is without S_σ, then also $(E, \Im)$ is without S_σ.

Proof:

Let $(E_n)_{n\in N}$ be an increasing sequence of closed subspaces of $(E, \Im)$ with $E = \bigcup_{n\in N} E_n$. Then $(E_n \cap L)_{n\in N}$ is an increasing sequence of closed subspaces in $(L, \Im \cap L)$ with $L = \bigcup_{n\in N} E_n \cap L$. Thus $L = E_n \cap L$ for some $n \in$ N and since L is dense in $(E, \Im)$ it is E $:= E_n$. □

Of course the other direction is not true (see $\varphi \subset \omega$). As a consequence we get that the completion $\tilde{E}$ is without S_σ, if E is without S_σ.

d) Let $(E, \Im)$ = $\text{ind}_{n\in N}(E_n, \Im_n)$ be a countable inductive limit of lcs without S_σ. Then the following are equivalent:

i) $(E, \Im)$ is without S_σ.

ii) $E = E_n$ for some $n \in$ N.

Proof:

The implication i) ⇒ ii) follows directly from the definition of S_σ. To get ii) ⇒ i) let L be a closed subspace of countably codimension in $(E, \Im)$. Then $L \cap E_n$ is a countably codimensional subspace of E_n and since E_n is without S_σ it follows from Proposition 1.3 that $L \cap E_n$ is of finite codimension in E_n. Then $E_n = L \cap E_n \oplus M$ for some finite dimensional subspace $M \subset E_n$. From this we get

$$E = E_n = L \cap E_n + M \subset L + M$$

$$\subset L + M + \{0\} \subset L + M + L \subset L + M$$

Thus L is of finite codimension and we are done. □

Part d) of Proposition 1.5 is an extension of a result of [36, page 67] where d) is mentioned for LF-spaces. As an application of this to inductive limits of Moscatelli type we get the following corollary.

1.6 Corollary.

Let $(\lambda, \|\,\|)$ be a nBss, Y be a barrelled metrizable lcs and X be a quasi-Baire space with continuous inclusion Y $\hookrightarrow$ X. For the corresponding inductive limit of Moscatelli type $(E, \Im) = E(Y \hookrightarrow X, \lambda)$ the following are equivalent:

i) $(E, \Im)$ is quasi-Baire

ii) $\overline{Y}^X$ = X

Proof:

Since Y is metrizable and barrelled also h(Y) is barrelled [16, Corollary 6.3], such that $E_n = \prod_{k<n} X \times \lambda((Y)_{k\geq n})$ is barrelled for all $n \in$ N. Consequently $(E, \Im)$ is barrelled and thus $(E, \Im)$ is quasi-Baire if and only if $(E, \Im)$ is without S_σ. From Proposition 1.4 d) it follows that this holds if and only if E = $\overline{E_n}$ for some $n \in$ N. This holds if and only if $\overline{Y}^X$ = X, as it is immediately seen. □

In connection with Proposition 0.6 we get with the previous corollary the following examples which distinguishes between Bairelike and quasi-Baire in the class of LF-spaces of Moscatelli type.

1.7 Corollary.

Let $(\lambda, \|\,\|)$ be a nBss and $(E, \Im) := E(Y \hookrightarrow X, \lambda)$ be an LF-space of Moscatelli type. Then the following are equivalent:

i) $(E, \Im)$ is quasi-Baire but not Bairelike.

ii) $\overline{Y}^X = X$ and $Y \neq X$.

The following example shows that the properties to be barrelled or to be without S_σ are independent of each other.

1.8 Example.

a) The space φ is and example of a barrelled space which has S_σ.

b) An example of a countably barrelled space which is without S_σ and is not barrelled is the following. Let Y, X be Banach space\, such that $Y \subset X$ and the inclusion $Y \hookrightarrow X$ is continuous, has proper dense range but is not open onto the range. Furthermore we suppose that the inclusion $(X', \beta(X', X)) \hookrightarrow (Y', \beta(Y', Y))$ has dense range. Let $(F, \Im) = F(Y \hookrightarrow X, \ell_1)$ be the corresponding projective limit of Moscatelli type. Then $(F', \beta(F', F))$ is countably barrelled, without S_σ but not barrelled. Moreover F is not distinguished and the bidual $(F'', \beta(F'', F'))$ admits a continuous norm.

Proof:

As F is metrizable, $(F', \beta(F', F))$ is countably barrelled and not barrelled since F is not distinguished [6, Corollary 2.5]. As the inclusion $(X', \beta(X', X)) \hookrightarrow (Y', \beta(Y', Y))$ has dense range, it follows from Corollary 1.6 that the LF-space of Moscatelli type $E := E((X', \beta(X', X)) \hookrightarrow (Y', \beta(Y', Y)), \ell_\infty)$ is quasi-Baire and as the identity $id : E \longrightarrow (F', \beta(F', F))$ is continuous, it follows by Proposition 1.5 b) that $(F', \beta(F', F))$ is without S_σ. Since the inclusion $Y \hookrightarrow X$ has dense range, it follows by [6, Corollary 2.16] that $(F'', \beta(F'', F))$ admits a continuous norm. ❑

The hypothesis supposed in the b) are satisfied for example for $Y := (\ell_2, \|\ \|_2)$ and $X := (c_0, \|\ \|_\infty)$. The following example shows that there is even a weak metrizable space without S_σ which is not barrelled.

c) Let $(X, \Im)$ bea Fréchet space such that $(X', \sigma(X', X))$ is separable and let $Y \subset X'$ with $Y \neq X'$ a countably dimensional and dense linear subspace. Then $(X, \sigma(X, Y))$ is metrizable, without S_σ and not barrelled.

Proof:

As $Y \neq X'$ and $(X, \Im)$ is a Fréchet space the identity $id : (X, \sigma(X, Y)) \longrightarrow (X, \Im)$ is not continuous, such that $(X, \sigma(X, Y))$ cannot be barrelled. Since Y is countably dimensional $(X, \sigma(X, Y))$ is metrizable and without S_σ as $(X, \Im)$ is without S_σ. ❑

The hypothesis in c) are true for example for $(X, \Im) = (\ell_2, \|\cdot\|_2)$ and $Y = \varphi$. In this context we rema-k now some interesting connections between the property S_σ and the existence of continuous norms on the dual space.

1.9 Proposition.

Let $(X, \Im)$ be a lcs with S_σ, then $(X', \beta^*(X', X))$ does not admit a continuous norm.

Proof:

As $(X, \Im)$ has S_σ it follows from Proposition 1.3 that there exists a continuous, linear surjection $f : (X, \Im) \longrightarrow (\varphi, \sigma(\varphi, \varphi))$. Thus we get that the transpose $f^t : (\varphi, \sigma(\varphi, \varphi)) \longrightarrow (X', \sigma(X', X))$ is injective, linear and continuous. Since $(\varphi, \sigma(\varphi, \varphi))$ is metrizable and consequently quasibarrelled, we obtain the continuity of $f^t : (\varphi, \sigma(\varphi, \varphi)) \longrightarrow (X', \beta^*(X', X))$. As $(\varphi, \sigma(\varphi, \varphi))$ does not admit a continuous norm also $(X', \beta^*(X', X))$ does not admit a continuous norm. ❑

1.10 Remark.

The other direction in the previous proposition does not hold, since Y. Kōmura constructed in [22] a Fréchet space X which is not separable, but in which every bounded subset is separable. In particular for every $B = \Gamma B \subset X$ bounded, one has $\overline{[B]}^X \neq X$ and $(X', \beta(X', X)) = (X', \beta^*(X', X))$ does not admit a continuous norm. In fact, if $(X', \beta(X', X))$ admitted a continuous norm, there would exist a bounded subset $B = \overline{\Gamma B}^X \subset X$, such that $\bigcap_{n\in N} \frac{1}{n} B^\circ = \{0\}$, hence

$$X = \{0\}^\circ = \left(\bigcap_{n\in N} \frac{1}{n} B^\circ\right)^\circ = \overline{\Gamma \bigcup_{n\in N} nB}^X = \overline{[B]}^X$$

which is a contradiction.

Furthermore one cannot substitute $\beta^*(X', X)$ by $\beta(X', X)$ in Proposition 1.5, since $(\varphi, \|\cdot\|)$, where $\|\cdot\|$ is an arbitrary norm on φ, is a DF-space which has S_σ, but $((\varphi, \|\cdot\|)', \beta((\varphi, \|\cdot\|)', \varphi))$ is a Banach space. In the following proposition we presenta positive result for an equivalence.

1.11 Proposition.

Let $(X, \Im)$ be a locally complete space, such that the strong dual $(X', \beta(X', X))$ is a Fréchet space. (Such spaces are called df-spaces, see [12.4.].) Then the following are equivalent:

i) $(X, \Im)$ is without S_σ.
ii) $(X', \beta(X', X))$ admits a continuous norm.
iii) $(X', \beta^*(X', X))$ admits a continuous norm.

Proof:

Because of [Corollary 10.2.2] and Proposition 1.9 it is enough to prove i) $\Rightarrow$ ii). So we suppose $(X', \beta(X', X))$ not to admit a continuous norm. As $(X, \Im)$ is a df-space, $(X', \beta(X', X))$ is a Fréchet space without continuous norm, such that it follows from [18, Theorem 7.2.7] that $(X', \beta(X', X))$ contains ω as a topologica1 subspace. Since ω is minimal, it is a topologica1 subspace of $(X', \sigma(X', X))$ and even complemented in $(X', \sigma(X', X))$. Thus $(X, \Im)$ contains a closed subspace of countably infinite codimension and hence by Proposition 1.3 it has S_σ, which is a contradiction to i). ❑

As an application of Proposition 1.11 we can supplement Example 1.8 b) by more countably barrelled spaces without S_σ, which al-e not barrelled.

1.12 Proposition.

Let $(E, \Im)$ be a Fréchet space which is not distinguished, such that $(E'', \beta(E'', E))$ admits a continuous norm. Then $(E', \beta(E', E))$ is countably barrelled, without S_σ, but not barrelled and hence not quasi-Baire.

Proof:

Since $(E, \Im)$ is not distinguished, $(E', \beta(E', E))$ is not barrelled, but countably barrelled since $(E, \Im)$ is metrizable. As $(E'', \beta(E'', E'))$ admits a continuous norm from Proposition 1.11 it follows that $(E', \beta(E', E))$ is without S_σ. ❑

In the next example we will show that S_σ is neighter inherited from a space by its bidual, nor from the bidual by the origina1 space.

1.13 **Example.**

Let $(E, \Im) = E(\ell_2 \hookrightarrow c_0, \ell_2)$ be the LB-space of Moscatelli type with respect to $(\ell_2, \| \|)$ and $(c_0, \| \|_\infty)$. Then (E, 3) is without S_σ, but its bidual $(E'', \beta(E'', E'))$ has S_σ.

Proof:

Since ℓ_2 is dense in $(c_0, \| \|_\infty)$ it follows from Corollary 1.6, that (E, $\Im$) is quasi-Baire and hence without S_σ. Moreover follows from [6, Proposition 2.6] that the strong dual $(E', \beta(E', E))$ is topologically isomorphic to the projective limit of Moscatellli type $F(\ell_1 \hookrightarrow \ell_2, \ell_2) =: F$ with respect to $(\ell_1, \|\cdot\|_1)$ and $(\ell_2, \|\cdot\|_2)$, which isdistinguished by loc. at., whence $(E'', \beta(E'', E')) = (F', \beta(F', F))$ is an LB-space. Thus the identity $id : (E'', \beta(E'', E')) \longrightarrow (F', \beta(F', F))$ is a continuous, linear bijection between LB-spaces and hence open by the open mapping theorem for LF-spaces [29, Theorem 8.4.11]. Now it follows from [6, Proposition 2.31. that $(F', \beta(F', F))$ is topologically isomorphic to the inductive limit of Moscatelli type $E(\ell_2 \hookrightarrow \ell_\infty, \ell_2)$ with respect to $(\ell_2, \|\cdot\|_2)$ and $(\ell_\infty, \|\cdot\|_\infty)$, such that altogether $(E'', \beta(E'', E'))$ is topologically isomorphic to $E(\ell_2 \hookrightarrow \ell_\infty, \ell_2)$. But since ℓ_2 is not dense in $(\ell_\infty, \| \|_\infty)$, we get from Corollary l.6, that $E(\ell_2 \hookrightarrow \ell_\infty, \ell_2)$ has S_σ. ❑

Furthermore $(\varphi, \sigma(\varphi, \varphi))$ is an example of a space which has S_σ, but its biduel $(\omega, \sigma(\omega, \varphi))$ is a Baire space, hence without S_σ.

By [6, Proposition 2.14] a projective limit of Moscatelli type $F(Y \xrightarrow{f} X, \lambda)$ with respect to Banach spaces Y, X admits a continuous norm if und only iff is injective. In the next proposition we extend this characterization to Ics Y and X.

1.14 Proposition.

Let X, Y be lcs, $f : Y \longrightarrow X$ be a linear continuous map and Ih, $\| \|$) be an arbitrary nBss. Then for $F := F(Y \xrightarrow{f} X, \lambda)$ the following are equivalent:

i) $f(Y)$ admits a continuous norm and f is injective.

ii) F admits a continuous norm.

Proof:

We may assume that X $= f(Y)$. Let p be a continuous norm on $f($ Y). Then $\hat{p}(x) := \|(p(x_n))_{n\in N}\|$ is a continuous norm on h(X). Furthermore the map $\hat{f} : F \longrightarrow h(X)$, $(y_n)_{n\in N} \mapsto (f(y_n))_{n\in N}$ is continuous, linear and injective, such that we get altogether that $\hat{p} \circ \hat{f}$ is a continuous norm on F.

Now let p be a continuous norm on F. Since $W := \{y \in F : p(y) \leq 1\} \in \mathcal{U}_0(F)$, there is $n \in$ N and $U \in \mathcal{U}_0(X)$ such that

$$\{((0)_{k<n}, (y_k)_{k\geq n}) \in F : \|((0)_{k<n}, (p_U(f(y_k)))_{k\geq n})\| \leq 1\} \subset W.$$

Assume that there is $y \in Y \setminus \{0\}$, such that $f(y) = 0$. Then for all $(\alpha_k)_{k\in N} \in \varphi$ it is $((0)_{k<n}, (p_U(f(\alpha_k y)))_{k\geq n}) = 0$, hence $\{((0)_{k<n}, (\alpha_k y)_{k\geq n}) : (\alpha_k)_{k\in N} \in \varphi\}$ is a linear subspace of F which is contained in W, a contrudiction to $y \neq 0$. This proves the injectivity of f. Moreover p_U is a continuous seminorm on X, which is a norm. In fact, if $p_U(f(y)) = 0$ for some $y \in$ Y, then $\rho(\delta_{nk} y)_{k\in N} \in W$ for every $\rho \in K$. But since W contains no linear subspace different from {0}, we get $y = 0$. ❑

1.15 Remark.

Condition i) in Proposition 1.14 cannot be substituted by
[i′)] X admits a continuous norm and f is injective.
even if we assume $f(Y)$ to be dense in X. In fact it suffices to present a (metrizable) lcs Y with continuous norm, whose completion X := $\tilde{Y}$ does not admit a continuous norm (see Example 1.16). Then $F(Y \hookrightarrow X, \lambda)$ has a continuous norm, but X not.

1.16 Example.

There exists a Fréchet space X without continuous norm, which contains a dense linear subspace Z ⊂ X which admits a continuous norm.

Proof:

Let $\Im$ be the usual Fréchet space topology on ω and let $\wp$ be a Banach topology on ω. This is possible since ω and ℓ_∞ both have (algebraic) dimension card(R) [29, Theorem 2.2.4]. We denote by $\Im \wedge \wp$ the final locally convex topology on ω with respect to $(id : (\omega, \Im) \longrightarrow \omega)$ and $(id : (\omega, \wp) \longrightarrow \omega)$.

First of all we prove that $L := \overline{\{0\}}^{\Im\wedge\wp}$ is finite codimensional in ω. In fact from [12, Lemma 1] we obtain that

$$q : (\omega, \wp) \times (\omega, \Im) \longrightarrow (\omega, \wp \wedge \Im)(x, y) \mapsto x + y$$

is linear, continuous and open. As $(\omega, \wp) \times (\omega, \Im)$ is complete and metrizable, $(\omega, \Im \wedge \wp)$ is complete and semimetrizable. Now let M be an algebraic complement of L in ω. Then we get that for the quotient map $q_L : \omega \longrightarrow \omega / L$ the bijective restriction $q_L : (M, (\Im \wedge \wp) \cap M) \longrightarrow (\omega, \Im \wedge \wp) / L$ is a topological isomorphism. Thus $\omega = L \oplus M$ and since L is closed in $(\omega, \wp)$ and in $(\omega, \Im)$, $(\omega, \wp) / L$ and $(\omega, \Im) / L$ are also complete and metrizable. By the closed graph theorem we get that the identities $id : (\omega, \wp) / L \longrightarrow (\omega, \Im \wedge \wp) / L$ and $id : (\omega, \Im) / L \longrightarrow (\omega, \Im \wedge \wp) / L$ are topologically isomorphic, such that altogether

$$\Im / L = \wp / L = (\Im \wedge \wp) / L.$$

As $\wp / L$ is normed and $\Im / L$ is weak topologically, is follows that $(\omega, \Im) / L$ is finite dimensional.

Now we define $\Delta_- := \{(x, -x) \in \omega \times \omega : x \in \omega\}$ and $Z := \Delta_- + M \times M$ and we prove, that Z is dense in X := $(\omega, \Im) \times (\omega, \wp)$. So let $(a, b) \in \omega \times \omega$, $U \in \mathcal{U}_0(\omega, \wp)$ and $V \in \mathcal{U}_0(\omega, \Im)$ be given. Since q is a quotient map, $U + V$ is a 0-nbhd in $(\omega, \Im \wedge \wp)$. As M is dense in $(\omega, \Im \wedge \wp)$, we get that $(a + b + U + V) \cap M \neq \emptyset$. Thus we can find $z \in M$, $u \in U$, $v \in V$ with $z = a + b + u + v$. Consequently

$$(a, b) = (a + u + z, -(a + u + z)) + (-z, 2z)$$

and hence $((a, b) + U \times V) \cap Z \neq \emptyset$.

Finally we have to show that there exist a continuous norm on Z, but not on X. Since $(\omega, \Im)$ is a complemented subspace of X, there can't exist a continuous norm on X. Now let $p : X \longrightarrow [0, \infty[, (x, y) \mapsto \|y\|_\wp$, where $\| \|_\wp$ is a continuous norm on $(\omega, \wp)$. It is easy to see that $p|_{\Delta_-}$ is a continuous norm. As we have that $(M, \Im \wedge \wp \cap M)$ is Hausdorff and finite dimensional, there exists a continuous norm $\|\ \|$ on $(M, \Im \wedge \wp \cap M)$. Thus there is a 0-nhhd $\Gamma V = V$ in $(\omega, \Im \wedge \wp)$ with $V \cap M \subset \{x \in M : \|x\| \leq 1\} =: U$. Consequently p_V is a continuous seminorm on $(\omega, \Im \wedge \wp)$ and since $p_U(x) = \|x\|$ forali $x \in M$, it is a continuous norm on $(M, 3 \wedge \wp \cap M)$. Thus $p_V \circ q$ is a continuous seminorm on X and hence also

$$l : X \longrightarrow [0, \infty[(x, y) \mapsto (p_V \circ q)(x, y) + p(x, y)$$

is a continuous seminorm on X. Now we will show that it is a continuous norm on Z. Clearly it is a continuous seminorm on Z. So let $(0.0) \neq (x, y) \in Z$ he given. Then there exists $(u, v) \in M \times M$ und $(w, -w) \in \Delta_-$ with $(x, y) = (u, v) + (w, -w)$. If $(u, v) = (0, 0)$ we get $w \neq 0$ and hence $l(x, y) \geq p(w, -w) > 0$. If $(u, v) \neq (0, 0)$ we may assume $(u, v) \notin$ A- such that $q(x, y) = u + v \neq 0$ and hence $l(x, y) \geq p_V(u + v) > 0$. Thus l is a continuous norm on Z and we are done. □

2. PROJECTIVE LIMITS

In this section we examine the stability of the Baire properties defined in Definition 0.3 under projective limits. As every complete Ics is the projective limit of Banach spaces, these properties are not inherited hy projective limits in general. But even for countable projective limits this is not true, since W. Roelke et. al. [32] constructed examples of countable projective limits of separable, normable Baire spaces, which are not barrelled. Analogous to these examples we construct in the following example countable projective limits of Baire spaces, which has S_σ and are not ℓ_∞-barrelled.

2.1 **Example** (cf. **[5,** 2.11)

Let $(X, \Im)$ be a Hausdorff Ics containing a dense linear subspace Y of countably infinite codimension, such that $(Y, \Im \cap Y)$ is a Baire space.

Such a space can be found in every infinite dimensional Hausdorff locally convex Baire space X. In fact, let X be an arbitrary locally convex Baire space of infinite dimension. We choose a linear subspace Z io X, such that $\dim X / Z$ is countably infinite and a cobasis $(x_n)_{n \in N}$ of Z in X. Thus X = $\bigcup_{n \in N} L_n$, where $L_n = Z + [x_1, \ldots, x_n]$ for all $n \in$ N, whence L_n is not meagre in X for a suitable $n \in$ N. Consequently L_n is a dense Baire subspace of X, cf. [32,1].

Let again $(x_n)_{n \in N}$ he a cobasis of Y in X. Then for every $n \in$ N the topology $\mathcal{D}_n$ of the topological direct sum $(Y + [x_m : m > n]) \oplus [x_1, \ldots, x_n]$ is a Ics Baire space topology on X und $\mathcal{D}_{n+1} \supset \mathcal{D}_n$. The projective limit of the projective sequence $((X, \mathcal{D}_n)_{n \in N}, (id_X)_{m \geq n})$ of Baire spaces is topologically isomorphic to X provided with the supremum topology $\mathcal{D} := \bigvee_{n \in N} \mathcal{D}_n$, which has S_σ, hecause Y is a closed linear subspace of $(X, \mathcal{D})$ of countably infinite codimension.

Furthermore (X, $\mathcal{D}$) is not ℓ_∞-barrelled. For each $n \in$ N there is $\Psi_n \in (X, \mathcal{D}_n)'$, such that $\Psi_n |_{(Y+[x_m : m > n])} = 0$ and $\Psi_n(x_j) = 1$ for all $1 \leq j \leq n$. Then clearly $\{\Psi_n : n \in \mathrm{N}\} \subset$ (X, $\mathcal{D})'$ is pointwise bounded, but not equicontinuous. In fact the space $L := [x_m : m \in N]$ provided with $\mathcal{D}$ n L is topologically isomorphic to $(\varphi, \sigma(\varphi, \varphi))$ (transitivity of the initial topology), such that for all $x = \sum_{n \in N} \alpha_n x_n \in L$ it is true that $\lim_{n \to \infty} \Psi_n(x) = \sum_{n \in N} \alpha_n$ and $(1)_{n \in N} \notin \varphi$ and thus we are done.

Now we examine a special class of projective limits. which are called strict.

2.2 Definition.

We call a projective system of lcs $((X_i,\Im_i)_{i\in I},(f_{ki})_{k<i})$ strict, if the linking maps f_{ki} are open for all $k<i$, $i\in I$.

For countable I this definition coinsides with the definition of a strict projective sequence $((E_n)_{n\in N},(p_{n,m})_{m\geq n})$ of [13]. In the following we present some connections between projective systems with open linking maps and such with open projections

2.3 **Remark**

Let $(X,\Im)=\text{proj}_{i\in I}((X_i,\Im_i),(f_{ki})_{k<i})$ be the projective limit of a projective system of lcs $(X_i,\Im_i)_{i\in I}$. We consider the following statements:

i) the canonica1 projections $pr_i:(X,\Im)\longrightarrow(X_i,\Im_i)$ are open for all $i\in I$.

ii) the linking maps $f_{ki}:(X_i,\Im_i)\longrightarrow(X_k,\Im_k)$ are open for all $k<i$ with $i,k\in I$.

Then i) $\Rightarrow$ ii) is always true whereas ii) $\Rightarrow$ i) is true if the projections pr_i are surjective for all $i\in I$.

Proof:

i) $\Rightarrow$ ii)

Let $k,i\in I$ with $k<i$ be given. Furthermore let U be a 0-nbhd in $(X_i,\Im_i)$. As pr_i is continuous, we get $pr_i^{-1}(U)\in\mathcal{U}_0(X,\Im)$ and since pr_k is open we get $pr_k(pr_i^{-1}(U))\in\mathcal{U}_0(X_k,\Im_k)$. Now let $y\in pr_i^{-1}(U)$ be given. As $pr_k(y)=f_{ki}(pr_i(y))$ and $pr_i(y)\in U$ we get $pr_k(y)$ ● ,fn,(U) and thus $f_{ki}(U)\supset pr_k(pr_i^{-1}(U))$. such that f_{ki} is open.

ii) $\Rightarrow$ i)

Let the canonical projections be surjective and the linking maps be open. Furthermore let $i\in I$ and $U\in\mathcal{U}_0(X,\Im)$ be given. As $\Im$ is the initial topology with respect to $(pr_j)_{j\in I}$ we obtain that there is $k\in I$ and $V\in\mathcal{U}_0(X_k,\Im_k)$ with $pr_k^{-1}(V)\subset U$. Without loss of generality we may assume $k\geq i$. As f_{ik} is open we get $f_{ik}(V)\in\mathcal{U}_0(X_i,\Im_i)$ and since $pr_i(pr_k^{-1}(V))=f_{ik}(pr_k(pr_k^{-1}(V)))$ we get with the surjectivity of pr_k that

$$\begin{aligned}pr_i(U)\supset pr_i(pr_k^{-1}(V))&=f_{ik}(pr_k(pr_k^{-1}(V)))\\&=f_{ik}(V)\in\mathcal{U}_0(X_i,\Im_i)\end{aligned}$$

and thus pr_i is open. ❑

It is easy to see that the linking maps f_{ik} are surjective if the canonical projections pr_i are surjective. For a projective sequence $(X,\Im)=\text{proj}_{n\in N}(X_k,\Im_k)$ of lcs the statements i) and ii) are equivalent. In fact it is enough to prove that the canonica1 projections are surjective. So let $y\in X_k$ be given. As the linking maps are surjective, we can find $x_{k+1}\in X_{k+1}$ with $f_{k,k+1}(x_{k+1})=x_k$. Inductively we obtain an $x\in X$ with $pr_n(x)=x_n=f_{n,n+1}(x_{n+1})$ for all $n\in N$ and altogether pr_k is surjective. This shows that the Definition 2.2 is compatible to the definition of a strict projective sequence given in [29, Definition 8.4.27], cf. also [13, page 549/550]. For an uncountable set I the statements i) and ii) are not equivalent, as the following example shows.

2.4 Example.

There exists a projective limit $Z = \text{proj}_{L\in K}(Y_K, p_{KL})$ of Banach spaces Y_K with open linking maps, such that the canonical projections are not surjective.

Proof:

Let X be a locally compact Hausdorff space which is not normal. Such a space can be found in [14, Example 3.3.14]. As X is not normal, there exists closed not empty subsets A, $B \subset X$ with $A \cap B = \emptyset$, such that for every open subset U and V in X with $A \subset U$ and $B \subset V$, we have $U \cap V \neq \emptyset$. Consequently there exists no continuous function $f : X \longrightarrow R$ with $f\,|_A = 1$ and $f\,|_B = 0$. For every compact subset $K \subset X$ we define

$$Y_K := \{f : K \longrightarrow R : f \text{ is continuous and } \exists_{c\in R} f\,|_{A\cap K} = c, f\,|_{B\cap K} = 0\}.$$

A standard proof shows that Y_K is a closed, linear subspace of $(C(K), \|\,\|_\infty)$ and hence $(Y_K, \|\,\|_\infty)$ is a Banach space for every compact subset $K \subset X$.

Now let $I := \{K \subset X : K \text{ is compact}\}$. I is directed by inclusion "C". Furthermore for every $K, L \in I$ with $K \subset L$ the map

$$p_{KL} : (Y_L, \|\,\|_\infty) \longrightarrow (Y_K, \|\,\|_\infty) f \longmapsto f\,|R$$

is welldefined, linear and continuous as it is norm decreasing. Moreover $p_{LL} = id$ and for $K \subset L \subset M$ with $K, L, M \in I$ it is true that $p_{KL} \circ p_{LM} = p_{KM}$.

The maps p_{KL} are surjective. In fact, let $f : K \longrightarrow R$ be a continuous map with $f\,|_{A\cap K} = c$ and $f\,|_{B\cap K} = 0$ for $c \in R$. Then the function

$$g : K \cup (A\cap L) \cup (B \cap L) \longrightarrow R x \mapsto \begin{cases} f(x) & , x \in K \\ c & x \in A \cap L \\ 0 & x \in B \cap L \end{cases}$$

is welldefined and continuous [Proposition 2.1.13]. As L is normal und $K \cup (A\cap L) \cup (B \cap L)$ is closed in L it follows by the extension theorem of Tietze-Urysohn [14, Theorem 2.1.8], that there exists a continuous extension $h : L \longrightarrow R$ of g. Thus $h \in Y_L$ and $p_{KL}(h) = f$, such that p_{KL} is surjective.

As $(Y_L, \|\cdot\|_\infty)$ and $(Y_K, \|\cdot\|_\infty)$ are Banach spaces we get from the open mapping theorem, that p_{KL} is also open. Altogether the spaces $(Y_K, \|\,\|_\infty)$ with the linking maps p_{KL} for $K \subset L$ with $K, L \in I$ define a projective system of Banach spaces with open linking maps.

Finally we show that the canonical projections $Z := \text{proj}_{L\in I}((Y_L, \|\,\|_\infty), (p_{KL})_{K\subset L}) \longrightarrow (Y_K, \|\,\|_\infty)$ are not surjective. So let $a \in A$ and $b \in B$. As X is Hausdorff $K := \{a, b\}$ is compact in X and

$$h : K \longrightarrow R \qquad \begin{matrix} a \longmapsto 1 \\ b \longmapsto 0 \end{matrix}$$

is welldefined and continuous with $h\,|_{A\cap K} = 1$ and $h\,|_{B\cap K} = 0$, such that $h \in Y_K$. Suppose that the canonical projection $p_K : Z \longrightarrow Y_K, (f_L)_{L\in I} \mapsto f_K$ is surjective. Thus there exists $(f_L)_{L\in I} \in Z$ with $f_K = h$. Clearly

$$f : X \longrightarrow R x \longmapsto f_L(x) \text{for} x \in L$$

is welldefined and also continuous, as X is locally compact. It is easy t" see that $f|_A = 1$ and $f|_B = 0$. But this is a contradiction to the choice of A and B. ❑

We would like t" remark that Z is topologically isomorphic to C(X) with the compact open topology, cf. also [18, Proposition 3.6.3], where for a completely regular Hausdorff space X such a projective limit is contructed in order t" represent the completion of C(X) with the compact open topology.

In the following example we present a projective limit of Banach spaces with open canonical projections, which is not barrelled. Thus in general we cannot expect "good" permanence properties for strict projective limits.

2.5 Example (cf. **[5, 2.2]**.

Let I be an uncountable set. We consider the projective system of Banach spaces $(\ell_1(J))_{J\subset I, J \text{countable}}$ with respect to the natural projections

$$\ell_1(K) \longrightarrow \ell_1(J)(x_i)_{i\in K} \longmapsto (x_i)_{i\in J}(J \subset K \subset I, \text{K countable})$$

as linking maps. Let 3 denote the initial topology on $\ell_1(I)$ with respect t" the natural projections $p_J : \ell_1(I) \longrightarrow (\ell_1(J), \| \|_1)(J \subset I\text{countable})$. According t" [8] and [IO], $(\ell_1(I), \Im)$ is a complete Hausdorff DF-space, which is not barrelled and whose strong dual is a Banach space. Now

$$\eta : (\ell_1(I), \Im) \longrightarrow X := \text{proj}_{J\subset I, \text{co un table}}(\ell_1(J), \| \|_1)$$

$$(x_i)_{i\in I} \longmapsto ((x_i^J)_{i\in J})_{J\subset I, \text{countable}}$$

is welldefined, linea and injective. Because of the transitivity of initial topologies it is also continuous and open onto the range. Now we show thut the range is dense in X. So let $x = ((x_i^J)_{i\in J})_{J\subset I, \text{countable}} \in$ X and $U \in \mathcal{U}_0(X)$ be given. Then there exists a countable subset $J_0 \subset I$ with $\ker pr_{J_0} \subset U$ for the canonical projections pr_J : X $\longrightarrow (\ell_1(J), \| \|_1)$. Now let $z := ((x_i^{J_0})_{i\in J_0}, (0)_{i\in I\setminus J_0})$. Then $z \in \ell_1(I)$ und

$$pr_{J_0}(\eta(z) - x) =$$

$$pr_{J_0}((x_i^J)_{i\in J_0\cap J}, (0)_{i\in I\setminus J_0})_{J\subset I, \text{co un table}}) - pr_{J_0}(((x_i^J)_{i\in J})_{J\subset I, \text{coun table}})$$

$$= (\text{A t a}, \ - (x_i^{J_0})_{i\in J_0}$$

$$= (0)_{i\in J_0}$$

Thus $\eta(z) - x \in \ker pr_{J_0} \subset U$ and consequently $\eta(\ell_1(I))$ is dense in X. Altogether η is a surjective topological isomorphism and hence X is not barrelled.

Moreover for every countable subset $J \subset I$ the natural canonical projection $p_J : (\ell_1(I), 3) \longrightarrow (\ell_1(J), \|\cdot\|_1)$ is open, as $\Im$ is coarser than the $\|\cdot\|_1$-topology on $\ell_1(I)$ and since $(\ell_1(I), \|\cdot\|_1)$ is a Banach space we obtain from Proposition 1.5 b) that also $(\ell_1(I), \Im)$ is without S_σ.

In contrast t" Example 2.5 we get positive results for strict projective sequences. But before proving them we need some technical preparation.

2.6 Proposition.

Let $(X_n, \Im_n)_{n\in N}$ be a projective sequence of Ics with respect to linear continuous linking maps $p_{n,n+1} : (X_{n+1}, \Im_{n+1}) \longrightarrow (X_n, \Im_n)$, let $(X, \Im) := \mathrm{proj}_{n\in N}(X_n, \Im_n)$ be the corresponding projective limit and for all $n \in$ N let p_n : X $\longrightarrow$ X,, denote the canonical projection. If $X = \bigcup_{n\in N} A$,,, where each A,, is a closed balanced subset of $(X, \Im)$, then there are $m, n \in$ N, such that ker$p_m \subset A$,,.

Proof:

For each $n \in$ N let $\mathcal{D}_n$ denote the discrete topology on X_n, and let $\mathcal{D}$ be the initial topology on X with respect to $(p_n$: X $\longrightarrow (X_n, \mathcal{D}_n))_{n\in N}$. Then the embedding $(X, \mathcal{D}) \hookrightarrow \prod_{n\in N}(X_n, \mathcal{D}_n)$ is a linear homomorphism onto a closed subset of $\prod_{n\in N}$ (X,,, $\mathcal{D}_n$), which is completely metrizable as a countable product of completely metrizable spaces. Thur $(X, \mathcal{D})$ is a Baire space.

As $\mathcal{D} \supset \Im$ all A,, are closed in (X, $\mathcal{D}$). Therefore there are $n \in N, x \in$ A,, and $m \in$ N, such that x + ker$p_m \subset A$,,. Let $y \in$ kerp_m be given. Then for all $k \in N$ it is true that $x + ky \in x +$ ker$p_m \subset A$,,, hence, as A,, is balanced we get $\frac{1}{k}x + y = \frac{1}{k}(x + ky) \in A$,,. As $\lim_{k\to\infty}(\frac{1}{k}x + y) = y$ in $(X, \Im)$ and as A,, is closed in $(X, \Im)$, we obtain that $y \in A$,,, which proves that ker$p_m \subset A$,,. □

For the following two results see also [5, Proposition 2.3].

2.7 **Lemma.**

Let $(X_i, \Im_i)_{i\in I}$ be a projective system of Ics with respect to linear continuous and open maps $p_{ik} : (X_k, \Im_k) \longrightarrow$ (X,, $\Im_i)(i \leq$ k), let $(X, \Im) := \mathrm{proj}_{i\in I}(X_i, \Im_i)$ be the corresponding projective limit and assume that for all $i \in I$ the canonical projection p_i : $(X, 3) \longrightarrow (X_i, \Im_i)$ is open (see Remark 2.3). Let $(A_n)_{n\in N}$ be an increasing sequence of absolutely convex closed sets in $(X, \Im)$, such that X $= \bigcup_{n\in N}$ A,,. Furthermore assume that

(1) for each $i \in I$, there is *n(i)* $\in N$ such that $\overline{p_i(A_{n(i)})}^{(X_i,\Im_i)}$ is a 0-nbhd in (X,, $\Im_i$) and
(2) there are $n \in$ N and $i \in I$ such that ker$p_i \subset A$,,.

Then there is $m \in$ N. such that A,,, is a O-nbhd in $(X, \Im)$.

Proof:

Let n, i be as in (2). By (1) there is $m \in$ N such that $\overline{p_i(A_m)}^{(X_i,\Im_i)} \in \mathcal{U}_0(X_i, 3,)$. As the $(A_k)_{k\in N}$ are increasing, we may assume that $n \leq m$, whence ker$p_i \subset A$,,,. *Now C* := $\overline{A,,, + \mathrm{ker}p_i}^{(X,\Im)}$ is a closed subset of $(X, 3)$ satisfying C = C+ ker$p_i = p_i^{-1}(p_i(C))$. Since p_i is open, $p_i(C)$ is a closed subset of (X,, 3,). Moreover as A,,, $\subset$ C, we obtain that $\overline{p_i(A_m)}^{(X_i,\Im_i)} \subset p_i(C)$, whence $p_i(C) \in \mathcal{U}_0(X_i, \Im_i)$ and C = $p_i^{-1}(p_i(C)) \in \mathcal{U}_0(X, \Im)$. Finally C $\subset \overline{A_m + A_m}^X \subset 2A_m$, which implies that $2A_m$ and hence A,,, are 0-nbhds in $(X, \Im)$. □

From Proposition 2.6 and Lemma 2.7 we immediately obtain the following proposition.

2.8 Proposition.

Let $((X_n, \Im_n)_{n\in N}, (p_{n,n+1})_{n\in N})$ be a strict projective sequence of Ics and let $(X, 3) := \mathrm{proj}_{n\in N}(X_n, \Im_n)$. Then the following statements hold:

a) All $(X_n, \Im_n)$ are barrelled if and only if $(X, \Im)$ is barrelled.
b) All $(X_n, \Im_n)$ are without S_σ if and only if $(X, 3)$ is without S_σ.
c) All $(X_n, \Im_n)$ are quasi-Baire if and only if $(X, \Im)$ is quasi-Baire.
d) All $(X_n, \Im_n)$ are Bairelike if and only if (X, $\Im$) is Bairelike.

Proof:

As the canonical projections are open, only "$\Rightarrow$" needs a proof in (a)-(d).

a) If A is a barrel in $(X, \Im)$, put $A_n := nA$ for all $n \in$ N. Then by Proposition 2.6 condition (2) in Lemma 2.1 is satisfied and condition (1) of Lemma 2.1 follows from the barrelledness of the $(X_n, \Im_n)(n \in N)$, such that mA and hence A is a 0-nbhd in $(X, \Im)$. Thus (X, 3) is barrelled.

b) If X is the increasing union of closed subspaces $A_n(n \in N)$, then again by Proposition 2.6 condition (2) in Lemma 2.7 is satisfied and condition (1) of Lemma 2.7 is true as all $(X_n, \Im_n)$ are without S_σ. By Lemma 2.7 we obtain that there is $m \in$ N, such that A,,, is a 0-nbhd and hence we get that $(X, \Im)$ is without S_σ.

c) This follows immediately by (a) and(b).

d) If X is the increasing union of closed absolutely convex subsets, we obtain by Proposition 2.6 that condition (2) in Lemma 2.1 is satisfied and as all (X,,, $\Im_n$) are Bairelike also condition (1) in Lemma 2.7 is true. Thus by Lemma 2.7 we get that $(X, \Im)$ is Bairelike.

□

Unfortunately we cannot present an analogous result for "db" or "unordered Bairelike". The crucial point which we couldn't overcome is the following. If X = $\bigcup_{n \in N} A_n$ with A,, = $\Gamma A_n (n \in N)$ then the Proposition 2.6 yields $n, m \in$ N such that ker$p_m \subset A$,,. But we would need: there is $m \in$ N, such that $\bigcup_{ker\ p_m \subset A_n} p_m(A_n) = X_m$.

We also cannot present an analogous result for "Baire". In fact, we might start with $X = \bigcup_{n \in N} nC$, where C is a closed balanced absorbing subset of $(X, \Im)$, see [35, Theorem 1]. But from Proposition 2.6 and the proof of Lemma 2.1 we only obtain that $\overline{\bigcap_{\varepsilon > 0} C + \varepsilon C}^X$ is a 0-nbhd in $(X, \Im)$, whereby we have to show that C is a 0-nbhd in (X, $\Im$).

However we can present a negative result concering uncountable strict projective limits.

2.9 Example (cf. [5, 2.5]).

As in Example 2.5 let I be an uncountable set and moreover let $(\ell_1(I), 3) = \text{proj}_{J \subset I.\text{countable}}$ (P,(J), $\|\ \|_1$). For every $J \subset I$ countable let $\mathcal{D}_J$ denote the discrete topology on C,(J) and let $\mathcal{D}$ denote the initial topology on $\ell_1(I)$ with respect to the canonica1 projections $(pr_J : \ell_1(I) \longrightarrow ((\ell_1(J), \mathcal{D}_J)_{J \subset I.\text{countable}})$. Then in particular $(\ell_1(I), \mathcal{D})$ is an abelian topologica1 group which is the projective limit with respect to discret abelian topological groups, which are of cource complete metrizable, hence Baire.

We show that $(\ell_1(I), \mathcal{D})$ is not Baire. In fact, we know from Example 2.5 that $(\ell_1(I), \Im)$ is not barrelled. Therefor there exists a barrel A in $(\ell_1(I), \Im)$, which is not a 0-nbhd. Moreover $\ell_1(I) = \bigcup_{n \in N} nA$. If (X, $\mathcal{D}$) were Baire, we would obtein $n \in N, x \in nA$ and $J \subset I$ countable, such that x + ker$pr_J \subset nA$. As A is absorbing, there is $m \in$ N, such that $x \in mA$, whence ker$pr_J \subset (m + n)A$, which proves that condition (2) in Lemma 2.1 is satisfied. Condition (1) is also satisfied, as all $(\ell_1(J), \|\ \|_1)$ are Banach spaces. Consequently, Lemma 2.7 would yield that A is a 0-nbhd in $(\ell_1(I), \Im)$, a contradiction.

Thus we have proved, that the projective limit of discrete abelian topologica1 groups need not be Baire. Moreover an analogue to Proposition 2.X does not hold for arbitrary projective limits.

In contrast to what was said after Proposition 2.8, we can offer a positive result concerning (db). unordered Bairelike and Baire for countable strict projective limits of a special shape.

2.10 **Remark.**

Let $(Y, \Im)$ be a lcs and $(L_n)_{n\in N}$ be a decreasing sequence of closed linear subspaces, such that $\bigcap_{n\in N} L_n = \{0\}$. Then the sequence of quotient spaces $(Y, \Im) / L_n$ with the linking maps $q_{n,m} : Y/L_m \longrightarrow Y/L_n, x + L_m \mapsto x + L_n$ is a strict projective sequence. Let $(Z, \wp) := \text{proj}_{n\in N}((Y, \Im) / L_n, (q_{n,n+1}))$ be the corresponding projective limit. Moreover let $\mathcal{D}_n$ be the discrete topology on Y / L_n and $(Z, \mathcal{D}) := \text{proj}_{n\in N}((Y / L_n, \mathcal{D}_n), (q_{n,n+1}))$. Furthermore let

$$\eta : (Y, \Im) \longrightarrow (Z, \wp) y \longmapsto (y + L_n)_{n\in N} \text{ and}$$

$$h : (Y, \Im) \times (Z, \mathcal{D}) \longrightarrow (Z, \wp)(y, z) \longmapsto \eta(y) + z$$

a) h is open.

b) Let $i : (Y, \Im) \longrightarrow (Y \times Z, \Im \times \mathcal{D}) y \longmapsto (y, 0)$ and let $(B_n)_{n\in N}$ a sequences of closed absolutely convex subsets of $(Y \times Z, \Im \times \mathcal{D})$ with $Y \times Z = \bigcup_{n\in N} B_n$. Moreover let $J := \{n \in N : i^{-1}(B_n) \in \mathcal{U}_0(Y, \Im)\}$. If $(Y, \Im)$ is unordered Bairelike, then it is true that $Y \times Z = \bigcup_{\substack{n\in J\\ m\in N}} mB_n$.

Proof:

a) η is clearly welldefined, linear and continuous. It is injective as $\bigcap_{n\in N} L_n = \{0\}$.

It is immediate that h is welldefined, linear, continuous and surjective. To show that h is open let $U \in \mathcal{U}_0(Y, \Im)$ and $W \in \mathcal{U}_0(Z, \mathcal{D})$ be given. Then there is $n \in N$, such that $\ker p_n \subset W$ for the canonical projection $p_n : Z \longrightarrow Y / L_n$. It is true that

$$\eta(U) + \ker p_n = p_n^{-1}(q_n(U)). \, (*)$$

In fact, let $u \in U$ and $v \in \ker p_n$ be given. Then $\eta(u) = (q_k(u))_{k\in N}$ with the canonical quotient map $q_n : (Y, \Im) \longrightarrow (Y, \Im) / L_n$. This proves **"⊂"**. To get "⊃" let $w = (w_k)_{k\in N} \in Z$ such that $p_n(w) \in q_n(U)$ be given. Then there is $u \in U$ with $w_n = q_n(u)$ and hence $p_n(\eta(u) - w) = p_n((q_k(u))_{k\in N}) - q_n(u) = 0$. Thus $\eta(u) - w \in \ker p_n$ and hence $w \in \eta(U) + \ker p_n$, which proves $(*)$.

As q_n is open. $q_n(U) \in \mathcal{U}_0((Y, \Im) / L_n)$ and since p_n is continuous we get that $p_n^{-1}(q_n(U)) \in \mathcal{U}_0(Z, \wp)$. Thus we obtain

$$h(U \times W) = \eta(U) + W \supset \eta(U) + \ker p_n$$

$$= p_n^{-1}(q_n(U)) \in \mathcal{U}_0(Z, \wp)$$

and hence h is open.

b) Suppose this is not true. As for all $n \in N$ $\bigcup_{m\in N} mB_n$ is a linear subspace of $Y \times Z$ we obtain by [29, Lemma 9.1.32] that

$$Y \times Z = \bigcup_{\substack{n\in N\setminus J\\ m\in N}} mB_n$$

and hence

$$Y = \bigcup_{\substack{n\in N\setminus J\\ m\in N}} mi^{-1}(B_n).$$

As $(Y, \Im)$ is unordered Bairelike, there exists $m \in N, n \in N\backslash J$, such thut $mi^{-1}(B_n) \in \mathcal{U}_0(Y, \Im)$ and hence $i^{-1}(B_n) \in \mathcal{U}_0(Y, \Im)$. But this is a contradiction to $n \notin J$. □

2.11 Proposition.

Let $(Y, \Im)$ bea lcs and $(L_n)_{n\in N}$ bea decreasing sequence of closed linear subspaces of $(Y, \Im)$, such that $\bigcap_{n\in N} L_n = \{0\}$. Moreover let $q_{n,n+1} : Y / L_{n+1} \longrightarrow Y / L_n, x + L_{n+1} \longmapsto x + L_n$, and let $(Z, \wp) := \text{proj}_{n\in N}((Y, \Im) / L_n, (q_{n\ n+1}))$ be the corresponding strict projective limit. Then the following statements are true:

a) If $(Y, \Im)$ is a Baire space, then $(Z, \wp)$ is a Baire space. (cf. [2.4])
b) If $(Y, \Im)$ is a unordered Bairelike, then $(Z, \wp)$ is unordered Bairelike.
c) If $(Y, \Im)$ is a db-space, then $(Z, \wp)$ is a db-space.

Proof:

Let $(Z, \mathcal{D})$, h and η be as in Remark 2.10. Then $(Z, \mathcal{D})$ is a complete metrizable topological group and hence a Baire space.

a) If $(Y, \Im)$ is a Baire space, we obtain by [1, Theorem 4.2] and [28, 5.] that $(Y \times Z, \Im \times \mathcal{D})$ is a Baire space. As $h : (Y \times Z, \Im \times \mathcal{D}) \longmapsto (Z, \wp)$ is linear, continuous, what is immediate and open by Remark 2.10, this implies that also $(Z, \wp)$ is a Baire space.

b) Let $(A_n)_{n\in N}$ be a sequence of closed absolutely convex subsets of $(Z, \wp)$ with $Z = \bigcup_{n\in N} A_n$. As h is continuous and linear, $(h^{-1}(A_n))_{n\in N}$ is a sequence of closed absolutely convex subsets of $(Y \times Z, \Im \times \mathcal{D})$ with

$$Y \times Z = \bigcup_{n\in N} h^{-1}(A_n)$$

Let $i : (Y, \Im) \longrightarrow (Y \times Z, \Im \times \mathcal{D}), y \longmapsto (y, 0)$ and $J := \{n \in \mathbf{N} : i^{-1}(h^{-1}(A_n)) \in \mathcal{U}_0(Y, \Im)\}$. Then it follows from Remark 2.10 b), that

$$\bigcup_{\substack{n\in J\\ m\in \mathbb{N}}} mh^{-1}(A_n) = Y \times Z. \quad (*)$$

Now let $j : (Z, \mathcal{D}) \longrightarrow (Y \times Z, \Im \times \mathcal{D}), y \longmapsto (0, y)$. From $(*)$ we obtain

$$Z = \bigcup_{\substack{n\in J\\ m\in \mathbb{N}}} mj^{-1}(h^{-1}(A_m))$$

and as $(Z, \mathcal{D})$ is a Baire space, there exists $n \in J$ and $m \in \mathbf{N}$, such that the set $\overline{j^{-1}(h^{-1}(mA_n))}^{\mathcal{D}}$ has nonempty interior. Thus we get that the difference $mj^{-1}(h^{-1}(A_n)) - mj^{-1}(h^{-1}(A_n))$ is a 0-nbhd in $(Z, \mathcal{D})$. Consequently the set $2mj^{-1}(h^{-1}(A_n))$ $(A_n$ is absolutely convex) and as the multiplication $x \longmapsto ax, a \in K$ is continuous, also $j^{-1}(h^{-1}(A_n))$ is a 0-nbhd in $(Z, \mathcal{D})$. As $n \in J$ also $i^{-1}(h^{-1}(A_n)) \in \mathcal{U}_0(Y, \Im)$ and altogether it is $i^{-1}(h^{-1}(A_n)) \times j^{-1}(h^{-1}(A_n))$ a 0-nbhd in $(Y \times Z, \Im \times \mathcal{D})$. As

$$i^{-1}(h^{-1}(A_n)) \times j^{-1}(h^{-1}(A_n)) = i^{-1}(h^{-1}(A_n)) \times \{0\} + \{0\} \times j^{-1}(h^{-1}(A_n))$$
$$\subset h^{-1}(A_n) + h^{-1}(A_n) \subset 2h^{-1}(A_n)$$

we obtain that $2h^{-1}(A_n)$ and hence $h^{-1}(A_n)$ is a 0-nbhd in $(Y \times Z, \Im \times \mathcal{D})$. Since h is open this implies that $A_n = h(h^{-1}(A_n))$ is a 0-nbhd in $(Z, \wp)$.

c) Let $(Z_n)_{n\in N}$ be an increasing sequence of linear subspaces of Z, such that $Z = \bigcup_{n\in N} Z_n$. As $(Y, \Im)$ is a db-space, also $(Y, \Im) / L_n$ is a db-space and hence Bairelike for all $n \in$ N. Since (Z, $\wp$) is a strict projective limit, Proposition 2.X implies that $(Z, \wp)$ is Bairelike, such that there exists $n \in$ N with $\overline{Z_n}^{\wp} =$ Z. Therefore we may assume that $\overline{Z_n}^{\wp} =$ Z for all $n \in$ N. Now we must show that there exists $n \in$ N, such that $(Z_m, \wp \cap Z_m)$ is barrelled. Suppose this is not true. Then for all $n \in$ N there is a barrel C_n in $(Z_n, \wp \cap Z_n)$ such that $\overline{C_m}^{\wp} \notin \mathcal{U}_0(Z, \wp)$. Now let $Y_m := h^{-1}(Z_m)$ for all $n \in$ N. Thus $Y \times Z = \bigcup_{m\in N} Y_m$ and as h is continuous we get that also the restriction of h to Y_m, $\tilde{h}|_{Y_m} : (Y_m, \Im \times \mathcal{D}) \longrightarrow (Z_m, \wp \cap Z_m), y \longmapsto h(y)$ is continuous, such that $h^{-1}(C_m)$ is closed in $(Y_m, \Im \times \mathcal{D} \cap Y_m)$.

Let i and j as in b). Then $(i^{-1}(Y_m))_{m\in N}$ is an increasing sequence of linear subspaces of $(Y, \Im)$ with $Y = \bigcup_{m\in N} i^{-1}(Y_m)$ and we define $X_m := i^{-1}(Y_m) = i^{-1}(h^{-1}(Z_m))$ for all $m \in$ N. As i is continuous and $h^{-1}(C_m)$ is closed in $(Y_m, \Im \times \mathcal{D} \cap Y_m)$ also $i^{-1}(h^{-1}(C_m))$ is closed in $(X_m, \Im \cap X_m)$ for all $m \in$ N. Moreover $i^{-1}(h^{-1}(C_m))$ is absolutely convex and absorbing, hence a barrel in $(X_m, \Im \cap X_m)$ for all $m \in$ N. Since

$$Y = \bigcup_{n\in N} i^{-1}(h^{-1}(Z_m)) = \bigcup_{m\in N} X_m$$

and $(Y, \Im)$ is a db-space, we obtain that there exists $n \in$ N such that X_n is dense in $(Y, \Im)$ and $(X_n, \Im \cap X_n)$ is barrelled. Thus for all $m \geq n$ we have that $(X_m, \Im \cap X_m)$ is barrelled and X_m is dense in $(Y, \Im)$. Therefore $i^{-1}(h^{-1}(C_m))$ is a 0-nbhd in $(X_m, \Im \cap X_m)$ for all $m \geq n$. As $\overline{X_m}^{\Im} =$ Y we obtain for all $m \geq n$ that $\overline{i^{-1}(h^{-1}(C_m))}^{\Im} \in \mathcal{U}_0(Y, \Im)$. Moreover we have

$$Z = \bigcup_{\substack{m\geq n \\ k\in N}} kj^{-1}(h^{-1}(C_m))$$

and as $(Z, \mathcal{D})$ is a Baire space there is $m \geq n, k \in N$ such that $\overline{kj^{-1}(h^{-1}(C_m))}^{\mathcal{D}}$ has nonempty interior. As $C_m = \Gamma C_m$, it follows that

$$\overline{kj^{-1}(h^{-1}(C_m))}^{\mathcal{D}} - \overline{kj^{-1}(h^{-1}(C_m))}^{\mathcal{D}} \in \mathcal{U}_0(Z, \mathcal{D})$$

and hence $\overline{j^{-1}(h^{-1}(C_m))}^{\mathcal{D}} \in \mathcal{U}_0(Z, \mathcal{D})$. Altogether we obtain that

$$\overline{i^{-1}(h^{-1}(C_m))}^{\Im} \times \overline{j^{-1}(h^{-1}(C_m))}^{\mathcal{D}} \in \mathcal{U}_0(Y \times Z, \Im \times \mathcal{D})$$

and as

$$\overline{i^{-1}(h^{-1}(C_m))}^{\Im} \times \overline{j^{-1}(h^{-1}(C_m))}^{\mathcal{D}} \subset$$

$$\overline{i^{-1}(h^{-1}(C_m))}^{\Im} \times \{0\} + \{0\} \times \overline{j^{-1}(h^{-1}(C_m))}^{\mathcal{D}} \subset$$

$$\overline{h^{-1}(C_m)}^{\Im\times\mathcal{D}} + \overline{h^{-1}(C_m)}^{\Im\times\mathcal{D}} \subset \overline{2h^{-1}(C_m)}^{\Im\times\mathcal{D}}$$

also $2\overline{h^{-1}(C_m)}^{\Im\times\mathcal{D}}$ *and hence* $\overline{h^{-1}(C_m)}^{\Im\times\mathcal{D}}$ is a 0-nbhd in $(Y\times Z, \Im\times\mathcal{D})$. Since h is open by Remark 2.10 a) and continuous, $h\left(\overline{h^{-1}(C_m)}^{\Im\times\mathcal{D}}\right)\subset\overline{h(h^{-1}(C_m))}^{\wp}=\overline{C_m}^{\wp}$ such that $\overline{C_m}^{\wp}$ is a 0-nbhd in $(Z,\wp)$, which is a contradiction to the choice of C,,, and we are done. □

2.12 Remark.

We would like to remark that in the construction of the projective limit in Proposition 2.1 1 the "inclusion" $\eta:(Y,\Im)\longrightarrow \mathrm{proj}((Y,\Im)/L_n, q_{n,n+1}) =: (Z,\wp)$ need not be open onto the range or surjective. In fact let Y be a Baire subspace of ω containing φ such that dimw / Y is infinite (see Example 2. l or [32,1.]) provided with its relative topology 3 and let $L_n := \left(\prod_{k\le n}\{0\}\times\prod_{k>n}K\right)\cap$ Y. Then $L_n\cong$ K" and $(Z,\wp)=\omega$. The inclusion Y $\hookrightarrow\omega$ is topologically isomorphic onto the range but not surjective.

For another example let $(\lambda, \|\ \|)$ be a nBss and let L_n as above. Then it is true that (Z, $\wp$) $=\omega$ and the inclusion $(\lambda,\|\ \|)\hookrightarrow$ ω is not open onto the range, as ω admits no continuous norm and clearly not surjective.

We want to supplement this section about projective limits with a result about projective limits which need not he strict.

2.13 Proposition.

Let $((X_n)_{n\in N},(p_{n,m})_{m\ge n})$ be a projective sequence of df-spaces X,, (cf. Proposition 1. Il) without S_σ, let X := $\mathrm{proj}_{n\in N}X_n$ and p_n : X $\longrightarrow$ X,, be the canonical projections. We suppose that X is reduced and ℓ_∞- barrelled. Then X is without S_σ.

Proof:

Suppose X has S_σ. Then from Proposition l.4 it follows that X contains the space $(\varphi,\tau(\varphi,\omega))$ complemented, hence there exists a complemented linear subspace L in $(X',\sigma(X',X))$ which is topologically isomorphic to ω.

Moreover as X is reduced, all the transpose maps $p_n^t: X_n'\longrightarrow X'$ are injective. Consequently there exists a natural LF-space topology $\wp$ on X' which is finer than $\sigma(X',X)$. In fact $(X',\wp)=\mathrm{ind}_{n\in N}(X_n',\beta(X_n',X_n))$ and all (X,,, $\beta(X_n',X_n)$) are Fréchet spaces. By Grothendick's Theorem A [21, Theorem 19.5.4] $\wp$ is the ultrabornological topology associated with $\sigma(X',X)$ on X'. As ω is ultrabornological we obtain by the functorial property that $\wp\cap L=\sigma(X',X)\cap L$.

Now for all $n\in$ N it is $L_n := (p_n^t)^{-1}(L)$ a closed and hence Fréchet subspace of $(X_n',\beta(X_n', X_n))$ and $L=\bigcup_{n\in N}L_n$. The topology $\Re$ on L defined by $(L,\Re):=\mathrm{ind}_{n\in N}(L_n,\beta(X_n',X_n)\cap L_n)$ is an LF-space topology on L which is stronger than $\wp\cap L=\sigma(X',X)\cap L$ which is a Baire space. Again by Theorem A of Grothendieck there is $n\in$ N such that $p_n^t(X_n')\supset L$. By the barrelledness of L we obtain from the closed graph theorem that

$$p_n^t|_{(p_n^t)^{-1}(L)}: ((p_n^t)^{-1}(L),\beta(X_n',X_n)\cap(p_n^t)^{-1}(L))\longrightarrow(L,\wp\cap L)$$

is a topological isomorphism. We have proved that $(X_n',\beta(X_n',X_n))$ contains a copy of ω, hence does not admit a continuous norm, a contradiction to the fact that X_n is without S_σ (see Proposition 1.1 1). □

We complete this section with an application of Proposition 2.13 to a projective sequence which need not be strict.

2.14 **Example.**

Let $Z = \text{ind}_{n\in N} Z_n$ bea weakly acyclic LF-space (for a definition see [44, l. page 58]), such that each Z_n is a reflexive Fréchet space with continuous norm. Then the projective sequence $(X_n)_{n\in N} := (Z'_n, \beta(Z'_n, Z_n))_{n\in N}$, with respect to the natural transpose maps $Z'_m \longrightarrow Z'_n$ for $m \geq n$ consists of reflexive LB-spaces, which are without S_σ by Proposition 1.1 l. The projective limit X := $\text{proj}_{n\in N} X_n$ is reduced as the transpose of the inclusion $Z_n \hookrightarrow$ Z has weakly dense range, hence strongly dense range by the reflexivity of X,,. Moreover X is bornological by [44, Lemma 4.1] and [43, Theorem 5.6]. As X is also complete, we get that X is barrelled. From Proposition 2.13 we obtain that X is even quasi-Baire.

Easy examples of weak acyclic LF-spaces can be found in [26, Remark].

3. VECTOR VALUED SPACES

In this section we will investigate the behaviour of quasi-Baire and Bairelike with respect to the formation of vector valued sequence spaces $\lambda(X)$ and also of projective limits of Moscatelli type.

First of all we will present a quasi-Baire space E such that V,(E) has S_σ. For this purpose we start with a technical lemma. which is in fact partly known.

3.1 Lemma.

a) Let $(X_n)_{n\in N}$ be an inductive sequence of Ics and let X = $\text{ind}_{n\in N} X_n$. Then the following are equivalent:
  i) $\ell_\infty(X) = \bigcup_{n\in N} \ell_\infty(X_n)$
  ii) the inductive sequence $(X_n)_{n\in N}$ is regular.
  We owe this improved version to J. Wengenroth.
b) Let X. Y be Banach spaces with continuous inclusion $Y \hookrightarrow$ X and unit balls A in X and B in Y, respectively. Moreover let X be separable. If C,(Y) is dense in $\ell_\infty(X)$ then X = Y.
c) Let X. Y, G, H be Ics with continuous inclusions Y $\hookrightarrow$ X and G $\hookrightarrow$ H. If $\ell_\infty(Y \times G)$ is dense in $\ell_\infty(X \times H)$ then $\ell_\infty(Y)$ is dense in $\ell_\infty(X)$.
  Proof:
a) ii) $\Rightarrow$ i) is obvious. In order to prove i) $\Rightarrow$ ii) let $B \subset$ X be a bounded set. If B is not a subset of X,, for all $n \in$ N, there exists $x_n \in B \backslash X_n$ for all $n \in$ N. Then $(x_n)_{n\in N} \in \ell_\infty(X) \backslash \bigcup_{n\in N} \ell_\infty(X_n)$, a contradiction to i).
  So we may assume $B \subset X_1$. Now if B is unbounded in X,, for each $n \in$ N, we get that for all $n \in$ N there is $U_n \in \mathcal{U}_0(X_n)$, such thet for all $k \in$ N there exists $x_{nk} \in B \backslash kU_n$. Thus $((x_{nk})_{k\leq n})_{n\in N} \in \ell_\infty(X) \backslash \bigcup_{n\in N} \ell_\infty(X_n)$, which is a contradiction to i). This proves part a).
b) As X is separable there exists a sequence $(x_k)_{k\in N}$ in A with $\overline{\{x_k : k \in N\}} = A$. Thus $(x_k)_{k\in N} \in \ell_\infty(X)$ and as f,(Y) is dense in $\ell_\infty(X)$ we obtain that there exists $\rho > 0$, such that

$$\{x_k : k \in N\} \subset \rho B + \frac{1}{4} A$$

Thus $A = \overline{\{x_k : k \in N\}} \subset \rho B + \frac{1}{2} A$ and by induction we get that for all $k \in$ N it is true that

$$A \subset \frac{1}{2^k} A + (1 + \cdots + \frac{1}{2^{k-1}}) \rho B$$

From this it follows $A \subset \frac{1}{2^k}A + 2\rho B$ for all $k \in N$ and hence $A \subset 2\rho\overline{B}^X$. Consequently the inclusion $Y \hookrightarrow X$ is nearly open and by the open mapping theorem it is open, which implies X = Y.

c) Let $(x_k)_{k\in N} \in \ell_\infty(X)$ and $U \in \mathcal{U}_0(X)$ be given. Then $(x_k, 0)_{k\in N} \in \ell_\infty(X \times H)$ and as $\ell_\infty(Y \times G)$ is dense in $\ell_\infty(X \times H)$, there exists a bounded sequence $(y_k, z_k)_{k\in N}$ in $Y \times X$, such that $(x_k, 0) - (y_k, z_k) \in U \times H$ for all $k \in N$. Thus $x_k - y_k \in U$ for all $k \in N$ and altogether $\ell_\infty(Y)$ is dense in $\ell_\infty(X)$. □

3.2 Example.

Let Y, X be Banach spaces such that X is separable with continuous inclusion $Y \hookrightarrow X$ and unit balls B in Y and A in X, respectively. Moreover let $Y \subset X$ be a proper dense subset and B closed in X. For an arbitrary nBss $(\lambda, \| \|)$ it is true that $E := E(Y \hookrightarrow X, \lambda)$ is quasi-Baire but $\ell_\infty(E)$ has S_σ.

Proof:

As Y is dense in X we get by Corollary 1.6 that E is quasi-Baire. Moreover as B is closed in X, E is regular (see e.g. [24, Proposition 8]), hence we get by Lemma 3.1 a) $\ell_\infty(E) = \bigcup_{n\in N} \ell_\infty(E_n)$. Suppose $\ell_\infty(E_n)$ is dense in $\ell_\infty(E)$ for some $n \in$ N. Then by Lemma 3.1 c) we obtain that $\ell_\infty(Y)$ is dense in $\ell_\infty(X)$ and as X is separable we get from Lemma 3.1 b) that Y = X, which is a contradiction to the assumption. Thus $\ell_\infty(E)$ has S_σ. □

Spaces used in Example 3.2 are for example $Y = (\ell_2, \| \|_2)$ and $X = (\ell_3, \| \|_3)$. In contrast to tbis we get a positive result for projective limits of Moscatelli type for the property without S_σ, if X satisfies the following countable boundedness condition, see Definition 3.3.

3.3 Definition.

A lcs (X, 3) satisfies the countable boundedness condition, if for every sequence $(B_n)_{n\in N}$ of bounded sets there exists a sequence of scalars $(\alpha_n)_{n\in N}$ in $(0, \infty)$, such that $\bigcup_{n\in N} \alpha_n B_n$ is bounded

For example every metrizable lcs satisfies the countable boundedness condition.

3.4 Proposition.

Let X, Y be lcs, $f : Y \longrightarrow X$ be linear and continuous and we suppose that X satisfies the countable boundedness condition. Moreover let $(\lambda, \|\cdot\|)$ be a nBss with sectional convergence. If Y is without S_σ, then also the corresponding projective limit of Moscatelli type $F := F(Y \xrightarrow{f} X, \lambda)$ is without S_σ

Proof:

Let $L \subset F$ be a closed linear subspace of countable codimension. First of all we prove that there is an $n \in$ N and a linear subspace $M \subset F$ with $\dim M < \infty$, such that

$$L + M \supset F \cap \left(\prod_{k<n} \{0\} \times \prod_{k \geq n} Y \right).$$

Suppose this is not true. Then there is a sequence $(y^n)_{n\in N}$ in $\bigoplus_{n\in N}$ Y and a proper increasing sequence $(k_n)_{n\in N}$ in N with $k_1 = 1$, such that $y^n_k = 0$ for all $n \in$ N, $n > 1$ and all $k \in$

$[1, k_n[\, \cup\, [k_{n+1}, \infty[$ and

$$y^n \notin L + [y^1, \cdots, y^{n-1}]. \ (*)$$

As X satisfies the contable boundedness condition, there is a sequence $(\alpha_n)_{n\in N}$ in $(0, \infty)$, such that $B := \bigcup_{n\in N} \alpha_n \{f(y_k^n) : k_n \leq k < k_{n+1}\}$ is bounded in X. Thus the map

$$j : (\lambda, \| \|) \longrightarrow F(\beta_k)_{k\in N} \longmapsto ((\beta_k \alpha_n y_k^n)_{k_n \leq k < k_{n+1}})_{n\in N}$$

is welldefined, and linear and continuous, as it is immediately seen. As L is closed in F, $j^{-1}(L)$ is closed is $(\lambda, \| \|)$ and countably codimensional in λ. As $(\lambda, \| \|)$ is a Banach space and hence without S_σ, it follows from Proposition 1.3 that $j^{-1}(L)$ is of finite codimension in λ. SO let

$$z^n := ((1)_{k_n \leq k < k_{n+1}}, (0)_{k \notin [k_n, k_{n+1}[\cap N})$$

for all $n \in$ N. Then $z^n \in \varphi \subset \lambda$ and $(z^n)_{n\in N}$ is linearly independent. Since φ is dense in $(\lambda, \| \|)$ it is true that there exists $r \in$ N and $\mu_1, \cdot, \mu_r \in K$ with $\mu_r \neq 0$, such that

$$\mu_1 z^1 + \cdot + \mu_r z^r \in j^{-1}(L)$$

As j is linear, we get

$$\mu_1 j(z^1) + \ + \mu_r j(z^r) \in L. \ (**)$$

From the definition of j und y^n for $n \in$ N it follows that $j(z^l) = \alpha_l y^l$ for all $1 \leq l \leq r$ and with $(**)$ we obtain

$$\mu_1 \alpha_1 y^1 + \ + \mu_r \alpha_r y^r \in L\ .$$

As $\mu_r \neq 0$ we get that $y^r \in L + [y^1, \cdot, y^{r-1}]$ which is a contradiction to $(*)$.

As Y is without S_σ, also the product $\prod_{k<n} Y \times \prod_{k\geq n}\{0\}$ is without S_σ by Proposition 1.5 a). Thus by Proposition 1.3 there exists a linear subspace N $\subset \prod_{k<n} Y \times \prod_{k\geq n}\{0\}$ with $\dim N < \infty$, such that

$$\prod_{k<n} Y \times \prod_{k>n} \{0\} \subset L + N.$$

Altogether we obtain that

$$\begin{aligned} F &= F \cap \left(\prod_{k<n}\{0\} \times \prod_{k\geq n} Y + \prod_{k<n} Y \times \prod_{k\geq n}\{0\} \right) \\ &\subset L + M + N \end{aligned}$$

and as $\dim M < \infty$ und $\dim N < \infty$ we obtain from Proposition 1.3 that F is without S_σ. □

As a corollary we obtain also a positive result for spaces of type h(X).

3.5 Corollary.

Let X be a lcs which satisfies the countable boundedness condition and is without S_σ. Let $(\lambda, \| \|)$ be a nBss with sectional convergence. Then h(X) is without S_σ.

Especially if X is a metrizable lcs without S_σ, then for every nBss $(\lambda, \| \|)$ with sectional convergence h(X) is without S_σ.

Now we turn to the behaviour of Bairelike with respect to the formation of h(X). Unfortunately we couldn't find out, whether in general a space of type h(X) is always Bairelike, if X is Bairelike and λ a nBss. But we will present partial positive results. A first result follows immediately from [16, Corollary 6.3].

3.6 Proposition.

Let X be a barrelled, metrizable lcs. Then for every nBss $(\lambda, \| \|)$ the space h(X) is again barrelled and metrizable, hence Bairelike.

For normed barrelled spaces and $\lambda \in \{\ell^p : 1 \leq p \leq \infty\}$ this is also a result of the theorem in [19]. Moreover for $\lambda = c_0$ T. Gilsdorf and J. Kąkol hus proved in [17] that $c_0(E)$ is Bairelike if and only if E is barrelled and the strong dual $(E', \beta(E', E))$ of E is strong fundamentally ℓ_1-bounded, where strong fundamentally ℓ_1-bounded is a stronger property than the property (B) of Pietsch, see the Definition 3.X below.

As a barrelled space is Bairelike if and only if its completion is Bairelike, the weakly barrelled lcs, i. e. lcs X which carry a weak topology and are barrelled, form nnother fairely big class of Bairelike spaces. We will show that they behave well with respect to the formation X $\longrightarrow$ h(X) and start with a well known lemma, a" easy proof of which we present fora sake of completeness.

3.7 Lemma.

Let $(X_i)_{i\in I}$ he a family of lcs and let $(\lambda, \| \|)$ b e a nBss. Then $\lambda\left(\prod_{i\in I} X_i\right)$ is topologically isomorphic to $\prod_{i\in I} \lambda(X_i)$.

Proof:

We show that

$$\eta \;:\; \lambda\left(\prod_{i\in I} X_i\right) \longrightarrow \prod_{i\in I} \lambda(X_i)((x_i^k)_{i\in I})_{k\in N} \longmapsto ((x_i^k)_{k\in N})_{i\in I}$$

is topologically isomorphic. In fact, it is welldefined, linear, injective and continuous, as it is easy to see. Moreover if V is a 0-nbhd in $\lambda\left(\prod_{i\in I} X_i\right)$, there is a $E \subset I$ finite and there are $U_i \in \mathcal{U}_0(X_i)$, fori $\in E$ such that

$$\lambda\left(\prod_{i\in E} U_i \times \prod_{i\in I\setminus E} X_i\right) \subset V$$

(For the notation $\lambda(U_i)$ see Remark 0.2). Now

$$W := \prod_{i\in E} \lambda(U_i) \times \prod_{i\in I\setminus E} \lambda(X_i)$$

is a 0-nbhd in $\prod_{i\in I} \lambda(X_i)$ and $W \subset \eta(V)$, as can he verified without difficulty and we are done. ❑

Lemma 3.7 is a special case of Propositio" 2. 11 in [20], where the analogous statement is proved for arbitrary projective systemes of lcs. Following J. Schmets in [38, Remark IV 6.31 we give the following definition.

3.8 Definition.

A lcs E satisfies property (B) of Pietsch or is fundamentally ℓ_1-bounded (cf. [29, Definition 4.8.2 ii)]), if for every bounded subset B of ℓ_1(E) there is an absolutely convex bounded subset C of E. such that B is a bounded subset of $\ell_1([C], p_C)$.

For example every metrizable lcs has property (B) of Pietsch [Theorem 1.5.8].

The following proposition coincides in fact with (Proposition 3.2) in [5]; here we present a different proof, which is more elementary and avoids tensor products.

3.9 Proposition.

Let X be a weakly barrelled lcs and let $(\lambda, \|\,\|)$ be a nBss, such that either $(\lambda, \|\,\|) = (\ell_\infty, \|\,\|_\infty)$ or such that $(\lambda, \|\,\|)$ has the property of sectional convergence. Then h(X) is Bairelike.

Proof:

We want to apply the characterization of [29, Proposition 9.1.5] thut a barrelled dense subspace of a Bairelike space is again Bairelike. As the completion $\tilde{X}$ of X is topologically isomorphic to a product of copies of K, Lemma 3.7 implies, that $\lambda(\tilde{X})$ is topologically isomorphic to a product of Banach spaces, hence Bairelike. Therefore we must show that

h(X) is dense in $\lambda(\tilde{X})$ and that

h(X) is barrelled or equivalently - by [16, Lemma 6.1] quasibarrelled. For the density, the assertion is clear if $(\lambda, \|\,\|)$ has sectional convergence, because for the topology induced by h(X), $\oplus_{n\in N}X$ is dense in $\oplus_{n\in N}\tilde{X}$, $\oplus_{n\in N}\tilde{X}$ is dense in $\lambda(\tilde{X})$, and $\oplus_{n\in N}X \subset$ h(X).

In the case $\lambda = \ell_\infty$, we make use of the fact that in $\tilde{X}$ bounded sequences are precompact: Given $x = (x_n)_{n\in N} \in \ell_\infty(\tilde{X})$ and a 0-nbhd U in $\tilde{X}$, there is $y = (y_n)_{n\in N} \in \tilde{X}^N$ such that $\{y_n : n \in N\}$ is finite and such that $x - y \in U^N$; moreover there is $z = (z_n)_{n\in N} \in X^N$ such that $\{z_n : n \in \mathrm{N}\}$ is finite and $z - y \in U^N$; hence $z \in \ell_\infty(X)$ and $z - x \in (U+U)^N$.

We finally must establish the quasibarrelledness of h(X).

First, let $(\lambda, \|\|)$ satisfy sectional convergence and let $\mathcal{B}$ be a bounded subset of the strong dual (h(X)‘, $\beta(\lambda(X)', \lambda(X))$). By [16, Theorem 5.14 i)] we get that (h(X)‘, $\beta(\lambda(X)'$, h(X))) is continuously embedded into $\lambda^\alpha((X', \beta(X', X)))$, where $\lambda^\alpha := \{y \in \omega : \forall_{x\in\lambda} \sum_{n\in N} x_n y_n < \infty\}$ is the Köthe duul of λ. λ^α is a nBss with respect to the natural "dual" norm $\|\,\|_\alpha$. For more details see [16, page 11]. Consequently, $\mathcal{B}$ is a bounded subset of $\lambda^\alpha(X', \beta(X', X))$. For every $n \in \mathrm{N}$, put $\rho_n := \|(\delta_{nk})_{k\in N}\|_\alpha$. Then the map

$$h : \lambda^\alpha(X', \beta(X', X)) \longrightarrow \ell_\infty(X', \beta(X', X)),\ (y_n)_{n\in N} \longmapsto (\rho_n y_n)_{n\in N}$$

is welldefined, linear, injective and continuous. Therefore we get that the set $\{\rho_n y_n : (y_k)_{k\in N} \in \mathcal{B}, n \in N\}$ is bounded in (X’, $\beta(X', X)$), and thus finite dimensional. This implies that there is a finite dimensiona] linear subspace E of X' such that $\mathcal{B}$ is a bounded set in h(E). Let $F := X / E^\circ$ be the quotient space with respect to the polar space $E^\circ = \{x \in \mathrm{X} : \forall_{y\in E} < x, y > = 0\}$. Then by natural duality, F' = E. Moreover $(\lambda(F)', \beta(\lambda(F)', \lambda(F)))$ coincides topologically with $\lambda^\alpha(E)$, see e.g. [16, Remark 5.18]. As h(F) is a Banach space, the bounded subset $\mathcal{B}$ of h”(E) = *(h(F)‘*, $\beta(\lambda(F)', \lambda(F)))$ is an equicontinuous subset of $\lambda(F)'$. As the inclusion $\lambda(F)' = \lambda^\alpha(E) \hookrightarrow \lambda(X)'(\subset \lambda^\alpha(X', \beta(X', X)))$ is the transpose of the quotient map X $\longrightarrow$ X / $E^\circ = F$, $\mathcal{B}$ is also an equicontinuous subset of h(X)‘, such that altogether h(X) is barrelled, hence Bairelike for λ with sectional convergence.

Finally, let $(\lambda, \| \|) = (\ell_\infty, \| \|_\infty)$. As X is weakly barrelled, it is nuclear and from [38, IV. 7.10 c)] it follows that its strong dual $(X', \beta(X', X))$ has property (B) of Pietsch. Now we denote by βN the Stone-Cech-compactification of N. As $(X', \beta(X', X))$ has property (B) of Pietsch, we obtain hy the Mendoza-Marquina-Theorem [38, IV 7.7] that the space $C(\beta N, X)$ provided with the topology of uniform convergence is barrelled. Since the map

$$j : C(\beta N, X) \longrightarrow \ell_\infty(X), f \longmapsto (f(n))_{n\in N}$$

is a topological isomorphism onto its range, it suffices to prove thatj has dense range. But this follows again from the fact, that bounded sets in X are precompact : Let $x = (x_n)_{n\in N} \in$ C,(X) and let U be a 0-nbhd in X; then there is $y = (y_n)_{n\in N} \in X^N$ such that $\{y_n : n \in N\}$ is finite, hence compact. and such that $y - x \in U^N$. Consequently there is $f \in C(\beta N, X)$ satisfying $f(n) = y_n$ for all $n \in$ N and we are done. □

In particular we have supplemented Frerick's result in [16] about classes of barrelled spaces X for which h(X) is barrelled for nBss λ with sectional convergence by the class of weakly barrelled spaces. Having found a class of Ics X for which h(X) is Bairelike, we will now presenta class of Ics for which h(X) is not Baire. We will start with a technical lemma.

3.10 Lemma.

(see [5, Lemma 3.31; for the sake of completeness we present the short proof again.)
Let X be a Hausdorff Ics, let I be a set and let $m \in$ N. Then the set

$$C_m := \{(x_i)_{i\in I} \in X^I : \dim[x_i : i \in I] \leq m\}$$

is a closed subset of the topologica1 product X'

Proof:

Let $((x_i^{(\alpha)})_{i\in I})_{\alpha\in A}$ be a net in C_m converging to an element $(x_i)_{i\in I}$ in X'. We must show that for any $i_1, , i_{m+1} \in I$ the $(m+1)$-tuple $(x_{i_1}, , x_{i_{m+1}})$ is linearly dependent. For each $\alpha \in$ A there is $(\lambda_1^{(\alpha)}, , \lambda_{m+1}^{(\alpha)}) \in K^{m+1}\backslash\{(0, ,0)\}$ such that $\sum_{j=1}^{m+1} \lambda_j^{(\alpha)} x_{i_j}^{(\alpha)} = 0$, and we may assume that each $(\lambda_1^{(\alpha)}, , \lambda_{m+1}^{(\alpha)})$ belongs e. g. to the $\| \|_1$-unit sphere S in K^{m+1}. By the compactness of S, the net $(\lambda_1^{(\alpha)}, , \lambda_{m+1}^{(\alpha)})$ has an accumulation point $(\lambda_1, . . , \lambda_{m+1})$ in S. As for all j $\in \{1, , m+1\}$ the net $(x_{i_j}^{(\alpha)})_{\alpha\in A}$ converges to x_{i_j}, we obtain by the continuity of linear combinations $K^{m+1} \times X^{m+1} \longrightarrow$ X that $\sum_{j=1}^{m+1} \lambda_j x_{i_j}$ is an accumulation point of $(\sum_{j=1}^{m+1} \lambda_j^{(\alpha)} x_{i_j}^{(\alpha)})_{\alpha\in A}$ and hence equal to $0 \in$ X. □

3.11 Proposition (cf. [5, 3.4]).

Let X be an infinite dimensional Hausdorff Ics such that each hounded subset of X has finite dimensiona1 linear span (or – equivalently – that its weak dual $(X', \sigma(X', X))$ is barrelled), and let $(\lambda, \| \|)$ bea nBss. Thcn the space h(X) is not Baire.

Proof:

By Lemma 3. 10 we get that for each $m \in$ N, the set $C_m := \{(x_n)_{n\in N} \in X^N : \dim[x_n : n \in N] \leq m\}$ is closed in X^N with the product topology, hence the set $C_m \cap$ h(X) is closed in h(X). No $C_m \cap$ h(X) has an interior point in h(X). In fact, otherwise there would exist a sequence

$(U_n)_{n\in N}$ of 0-nbhds in X such that

$$\bigoplus_{n\in N} U_n \subset C_m - C_m \subset C_{2m}$$

As X is of infinite dimension, we can choose a linearly independent sequence $(x_n)_{n\in N} \in \prod_{n\in N} U_n$, from which we obtain that $((x_n)_{n\leq 2m+1}, (0)_{n>2m+1}) \in C_{2m}$, a contradiction. Thus it remains to show that

$$\lambda(X) = \bigcup_{m\in N} C_m$$

which is established by involving once again the continuous linear injection

$$h: \lambda(X) \longrightarrow \ell_\infty(X)(x_n)_{n\in N} \longmapsto (\rho_n x_n)_{n\in N}$$

with $\rho_n := \|(\delta_{nk})_{k\in N}\|$ for $n \in N$, which shows that for all $(x_n)_{n\in N} \in \lambda(X)$ the linear span $[x_n : n \in N]$ has finite dimension. ❑

3.12 Example (cf. [5, 3.5])

The incomplete Montel space X constructed by Amemiya and Kōmura in [4] satisfies both the assumptions of the Propositions 3.9 and 3.11. Thus, for every nBss $(\lambda, \|\cdot\|)$ with sectional convergence and for $(\lambda, \|\ \|) = (\ell_\infty, \|\ \|_\infty)$, the space h(X) is Bairelike but not Baire. (In particular, this is true for $\ell_p(X)(1 \leq p \leq \infty)$ and for $c_0(X)$.) Moreover all these spaces h(X) are quasicomplete by [16, Cor. 4.7] as X is clearly quasicomplete.

Thus we have obtained a class of quasicomplete Bairelike spaces which are not Baire. According to [29, 13.9.1] it is an open question of Valdivia, whether there exist complete non-Baire but Bairelike spaces. Obviously, the spaces in Example 3.12 are never complete. On the other hand, all the non-Baire Bairelike spaces we could find in the literature are not sequentially complete and not even locally complete.

Now we turn to the stability of Bairelike under projective limits of Moscatelli type. Unfortunately we could not prove an analogous result to Proposition 3.4 for Bairelike. But we can present partial results for weakly barrelled or metrizable barrelled spaces. We start with the following lemma.

3.13 Lemma.

Let X, Y be Hausdorff lcs, $f: Y \longrightarrow X$ be linear and continuous and let $(\lambda, \|\ \|)$ be a nBss. If Y is topologically isomorphic to a product of straight lines, then $F(Y \xrightarrow{f} X, \lambda)$ is topologically isomorphic to $(\ker f)^N \times \lambda(Y/\ker f)$.

Proof:

First of all since $\ker f$ is a closed subspace of a product of straight lines, there is a closed subspace L of Y such that $Y = \ker f \oplus L$. Furthermore, since L is also minimal as a closed subspace of Y, the restriction $f|_L : L \longrightarrow f(L)$ is a topological isomorphism onto the range and thus $F(L \xrightarrow{f|_L} f(L), \lambda) = h(L)$ and L is topologically isomorphic to $Y/\ker f$. Now we prove that

$$\rho: (\ker f)^N \times h(L) \longrightarrow F(Y \xrightarrow{f} X, \lambda)$$

$$((y_k)_{k\in N}, (z_k)_{k\in N}) \longmapsto (y_k + z_k)_{k\in N}$$

is a topological isomorphism ρ is clearly welldefined and linear. It is injective since if $(y_k + z_k)_{k\in N} = 0$ for $(y_k)_{k\in N} \in \ker f$ and $(z_k)_{k\in N} \in \lambda(L)$, it follows from $\ker f \cap h(L) = \{0\}$ that $y_k = z_k = 0$ for all $k \in N$.

To show that ρ is surjective let $(a_k)_{k\in N} \in F$ be given. Since $Y = \ker f + L$, we have that for all $k \in N$ there exists $y_k \in \ker f$ and $z_k \in L$ such that $a_k = y_k + z_k$. Then $(y_k)_{k\in N} \in (\ker f)^N$ implies that $(f(z_k))_{k\in N} = (f(a_k))_{k\in N} \in \lambda(X)$, such that from $(z_k)_{k\in N} \in L^N$ it follows that $(z_k)_{k\in N} \in A(Y)$. Thus ρ is surjective.

ρ is continuous as it is easy to see. As $\rho|_{(\ker f)^N}$ and $\rho|_{\lambda(L)}$ are topologically isomorphic onto the range, it suffices to show that $F(Y \xrightarrow{f} X, \lambda) = (\ker f)^N \oplus h(L)$ to get that ρ is open. So let $p : Y \longrightarrow \ker f$ be a continuous linear projector with $\ker p = L$. Then

$$\hat{p} : F(Y \xrightarrow{f} X, \lambda) \longrightarrow (\ker f)^N$$

$$(a_k)_{k\in N} \longmapsto (p(a_k))_{k\in N}$$

is a welldefined. linear projector with

$$\ker \hat{p} = L^N \cap F(Y \xrightarrow{f} X, \lambda) = h(L)$$

As $F(Y \xrightarrow{f} X, \lambda)$ is topologically isomorphic to the space $H := \{(y_n)_{n\in N} \in Y^N : (f(y_n))_{n\in N} \in h(X)\}$ provided with the initial topology with respect to the inclusion $H \hookrightarrow \prod_{n\in N} Y$ and $H \rightarrow h(X)$, $(x_n)_{n\in N} \mapsto (f(x_n))_{n\in N}$, $\hat{p}$ is continuous and thus we get that ρ is open. ☐

As a direct consequence we obtain the following corollary.

3.14 Corollary.

Let X, Y be Hausdorff lcs, $f : Y \longrightarrow X$ be linear and continuous and $(\lambda, \|\,\|)$ a nBss. If Y is topologically isomorphic to a product of straight lines, then the projective limit of Moscatelli type $F(Y \xrightarrow{f} X, \lambda)$ is a Baire space.

Proof:

Because of Lemma 3.13 $F(Y \xrightarrow{f} X, \lambda)$ is topologically isomorphic to $(\ker f)^N \times \lambda(Y/\ker f)$. Since Y is topologically isomorphic to a product of straight lines. $\ker f$ is also topologically isomorphic to a product of straight lines and thus topologically isomorphic to K^I for some index set I. Since $Y/\ker f$ is again topologically isomorphic to a product of straight lines, we get with Lemma 3.7 that $\lambda(Y/\ker f)$ is topologically isomorphic to λ^J for some index set J. Altogether $F(Y \xrightarrow{f} X, \lambda)$ is topologically isomorphic to a product of Banach spaces, hence Baire. ☐

With the help of the following lemma we will get a result for Bairelike.

3.15 Lemma.

Let Y, X be Hausdorff lcs, let $f : Y \longrightarrow X$ be linear and continuous and let $\tilde{Y}$ and $\tilde{X}$ be the completions of Y and X. respectively. Furthermore let $\tilde{f} : \tilde{Y} \longrightarrow \tilde{X}$ be the continuous linear extension from f to $\tilde{Y}$. If $(\lambda, \|\,\|)$ is a nBss with sectional convergence, then $F(Y \xrightarrow{f} X, \lambda)$ is a dense topological subspace of $F(\tilde{Y} \xrightarrow{\tilde{f}} \tilde{X}, h)$.

Proof:

Because of the transitivity of the initial topology $F(Y \xrightarrow{f} X, \lambda)$ is a toplogical subspace of $F(\tilde{Y} \xrightarrow{\tilde{f}} \tilde{X}, \lambda)$. Moreover $\bigoplus_{n\in N}$ Y is dense in $\bigoplus_{n\in N} \tilde{Y}$ with respect to the induced topology of $F(\tilde{Y} \xrightarrow{\tilde{f}} \tilde{X}, \lambda)$ and because of the sectional convergence, $\bigoplus_{n\in N} \tilde{Y}$ is dense in $F(\tilde{Y} \xrightarrow{\tilde{f}} \tilde{X}, \lambda)$, such that altogether $\bigoplus_{n\in N}$ Y is a dense subspace of $F(\tilde{Y} \xrightarrow{\tilde{f}} \tilde{X}, \lambda)$. As it is true that $\bigoplus_{n\in N}$ Y $\subset$ $F(Y \xrightarrow{f} X, \lambda)$ we are done. ❑

As a consequence of Lemma 3.15 and Corollary 3.14 the following corollary.

3.16 Corollary.

Let Y, X be Hausdorff lcs, such that Y carries a weak topology. let f : Y $\longrightarrow$ X be linear and continuous and let $(\lambda, \| \,\|)$ be a nBss with sectional convergence. Then $F(Y \xrightarrow{f} X, \lambda)$ is Bairelike if and only if it is barrelled.

Proof:

From Lemma 3.15 we get that $F(Y \xrightarrow{f} X, \lambda)$ is a dense subspace of $F(\tilde{Y} \xrightarrow{\tilde{f}} \tilde{X}, \lambda)$ and since $\tilde{Y}$ is topologically isomorphic to a product of straight lines, we get from Corollary 3.14 that $F(\tilde{Y} \xrightarrow{\tilde{f}} \tilde{X}, \lambda)$ is a Bairespace. Thus $F(Y \xrightarrow{f} X, \lambda)$ is Bairelike if and only if it is barrelled (see [29, Proposition 9.1.3]). ❑

Now the question arises under which conditions a projective limit of Moscatelli type is barrelled. We give a characterization with the keep of a generalization of Lemma 6.1 of [16]. In fact it is exactly Frerick's proof which works also in the more general situation.

3.17 Lemma.

Let X. Y be Hausdorff lcs, f : Y $\longrightarrow$ X be linear und continuous, and let $(\lambda, \| \,\|)$ bea nBss. Then the following are equivalent:

i) $F(Y \xrightarrow{f} X, \lambda)$ is barrelled.

ii) $F(Y \xrightarrow{f} X, \lambda)$ is quasibarrelled and Y is barrelled.

Proof:

Only ii) $\Rightarrow$ i) needs a proof. So let T be a barrel in $F := F(Y \xrightarrow{f} X, \lambda)$. Since F is quasibarrelled, it is sufficient to show that T is bornivorous. So let $B \subset F$ be a bounded set. Without loss of generality we may assume that for all $x = (x_k)_{k\in N} \in B$ and $J \subset N$ it is true that $((x_n)_{n\in J}, (0)_{n\in N\setminus J}) \in B$. It is immediate that F is topologically isomorphic to

$$\left(\prod_{k<n} Y \times \prod_{k\geq n} \{0\}\right) \oplus \left(F \cap \prod_{k<n} \{0\} \times \prod_{k\geq n} Y\right)$$

for every $n \in$ N. Since $\prod_{k<n}$ Y $\times \prod_{k\geq n}\{0\}$ is barrelled for all $n \in$ N as Y is barrelled, it is enough to show that T absorbs $B \cap \left(\prod_{k<n}\{0\} \times \prod_{k\geq n} \text{Y}\right)$ for some $n \in$ N. We assume this is not true. Then we obtain that for every $n \in$ N there exists some $z^{(n)} = (z_k^{(n)})_{k\in N} \in B \cap \left(\prod_{k<n}\{0\} \times \prod_{k\geq n} Y\right)$ such that $z^n \notin 2^{2n}T$. Now let $\tilde{f} : \tilde{Y} \longrightarrow \tilde{X}$ be the continuous extension of f lo the completion $\tilde{Y}$ and $\tilde{X}$ of Y and X, respectively, and let

$$h : (\ell_1, \| \,\|_1) \longrightarrow F(\tilde{Y} \xrightarrow{\tilde{f}} \tilde{X}, \lambda) =: F_1$$

$$(\alpha_k)_{k\in N} \longmapsto \sum_{k=1}^{\infty} \alpha_k \frac{1}{2^k} z^{(k)}$$

As $(\frac{1}{2^k} z^{(k)})_{k\in N}$ is bounded and $(\alpha_k)_{k\in N} \in \ell_1$, h is welldefined, it is also linea and we will show that $h: (\ell_1, \sigma(\ell_1, c_0)) \longrightarrow (F_1, \sigma(F_1, F_1'))$ is continuous. So let $\psi \in F_1'$ be given, then for all $\alpha = (\alpha_k)_{k\in N} \in \ell_1$ it is true that

$$\psi(h(\alpha)) = \psi\left(\sum_{k\in N} \alpha_k \frac{1}{2^k} z^{(k)}\right) = \sum_{k\in N} \alpha_k \frac{1}{2^k} \psi(z^{(k)})$$

and since $(z^{(k)})_{k\in N}$ is a bounded sequence, and ψ is linear and continuous $\{\psi(z^{(k)}) : k \in N\}$ is bounded in K. Thus $\psi \circ h$ is represented by $(\frac{1}{2^k}\psi(z^{(k)}))_{k\in N}$ which converges to zero and so ψ oh is $\sigma(\ell_1, c_0)$-continuous, which proves that the map $h : (\ell_1, \sigma(\ell_1, c_0)) \longrightarrow (F_1, \sigma(F_1, F'_1))$ is continuous.

As the closed unit ball B_1 in $(\ell_1, \|\cdot\|_1)$ is $\sigma(\ell_1, c_0)$-compact, the range $h(B_1)$ of B_1 is $\sigma(F_1, F_1')$ compact and thus a Banach disc in F_1 with its original topology. If $\alpha = (\alpha_n)_{n\in N} \in \ell_1$, then

$$\sum_{k\in N} \alpha_k \frac{1}{2^k} z^{(k)} = \sum_{k\in N} \alpha_k \frac{1}{2^k} (z_j^{(k)})_{j\in N}$$

$$= \left(\sum_{k=1}^{j} \alpha_k \frac{1}{2^k} z_j^{(k)}\right)_{j\in N} \in Y^N \cap F_1 = F.$$

Thus $h(B_1) \subset F$ such that $h(B_1)$ is a Banach disc in Fand hence T absorbs $h(B_1)$. It follows that there exists $\rho > 0$ such that for all $k \in$ N it is true that $\frac{1}{2^k} z^{(k)} \in \rho T$, such that for all $n \in$ N with 2" $\geq \rho$ it holds that $z^{(n)} \in 2^n 2^n T = 2^{2n} T$ which is a contradiction to the assumption. Consequently there is $n \in$ N and some $\rho > 0$ such that

$$B \cap \left(\prod_{k<n} \{0\} \times \prod_{k\geq n} Y\right) \subset \rho T,$$

whence we get that

$$\begin{aligned} B &\subset B \cap \left(\prod_{k<n} Y \times \prod_{k\geq n} \{0\}\right) + B \cap \left(\prod_{k<n} \{0\} \times \prod_{k\geq n} Y\right) \\ &\subset 2\mu T \end{aligned}$$

for some $\mu \geq \rho$, which completes the proof.

As a further result we obtain from this the following corollary.

3.18 Corollary.

Let Y, X be metrizable Ics, such that Y is barrelled. Then $F(Y \xrightarrow{L} X, \lambda)$ is barrelled, hence Bairelike.

Proof:

Since X and Y are metrizable also $F(Y \xrightarrow{f} X, \lambda)$ is metrizable and hence quasibarrelled. From Lemma 3.17 it follows that it is also barrelled, such that it is altogether metrizable and barrelled, hence Bairelike [29, Proposition 9.1.3 ii)]. □

A further positive result holds for weakly barrelled Hausdorff spaces, as we will show now.

3.19 Proposition.

Let X, Y be lcs, such that Y is Bairelike and X is weakly barrelled and Hausdorff. Moreover let $f : Y \longrightarrow X$ be linear, continuous and open map. If $(\lambda, \|\ \|)$ is a nBss with sectional convergence or $(\lambda, \|\cdot\|) = (\ell_\infty, \|\cdot\|_\infty)$, then the projective limit of Moscatelli type $F(Y \xrightarrow{f} X, \lambda)$ is Bairelike.

Proof:

Since Y is Bairelike also the finite product $\prod_{k<n} Y$ is Bairelike for all $n \in \mathbb{N}$. Furthermore by Proposition 3.9 we obtain that $\lambda((X)_{k\geq n})$ is also Bairelike for all $n \in \mathbb{N}$. Together we get that the steps $F_n := \prod_{k<n} Y \times \lambda((X)_{k\geq n})$ of $F(Y \xrightarrow{f} X, \lambda)$ are Bairelike, such that by Proposition 2.8 it is enough to show that the linking maps

$$g_{n+1\,n} : \prod_{k<n} Y \times Y \times \lambda((X)_{k\geq n+1}) \longrightarrow \prod_{k<n} Y \times X \times \lambda((X)_{k\geq n+1})$$

$$((y_k)_{k<n}, y_n, (y_k)_{k\geq n+1}) \longmapsto ((y_k)_{k<n}, f(y_n), (y_k)_{k\geq n+1})$$

are open, what is immediate, since f is open. □

We remark that in the previous proposition the condition on f to be open is essential, even if Y is chosen to be weakly barrelled. But before we give an example we show the following Lemma.

3.20 Lemma.

Let X, Y be lcs, $f : Y \longrightarrow X$ be linear and continuous, and $(\lambda, \|\ \|)$ be a nBss. Then the canonical map

$$\hat{f} : F(Y \xrightarrow{f} X, \lambda) \longrightarrow \lambda(X), (y_n)_{n\in\mathbb{N}} \longmapsto (f(y_n))_{n\in\mathbb{N}}$$

is surjective, if f is surjective and $\hat{f}$ is open, if f is open.

Proof:

Because of the definition of the projective limit of Moscatelli type, $\hat{f}$ is clearly welldefined, linear and continuous. Now let $(x_k)_{k\in\mathbb{N}} \in \lambda(X)$ be given. Since f is surjective, it is true that for all $n \in \mathbb{N}$ there is a $y_k \in Y$ such that $f(y_k) = x_k$. Thus $(f(y_k))_{k\in\mathbb{N}} = (x_k)_{k\in\mathbb{N}} \in \lambda(X)$, such that $(y_k)_{k\in\mathbb{N}} \in F(Y \xrightarrow{f} X, \lambda)$ and hence $\hat{f}$ is surjective.

Now let f be open and $V \in \mathcal{U}_0(Y)$, $n \in \mathbb{N}$ and $U \in \mathcal{U}_0(X)$ be given. Since $\hat{f}$ is surjective as we have just proved, we obtain that

$$\hat{f}\left(\prod_{k<n} V \times \{(y_k)_{k\geq n} \in \lambda((Y)_{k\geq n}) : \|((0)_{k<n}, (p_U(f(y_k)))_{k\geq n})\| \leq 1\}\right)$$

$$\supset \prod_{k<n} f(V) \times \{(x_k)_{k\geq n} \in \lambda((X)_{k\geq n}) : \|((0)_{k<n}, (p_U(x_k))_{k\geq n})\| \leq 1\}$$

and thus $\hat{f}$ is open, since f is open. □

3.21 Example.

There exist weakly barrelled Hausdorff spaces $(Y, \Im)$ and $(X, \wp)$ and a linear bijective, continuous map $f : (Y, \Im) \longrightarrow (X, \wp)$ such that $F(Y \xrightarrow{f} X, \lambda)$ is not ℓ_∞-barrelled for any nBss $(\lambda, \|\ \|)$.

Proof:

Let $(Y, \Im)$ bea weakly barrelled infinite dimensiona1 Hausdorff space, such that the bounded sets have finite dimensional linear span. An example of such a space is the incomplete Montelspace of I. Amemia and Y. Kōmura [4]. Then $(Y, \Im)$ is not complete, because otherwise it would be topologically isomorphic to the product of one dimensiona1 spaces, such that the bounded sets would not have finite dimensiona1 linear span.

Now let $\tilde{Y}$ be the completion of $(Y, \Im)$ and let $x \in \tilde{Y}\backslash Y$ be given. $\tilde{\Im}$ denote the topology on $Z := Y + [x]$ induced by $\tilde{Y}$. Now we prove that the bounded sets in $(Z, \tilde{\Im})$ also have finite dimensiona1 linear span. In fact, let B be a bounded subset of $(Z, \tilde{\Im})$, then for all $b \in B$ there exists some $y_b \in Y$ and $\lambda_b \in K$ such that

$$b = y_b + \lambda_b x. \ (*)$$

We will show that $K := \{\lambda_b : b \in B\}$ is bounded in K. Suppose this is not true. Then there is a sequence $(b_n)_{n\in N}$ in B such that $\lambda_{b_n} \overset{n\to\infty}{\Longrightarrow} \infty$ and $\lambda_{b_n} \neq 0$ for all $n \in N$. Now $(*)$ implies that for all $n \in N$

$$\frac{1}{\lambda_{b_n}}\, y_b = \frac{1}{\lambda_{b_n}} b_n - x$$

is true. Since $\{b_n : n \in N\}$ is bounded in $(Z, \tilde{\Im})$ and $\frac{1}{\lambda_{b_n}} \overset{n\to\infty}{\Longrightarrow} 0$ we get that $(\frac{1}{\lambda_{b_n}} y_{b_n})_{n\in N}$ is a sequence in Y which converges to $-x$. But since Y is quasicomplete and hence sequentially complete this would imply that $x \in Y$ which is a contradiction to $x \notin Y$. Thus K is a bounded subset of K and we get that $A := \{y_b : b \in B\} \subset B\ Kx$ is bounded in $(Z, \tilde{\Im})$. It follows that $A \subset Y$ is bounded in $(Y, \Im)$ and hence $\dim[A] < \infty$. Altogether we get that

$$\dim[B] \leq \dim[A \cup \{x\}] \leq \dim[A] + 1 < \infty$$

such that we have proved that the bounded sets in $(Z, \tilde{\Im})$ have finite dimensiona1 linear span.

Now we put $(X, \wp) := (Z, \tilde{\Im}) \,/\, [x]$ and denote by $q : (Z, \tilde{\Im}) \longrightarrow (X, \wp)$ the canonica1 quotient map. Since $(Y, \Im)$ is weakly barrelled $(Z, \tilde{\Im}) = Y + [x]$ is weakly barrelled and hence $(X, \wp)$ is also weakly barrelled. Furthermore $(X, \wp)$ is topologically isomorphic to a closed hyperplane in $(Z, \tilde{\Im})$ and thus the bounded sets in $(X, \wp)$ have finite dimensiona1 linear span. From the definition of $(Z, \tilde{\Im})$ and $(X, \wp)$ it follows directly that $f : (Y, 3) \longrightarrow (X, \wp), y \mapsto q(y)$ is linear. bijective and continuous but not open.

Now let $(\lambda, \|\ \|)$ be an arbitrary nBss. We prove that the projective limit of Moscatelli type $F((Y, \Im) \xrightarrow{f} (X, \wp), \lambda) =: F$ is not ℓ_∞-barrelled. Since $(Y, \Im)$ and $(X, \wp)$ are barrelled, they are in particular Mackey spaces, such that there exists a map $\psi \in (Y, \Im)'$ such that ψ of-' $\notin (X, \wp)'$. Let $\rho_n := \|(\delta_{nk})_{k\in N}\|$ and let $pr_n : F \longrightarrow Y$ be the canonical projection and let $B := \{\rho_n(\psi \circ pr_n) : n \in N\}$. Now we will show that B is pointwise bounded. In fact. let $(y_k)_{k\in N} \in F$ be given. Then $\{\rho_k f(y_k) : k \in N\}$ is a bounded set in $(X, \wp)$

such that $E := [\{f(y_k) : k \in N\}]$ is a finite dimensional subspace of X. Since $(X, \wp)$ is a Hausdorff space, we get by the uniqueness of finite dimensional linear Hausdorff topologies that$f|_{f^{-1}(E)}$: $(f^{-1}(E), \Im \cap f^{-1}(E)) \longrightarrow (E, \wp \cap E)$ is a topologica1 isomorphism. It follows that $(y_k)_{k \in N} \in \lambda(f^{-1}(E)) \subset$ h(Y) and thus

$$\{\rho_n(\psi(pr_n((y_k)_{k \in N}))) : n \in N\} = \{\rho_n(\psi(y_n)) : n \in N\} = \psi(\{\rho_n y_n : n \in N\})$$

is a bounded subset in K. We will show that B is not equicontinuous and hence $F(Y \xrightarrow{f} X, \lambda)$ is not ℓ_∞-barrelled. By the above argument we have also obtained that$f|_{\lambda(Y)}$: h(Y) $\longrightarrow$ A(X), $(y_k)_{k \in N} \mapsto (f(y_k))_{k \in N}$ is bijective. Thus if B were equicontinuous there would exist some $n_0 \in$ N, such that $\rho_n(\psi \circ pr_n \circ \hat{f}^{-1}) = \rho_n(\psi \circ f^{-1})$ would be continuous in $(X, \wp)$ for all $n \geq n_0$. But this would imply that ψ of-' $\in (X, \wp)'$ which is a contradiction. □

3.22 **Remark.**

The proof of Example 3.21 shows that for every weakly barrelled Hausdorff space $(Y, 3)$ such that the bounded sets have finite dimensional linear span, there exist a stictly weaker weakly barrrelled Hausdorff topology $\wp$ on Y, such that $F((Y, \Im) \xrightarrow{id} (Y, \wp), \lambda)$ is not ℓ_∞-barrelled.

Furthermore it seems worth mentioning that the question asked by Valdivia [29, 13.9.1] whether complete Bairelike spaces are Baire would get a negative answer, if one could find a complete locally convex Baire space Y admitting a quotient Y / L with weak topology, such that the bounded sets in Y / L have finite dimensional linear span. In fact, if q : Y $\longrightarrow$ Y / L is the quotient map and if $(\lambda, \|\cdot\|)$ is a nBss with sectional convergence or $(\lambda, \|\cdot\|) = (\ell_\infty, \|\cdot\|_\infty)$ the projective limit $F = F(Y \xrightarrow{q} Y / L, \lambda)$is complete, Bairelike by Proposition 3.19 but not Baire, since $\hat{f} : F \longrightarrow \lambda(Y / L)$, $(y_n)_{n \in N} \mapsto (q(y_n))_{n \in N}$ is linear, continuous, open by Lemma 3.20 and $\lambda(Y / L)$ is not Baire by Proposition 3. 11.

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