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A NEW APPROACH TO CONSTRAINED SYSTEMS WITH A CONVEX EXTENSION

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Abstract. *Systems S of N partial differential equations are considered, with M differential constraints and satisfying a convex supplementary conservation law. When $M = 0$, it is well known that these systems assume the symmetric hyperbolic form if the components of the mean field are taken as independent variables. To extend this property to the case $M \neq 0$, a new system S^* is here proposed with M supplementary variables x_A such that the solutions of S^* with $x_A = 0$ are those of the system S. Moreover S^* can be expressed in the symmetric hyperbolic form. This methodology is tested by applying it to the equations of the superfluid, modified from the classical Landau's formulation.*

1. INTRODUCTION

Some important physical problems are described by a set of field equations such as

$$\partial_t f_n + \partial_i f_n^i = p_n, \qquad \text{for } n = 1, \dots, N, \tag{1}$$

subject to differential constraints of the type

$$\partial_i g_A^i = q_A \qquad \text{with } A = 1, \dots, M. \tag{2}$$

By using a suitable invertible change of variables, we can take f_n as independent variables; f_n^i, g_A^i, p_n, q_A are given functions of f_n^i.

Usually this system is considered hyperbolic if the problem $-\lambda df_n + n_i\, df_n^i = 0$ has real eigenvalues λ and N *l. i.* eigenvectors df_n, for all n_i such that $n_i n^i = 1$.

Now this condition is too much restrictive; for example, it is not satisfied by the equations of the superfluid, as proved by Boillat and Pluchino [1]. They have also proved that the same equations assume the symmetric hyperbolic form if the components of the mean field are taken as independent variables.

Obviously, hyperbolicity must not be affected by invertible changes of variables; therefore a less restrictive definition of hyperbolicity for constrained systems is required. To this end we notice that eq. (1) can be substituted by the sum of eq. (1) and of eq. (2) multiplied by an arbitrary function T_n^A, i.e.,

$$\partial_t f_n + \partial_i f_n^i + T_n^A\, \partial_i g_A^i = p_n + T_n^A q_A. \tag{3}$$

We can now apply the above definition of hyperbolicity to this modified equation (3); we cannot expect that it will be satisfied for every choice of T_n^A, but only for a particular one. In other words, an appropriate definition of hyperbolicity for constrained systems is the following one, "The system (1), with the constraints (2), is hyperbolic if the functions T_n^A exist such that the problem

$$-\lambda df_n + n_i(df_n^i + T_n^A dg_A^i) = 0 \tag{4}$$

has real eigenvalues λ and N *l. i.* eigenvectors df_n, for all n_i such that $n_i n^i = 1$".

The constraints (2) have to be imposed only on the initial manifold Σ, after that they will be satisfied also off Σ [2].

In the section II it will be shown that the functions T_n^A can be easily found if the system (1,2) satisfies a convex supplementary conservation law, i.e. if the functions h, h^i, g exist such that the relation

$$\partial_t h + \partial_i h^i = g \tag{5}$$

holds for every solution of the system (1,2); moreover h is a convex function of the independent variables f_n.

In sect. III a new system is obtained of the type

$$\begin{aligned} &\partial_t f_n + \underset{i}{\partial} f_n^i + T_n^A \partial_i g_A^i = p_n + T_n^A q_A \\ &\partial_t x_A + \partial_i g_A^i = q_A, \end{aligned} \tag{6}$$

in the independent variables f_n, x_A; this system can be expressed in the symmetric hyperbolic form and reduces to the equations (3,2) for $x_A = 0$.

The equations (6) are more elegant than (3), because the differential constraints (2) appears also, even if in the transformed form $(6)_2$. The new variables x_A are only auxiliary quantities. The idea of considering an increased number of independent variables comes from a similar approach followed in extended thermodynamics (see, for example [3-5]).

This approach to constrained systems, has been already applied successfully to the cases of relativistic magnetofluid-dynamics [6] and to covariant Maxwell electrodynamics [7]. Here the general case is considered and another example of physical application is given, i.e., the equations of the superfluid.

However, I emphasize that the most important and original aspect of this paper is that it furnishes a general method to deal with the constrained systems with a convex extension, while the other published papers on this subject dealt only with particular physical problems. The equations of the superfluid furnish another physical example which confirm the potentiality of this method.

In sect. IV the problem (4) for the superfluid is investigated (Note that the problem (4) deals with the modified system, while the original one was $-\lambda df_n + n_i df_n^i = 0$).

I conclude this section noticing that $(6)_1$, by using $(6)_2$ becomes $\partial_t(f_n - T_n^A x_A) + x_A\, \partial_t T_n^A + \partial_i F_n^i = p_n$.

Now the hyperbolicity of the system (7), $(6)_2$ amounts to some inequalities and to some equations depending on T_n^A, x_A $\frac{\partial T_n^A}{\partial f_m}$, $\frac{\partial F_n^i}{\partial x_A}$, $\frac{\partial F_n^i}{\partial f_m}$, $\frac{\partial g_i^A}{\partial f_m}$; these inequalities and these equations hold for every value of the independent variables, also for $x_A = 0$. Therefore the system

$$\begin{aligned} &\partial_t(f_n - T_n^A x_A) + \partial_i F_n^i = p_n \\ &\partial_t x_A + \partial_i g_A^i = q_A \end{aligned} \tag{8}$$

is still hyperbolic, at least in a neighbourhood of $x_A = 0$. It has the advantage, with respect to system (6), to have preserved the conservative form.

2. CONSTRAINED SYSTEMS WITH A CONVEX EXTENSION

Let us consider the system (1,2) satisfying the supplementary conservation law (5); according to the results of refs. [8-10], the functions λ^n, μ^A exist such that

$$\frac{\partial h}{\partial f_n} = \lambda^n, \frac{\partial h^i}{\partial f_n} = \lambda^m \frac{\partial f^i_m}{\partial f_n} + \mu^A \frac{\partial g^i_A}{\partial f_n}, g = \lambda^n p_n + \mu^A q_A. \tag{9}$$

Eq. $(9)_1$ and the convexity of h, assures that λ^n are invertible functions of f^n; therefore we can take λ^n as independent variables, an idea already applied in other physical contexts [11-13]. The variables λ^n are also called the "mean field".

We can prove now that the system (4), with $T^A_n = \frac{\partial \mu^A}{\partial \lambda^n}$, is symmetric hyperbolic. In fact, eqs. $(9)_{1,2}$ can be written as

$$dh = \lambda^n df_n, dh^i = \lambda^m df^i_m + \mu^A dg^i_A \tag{10}$$

or, equivalently, as

$$dh' = f_n d\lambda^n, dh'^i = f^i_m d\lambda^m + g^i_A d\mu^A \tag{11}$$

with h', h'^i defined by

$$h' = -h + \lambda^n f_n, h'^i = -h^i + \lambda^m f^i_m + \mu^A g^i_A.$$

From eqs. (11) it follows $f_n = \frac{\partial h'}{\partial \lambda^n}, f^i_n + g^i_A \frac{\partial \mu^A}{\partial \lambda^n} = \frac{\partial h'^i}{\partial \lambda^n}$ which allow the system (4) to be written in the symmetric hyperbolic form

$$\partial_t \left(\frac{\partial h'}{\partial \lambda^n} \right) + \partial_i \left(\frac{\partial h'^i}{\partial \lambda^n} \right) - g^i_A \, \partial_i \left(\frac{\partial \mu^A}{\partial \lambda^n} \right) = p_n + T^A_n q_A. \tag{12}$$

(Note that $\frac{\partial^2 h'}{\partial \lambda^n \partial \lambda^m} d\lambda^n d\lambda^m = d\left(\frac{\partial h'}{\partial \lambda^n}\right) d\lambda^n = (df^n)\, d\left(\frac{\partial h}{\partial f^n}\right)$; therefore $\frac{\partial^2 h'}{\partial \lambda^n \partial \lambda^m}$ is positive definite).

Let use apply now this method to the case of the superfluid, whose equations [1,14] are

$$\begin{aligned}
&\partial_t \rho + \partial_k(\rho_S v_{Sk} + \rho_N v_{Nk}) = 0 \\
&\partial_t v_{Si} + \partial_k(\mu + v_S^2/2)\delta_{ki} = 0 \\
&\partial_t(\rho_S v_{Si} + \rho_N v_{Ni}) + \partial_k(P\delta_{ki} + \rho_N v_{Nk} v_{Ni} + \rho_S v_{Sk} v_{Si}) = 0 \\
&\partial_t(\rho S) + \partial_k(\rho S v_{Nk}) = 0
\end{aligned} \tag{13}$$

and the differential constraints are

$$\varepsilon_{akj}\, \partial_k\, v_{Sj} = 0. \tag{14}$$

In these equations the independent variables are P (pressure), T (temperature), $\underline{V}_S$ (superfluid component of the velocity) and $\underline{v}_n$ (normal component of the velocity).

The other functions are generated by the chemical potential $\mu = \mu\,(P, T, \alpha)$, by means of the relations

$$\alpha = (\underline{v}_N - \underline{v}_S)^2/2, \rho_S = \left(1 + \frac{\partial\,\mu}{\partial\,\alpha}\right)\left(\frac{\partial\,\mu}{\partial\,P}\right)^{-1}, \qquad (15)$$
$$\rho_N = -\frac{\partial\,\mu}{\partial\,\alpha}\left(\frac{\partial\,\mu}{\partial\,P}\right)^{-1}, \rho = \rho_S + \rho_N = \left(\frac{\partial\,\mu}{\partial\,P}\right)^{-1}, S = -\frac{\partial\,\mu}{\partial\,T},$$

ε_{kaj} is the Levi-Civita symbol.

Moreover the function μ is a concave function of P, T, α and such that [15-17]

$$1 + 2\alpha\left(\frac{1}{\rho_S}\frac{\partial\,\rho_S}{\partial\,\alpha} - \frac{\partial\,\rho_S}{\partial\,P}\right) > 0 \qquad (16)$$

The quantities ρ, v_{Si}, $\rho_S v_{Si} + \rho_N v_{Ni}$, ρS are invertible functions of P, T, v_{Si}, v_{Ni}; this property can be proved by considering the system

$$d\rho = 0, dv_{Si} = 0, d(\rho_S v_{Si} + \rho_N v_{Ni}) = 0, d(\rho S) = 0. \qquad (17)$$

From $(17)_{1-3}$ we obtain $0 = d(\rho_S v_{Si} + \rho_N v_{Ni} - \rho v_{Si}) = d[\rho_N\,(v_{Ni} - v_{Si})] = (v_{Ni} - v_{Si})\,d\rho_N + \rho_N\,d(v_{Ni} - v_{Si})$, from which it follows $2\alpha d\rho_N + \rho_N\,d\alpha = 0$, or equivalently,

$$d(-\rho_N/\rho) - \rho_N d\alpha/(2\alpha\rho) = 0. \qquad (18)$$

Eqs. $(17)_{1,4}$, (18) can be written as

$$\frac{\partial^2\,\mu}{\partial\,P^2}dP + \frac{\partial^2\,\mu}{\partial\,P\,\partial\,T}dT + \frac{\partial^2\,\mu}{\partial\,P\,\partial\,\alpha}d\alpha = 0$$
$$\frac{\partial^2\,\mu}{\partial\,P\,\partial\,T}dP + \frac{\partial^2\,\mu}{\partial\,T^2}dT + \frac{\partial^2\,\mu}{\partial\,T\,\partial\,\alpha}d\alpha = 0 \qquad (19)$$
$$\frac{\partial^2\,\mu}{\partial\,P\,\partial\,\alpha}dP + \frac{\partial^2\,\mu}{\partial\,\alpha\,\partial\,T}dT + \left(\frac{\partial^2\,\mu}{\partial\,\alpha^2} - \frac{\rho_N}{2\alpha\rho}\right)d\alpha = 0,$$

from which it follows $dP = dT = d\alpha = 0$

because $\mu(P, T, \alpha)$ is a concave function. From eqs. (19) it follows that the system (17) has only the solution $dP = dT = dv_{Si} = dv_{Ni} = 0$, which fact proves the invertibility of ρ, v_{Si}, $\rho_S v_{Si} + \rho_N\,v_{Ni}$, ρS.

The system (13,14) satisfies the supplementary conservation law (5), with

$$h = \rho ST - P + (\mu + v_S^2/2)\rho + \rho_N v_{Nj}(v_{Nj} - v_{Sj}) \qquad (20)$$
$$h^i = (\mu + v_S^2/2)(\rho_S v_S^i + \rho_N v_N^i) + \rho ST v_N^i + \rho_N v_N^i v_{Nj}(v_{Nj} - v_{Sj}).$$

In fact the relations (10) hold with $\lambda^n = (\theta, L^i_S, v^i_N, T)$,

$$\theta = \mu + v_S^2/2 - V_{Ni}v_{Si}, \underline{L}_S = -\rho_S(\underline{v}_{Ni} - \underline{v}_{Si}),$$
$$\mu^A = \rho_S \varepsilon_{abc} v_N^b v_S^c = \varepsilon_{abc} v_N^b L_S^c.$$

Moreover h is a convex function of the variables f_n; in fact

$$Q = \frac{1}{T}\frac{\partial^2 h}{\partial f_n \partial f_m} df_n df_m = \frac{1}{T} d\left(\frac{\partial h}{\partial f_n}\right) df_n = \frac{1}{T} d\lambda_n df_n =$$
$$\frac{1}{T}\left\{ d(\mu + v_S^2/2 - v_{Ni}v_{Si})d\rho + d[-\rho_S(v_{Ni} - v_{Si})]dv_{Si} + \right.$$
$$\left. + dv_{Ni} d(\rho_S v_S^i + \rho_N v_N^i) + dT d(\rho S) \right\} = \tag{21}$$
$$d\left[\frac{1}{T}(\mu + v_S^2/2 - v_{Ni}v_{Si})\right] d\rho + d\left[-\rho_S \frac{1}{T}(v_{Ni} - v_{Si})\right] dv_{Si} +$$
$$+ d\left(v_{Ni}\frac{1}{T}\right) d(\rho_S v_S^i + \rho_N v_N^i) - d\left(\frac{1}{T}\right) dh$$

and this quadratic form is positive definite, as proved in ref. [17]. We see now that $\frac{\partial \mu^A}{\partial \lambda^n} = (0, \varepsilon_{abi}\, v_N^b, -\varepsilon_{abi}\, L_S^b, 0)$; therefore the system (4) becomes

$$\begin{aligned}
&\partial_t \rho + \partial_k(\rho_S v_{Sk} + \rho_N v_{Nk}) = 0 \\
&\partial_t v_{Si} + \partial_k(\mu + v_S^2/2)\delta_{ki} + \underline{v_{Nj}(\partial_j v_{Si} - \partial_i v_{Sj})} = 0 \\
&\partial_t(\rho_S v_{Si} + \rho_N v_{Ni}) + \partial_k(P\delta_{ki} + \rho_N v_{Nk} v_{Ni} + \rho_S v_{Sk} v_{Si}) + \\
&\quad + \underline{\rho_S(v_{Nj} - v_{Sj})(\partial_j v_{Si} - \partial_i v_{Sj})} = 0 \\
&\partial_t(\rho S) + \partial_k(\rho S v_{Nk}) = 0,
\end{aligned} \tag{22}$$

where the identity $\varepsilon_{jik}\ \varepsilon_{jpq} = 2\delta_{p[i}\ \delta_{k]q}$ has been used, and the underlined terms are those responsible of the modification of the original system (13), where they don't appear. They have been obtained by adding, to the second and third equation, a linear combination of the differential constraints (14), following the general guidelines of the methodology exposed in this paper.

The eigenvalues and eigenvectors problem, for this system (22), will be exploited in sect. IV.

3. AN EXTENDED APPROACH FOR THE SYSTEM (4)

Let us consider the system (6) with

$$F_n^i = f_n^i + \frac{\partial g_A^i}{\partial \lambda^n} x^A, T_n^A = \frac{\partial \mu^A}{\partial \lambda^n}, \tag{23}$$

in the independent variables f_n, x^A. The solutions of this system, with $x^A = 0$, are those of the system (4). If we impose that $x^A = 0$, $\partial_i g^i_A = q_A$ hold on the initial manifold σ, they will propagate nicely off σ. Moreover the system (6) becomes symmetric hyperbolic, with a suitable change of variables. In fact, let

$$H' = -h + \lambda^n f_n + \frac{1}{2} x^A x_A, H'^i = -h^i + \lambda^m f^i_m + \mu^A g^i_A + g^i_A x^A. \tag{24}$$

By using eqs. (10) we obtain

$$dH' = f_n d\lambda^n + x^A dx_A, dH'^i = f^i_m d\lambda^m + g^i_A d\mu^A + g^i_A dx^A + x^A dg^i_A,$$

or, equivalently,

$$\frac{\partial H'}{\partial \lambda^n} = f_n, \frac{\partial H'}{\partial x^A} = x_A, \frac{\partial H'^i}{\partial \lambda^n} = f^i_n + g^i_A T^A_n + x^A \frac{\partial g^i_A}{\partial \lambda^n}, \frac{\partial H'^i}{\partial x^A} = g^i_A$$

(λ^n, x^A have been taken as new independent variables).

Therefore the system (6) becomes

$$\begin{aligned} &\partial_t \left(\frac{\partial H'}{\partial \lambda^n}\right) + \partial_i \left(\frac{\partial H'^i}{\partial \lambda^n}\right) - g^i_B \partial_i \frac{\partial \mu^B}{\partial \lambda^n} = p_n + T^B_n q_B, \\ &\partial_t \left(\frac{\partial H'}{\partial x^A}\right) + \partial_i \left(\frac{\partial H'^i}{\partial x^A}\right) - g^i_B \partial_i \frac{\partial \mu^B}{\partial x^A} = q_A \end{aligned} \tag{25}$$

where, obviously, $\frac{\partial \mu^B}{\partial x^A} = 0$.

This new system is symmetric hyperbolic, also because $H'(\lambda^n, x^A)$ is a convex function; in fact

$$\begin{aligned} &\frac{\partial^2 H'}{\partial \lambda^n \partial \lambda^m} d\lambda^n d\lambda^m + 2 \frac{\partial^2 H'}{\partial \lambda^n \partial x^A} d\lambda^n dx^A + \frac{\partial^2 H'}{\partial x^A \partial x^B} dx^A dx^B = \\ &= d\left(\frac{\partial H'}{\partial \lambda^n}\right) d\lambda^n + d\left(\frac{\partial H'}{\partial x^A}\right) dx^A = (df^n) d\lambda^n + dx_A dx^A = \\ &= \frac{\partial^2 h}{\partial f_n \partial f_m} df_n df_m + dx_A dx^A. \end{aligned}$$

Let us now apply this methodology to the case of the superfluid.

From the relations $\theta + \frac{1}{2} v^i_N v^i_N = \mu + \alpha$; $L_{Si} L_{Si} = \rho^2_S 2\alpha$ we obtain $\frac{1}{\rho}(dP + \rho_S d\alpha) = d\theta + v^i_N dv^i_N + S dT$,

$$2\alpha\rho_S \frac{\partial \rho_S}{\partial P} dP + \left(2\alpha\rho_S \frac{\partial \rho_S}{\partial \alpha} + \rho^2_S\right) d\alpha = L_{Si} dL_{Si} - 2\alpha\rho_S \frac{\partial \rho_S}{\partial T} dT \tag{26}$$

from which

$$d_\alpha = \frac{2\alpha}{\rho_S} K^{-1} \left[-\rho \frac{\partial \rho_S}{\partial P} d\theta + \frac{v_{Si} - v_{Ni}}{2\alpha} dL_{Si} - \rho \frac{\partial \rho_S}{\partial P} v^i_N dv^i_N + \right.$$

$$-\left(\frac{\partial \rho_S}{\partial T} + \rho S \frac{\partial \rho_S}{\partial P}\right) dT\Bigg],$$

$$dP = 2\alpha K^{-1}\left[\rho\left(\frac{\partial \rho_S}{\partial P} + \frac{K}{2\alpha}\right) d\theta + \frac{v_{Ni} - v_{Si}}{2\alpha} dL_{Si} + \right.$$

$$\left. + \rho\left(\frac{\partial \rho_S}{\partial P} + \frac{K}{2\alpha}\right) v_N^i dv_N^i + \left(\frac{\partial \rho_S}{\partial T} + \rho S\frac{\partial \rho_S}{\partial P} + \rho S \frac{K}{2\alpha}\right) dT\right],$$

with $K = 1 + 2\alpha\left(\frac{1}{\rho_S}\frac{\partial \rho_S}{\partial \alpha} - \frac{\partial \rho_S}{\partial P}\right)$.

From these relations and $2\alpha\rho_S^2 = L_{Si}L_{Si}$, we have also

$$d\rho_S = K^{-1}\left[\rho\frac{\partial \rho_S}{\partial P} d\theta + \frac{v_{Si} - v_{Ni}}{2\alpha}(K-1) dL_{Si} - \rho\frac{\partial \rho_S}{\partial P} v_N^i dv_N^i - \left(\frac{\partial \rho_S}{\partial T} - \rho S \frac{\partial \rho_S}{\partial P}\right) dT\right]. \tag{27}$$

Finally, from these relations and $g_i^A = \rho_S^{-1}\varepsilon_{Aij}L_{Sj} + \varepsilon_{Aij}v_{Nj}$, we have

$$\frac{\partial g_i^A}{\partial \theta} = -\rho_S^{-1}\varepsilon_{Aij}(v_{Sj} - v_{Nj})K^{-1}\rho\frac{\partial \rho_S}{\partial P},$$

$$\frac{\partial g_i^A}{\partial L_{Sk}} = \rho_S^{-1}\varepsilon_{Aij}(v_{Sj} - v_{Nj})K^{-1}\frac{1-K}{2\alpha}(v_{Sk} - v_{Nk}) + \rho_S^{-1}\varepsilon_{Aik},$$

$$\frac{\partial g_i^A}{\partial v_{Nk}} = \rho_S^{-1}\varepsilon_{Aij}(v_{Sj} - v_{Nj})K^{-1}\rho\frac{\partial \rho_S}{\partial P} v_{Nk} + \varepsilon_{Aik},$$

$$\frac{\partial g_i^A}{\partial T} = \rho_S^{-1}\varepsilon_{Aij}(v_{Sj} - v_{Nj})K^{-1}\left(\frac{\partial \rho_S}{\partial T} + \rho S\frac{\partial \rho_S}{\partial P}\right).$$

Therefore, the system (6), in the case of the superfluid, is

$$\partial_t \rho + \partial_k\left[\rho_S v_{Sk} + \rho_N v_{Nk} - \rho_S^{-1}\varepsilon_{Akj}(v_{Sj} - v_{Nj})K^{-1}\rho\frac{\partial \rho_S}{\partial P} x^A\right] = 0, \tag{28}$$

$$\partial_t v_{Si} + \partial_k\left\{(\mu + v_S^2/2)\delta_{ki} + \left[\rho_S^{-1}\varepsilon_{Aki} + \right.\right.$$

$$\left.\left.\rho_S^{-1}\varepsilon_{Akj}(v_{Sj} - v_{Nj})K^{-1}\frac{1-K}{2\alpha}(v_{Si} - v_{Ni}\right] x^A\right\} +$$

$$+ v_{Nj}(\partial_j v_{Si} - \partial_i v_{Sj}) = 0,$$

$$\partial_t(\rho_S v_{Si} + \rho_N v_{Ni}) + \partial_k\left\{P\delta_{ki} + \rho_N v_{Nk}v_{Ni} + \rho_S v_{Sk} v_{Si} + \right.$$

$$\left. + \left[\rho_S^{-1}\varepsilon_{Akj}(v_{Sj} - v_{Nj})K^{-1}\rho\frac{\partial \rho_S}{\partial P} v_{Ni} + \varepsilon_{Aki}\right] x^A\right\} +$$

$$+\rho_S(v_{Nj} - v_{Sj})(\partial_j v_{Si} - \partial_i v_{Sj}) = 0,$$

$$\partial_t(\rho S) + \partial_k \left[\rho S v_{Nk} + \rho_S^{-1}\varepsilon_{Akj}(v_{Sj} - v_{Nj})K^{-1}\left(\frac{\partial \rho_S}{\partial T} + \rho S\frac{\partial \rho_S}{\partial P}\right)x^A\right] = 0,$$

$$\partial_t x^A + \partial_k(\varepsilon_{Akj}v_{Sj}) = 0.$$

Similarly, the system (10) becomes

$$\partial_t \rho + \partial_k \left[\rho_S v_{Sk} + \rho_N v_{Nk} - \rho_S^{-1}\varepsilon_{Akj}(v_{Sj} - v_{Nj})K^{-1}\rho\frac{\partial \rho_S}{\partial P}x^A\right] = 0, \tag{29}$$

$$\partial_t(v_{Si} - \varepsilon_{Abi}x^A v_N^b) + \partial_k \Bigg\{(\mu + v_S^2/2)\delta_{ki} + \Bigg[\rho_S^{-1}\varepsilon_{Aki} +$$

$$+\rho_S^{-1}\varepsilon_{Akj}(v_{Sj} - v_{Nj})K^{-1}\frac{1-K}{2\alpha}(v_{Si} - v_{Ni})\Bigg]x^A\Bigg\} = 0,$$

$$\partial_t[\rho_S v_{Si} + \rho_N v_{Ni} + \rho_S\varepsilon_{Abi}x^A(v_N^b - v_S^b)] +$$

$$+\partial_k \Bigg\{P\delta_{ki} + \rho_N v_{Nk}v_{Ni} + \rho_S v_{Sk}v_{Si} +$$

$$+\left[\rho_S^{-1}\varepsilon_{Akj}(v_{Sj} - v_{Nj})K^{-1}\rho\frac{\partial \rho_S}{\partial P}v_{Ni} + \varepsilon_{Aki}\right]x^A\Bigg\} = 0,$$

$$\partial_t(\rho S) + \partial_k \left[\rho S v_{Nk} + \rho_S^{-1}\varepsilon_{Akj}(v_{Sj} - v_{Nj})K^{-1}\left(\frac{\partial \rho_S}{\partial T} + \rho S\frac{\partial \rho_S}{\partial P}\right)x^A\right] = 0,$$

$$\partial_t x^A + \partial_k(\varepsilon_{Akj}v_{Sj}) = 0.$$

4. WAVE SPEEDS AND EIGENVECTORS OF THE SYSTEM (22)

Let us consider now the system (22), where the underlined terms are those responsible of the modification respect to the original one. The wave speeds, associated to this system, are the eigenvalues λ of the problem

$$-\lambda d\rho + n_k d(\rho_S v_{Sk} + \rho_N v_{Nk}) = 0,$$
$$-\lambda d v_{Si} + n_i d(\mu + v_S^2/2) + v_{Nj}(n_j dv_{Si} - n_i dv_{Sj}) = 0,$$
$$-\lambda d(\rho_S v_{Si} + \rho_N v_{Ni}) + n_k d(P\delta_{ki} + \rho_N v_{Nk}v_{Ni} + \rho_S v_{Sk}v_{Si}) +$$
$$+\rho_S(v_{Nj} - v_{Sj})(n_j dv_{Si} - n_i dv_{Sj}) = 0,$$
$$-\lambda d(\rho S) + n_k d(\rho S v_{Nk}) = 0.$$

We will see that $\lambda = v_{Nk}n_k$ is one of these eigenvalues, and has multiplicity 4. The equation for the other eigenvalues will be indicated; in two special cases, this equation is a biquadratic one. The general case can be subdivided in the following particular cases:

Case 1: $\underline{v}_S = \underline{v}_N$. The system (30) becomes

$$\begin{aligned}
&(-\lambda + v_{Nk}n_k)d\rho + n_k(\rho_S dv_{Sk} + \rho_N dv_{Nk}) = 0,\\
&(-\lambda + v_{Nk}n_k)dv_{Si} + n_i d\mu = 0,\\
&(-\lambda + v_{Nk}n_k)(\rho_S dv_{Si} + \rho_N dv_{Ni}) + n_i dP = 0,\\
&(-\lambda + v_{Nk}n_k)\rho dS + S\rho_S n_k d(v_{Nk} - v_{Sk}) = 0,
\end{aligned} \tag{31}$$

where $(31)_3$ is the sum of $(30)_1$, multiplied by $-v_{Ni}$, and of $(30)_3$; similarly, $(31)_4$ is the sum of $(30)_1$, multiplied by $-S$, and of $(30)_4$. Moreover, $d\alpha = (v_{Nj} - v_{Sj})\, d(v_{Nj} - v_{Sj}) = 0$.

We see that $\lambda = v_{Nk}n_k$ is an eigenvalue with multiplicity 4; a set of 4 *l. i.* eigenvectors $(dP, dT, d\underline{v}_S, d\underline{v}_N)$ corresponding to this eigenvalue are

$$(0, 0, \underline{e}_1, \underline{0}), (0, 0, \underline{e}_2, \underline{0}), (0, 0, \underline{0}, \underline{e}_1), (0, 0, \underline{0}, \underline{e}_2)$$

where $\underline{e}_1$, $\underline{e}_2$ are 2 *l. i.* vectors orthogonal to $\underline{n}$.

If $\lambda \neq v_{Nk}n_k$, $(31)_{2,3}$ give $dv_{Si} = (\lambda - v_{Nk}n_k)^{-1} d\mu n_i$, $dv_{Ni} = (\lambda - v_{Nk}n_k)^{-1}\, (dP - \rho_S d\mu)\rho_N^{-1}$ n_i, and $(31)_{1,4}$ become

$$\left[(-\lambda + v_{Nk}n_k)^2 \frac{\partial^2 \mu}{\partial P^2} + \frac{1}{\rho^2}\right] dP + (-\lambda + v_{Nk}n_k)^2 \frac{\partial^2 \mu}{\partial P\, \partial T} dT = 0,$$

$$(-\lambda + v_{Nk}n_k)^2 \frac{\rho_N}{\rho} \frac{\partial^2 \mu}{\partial P\, \partial T} dP + \left[(-\lambda + v_{Nk}n_k)^2 \frac{\rho_N}{\rho} \frac{\partial^2 \mu}{\partial T^2} + S^2 \frac{\rho_S}{\rho}\right] dT = 0.$$

Therefore, we obtain 4 other real eigenvalues, given by

$$\begin{aligned}
(-\lambda + v_{Nk}n_k)^2 = \frac{1}{2}\left[\frac{\partial^2 \mu}{\partial P^2}\frac{\partial^2 \mu}{\partial T^2} - \left(\frac{\partial^2 \mu}{\partial P\, \partial T}\right)^2\right]^{-1} \Bigg\{ - \frac{\partial^2 \mu}{\partial P^2} S^2 \frac{\rho_S}{\rho_N} + \\
- \frac{\partial^2 \mu}{\partial T^2}\frac{1}{\rho^2} \pm \left[\left(\frac{\partial^2 \mu}{\partial P^2} S^2 \frac{\rho_S}{\partial_N} - \frac{\partial^2 \mu}{\partial T^2}\frac{1}{\rho^2}\right)^2 + 4\left(\frac{\partial^2 \mu}{\partial P\, \partial T}\right)^2 \frac{S^2}{\rho^2}\frac{\rho_S}{\rho_N}\right]^{1/2}\Bigg\},
\end{aligned}$$

these eigenvalues are distinct, except for the subcase $\frac{\partial^2 \mu}{\partial P\, \partial T} = 0$, $\frac{\partial^2 \mu}{\partial T^2} = \frac{\rho_S}{\rho_N}\rho^2\, S^2\, \frac{\partial^2 \mu}{\partial P^2}$. In this particular subcase, the eigenvalues distinct from $v_{Nk}\, n_k$ are $\lambda = v_{Nk}\, n_k \pm \left(\frac{\partial^2 \mu}{\partial P^2}\rho^2\right)^{-1/2}$ with multiplicity 2 and 2 *l. i.* eigenvectors (dP and dT are arbitrary).

In the next cases we shall have $\underline{v}_S \neq \underline{v}_N$. Therefore we can define $\underline{W}$ from $\underline{v}_S = \sqrt{2\alpha}\, \underline{W}$ $+\underline{v}_N$, from which it follows $W_i W_i = 1$.

In this way the system (30) becomes

$$\begin{aligned}
&- \lambda d\rho + n_k d(\rho v_{Sk} + \sqrt{2\alpha}\rho_N W_k) = 0,\\
&(v_{Nk}n_k - \lambda)dv_{Si} + n_i d(\mu - \sqrt{2\alpha} W_j dv_{Sj}) = 0,\\
&(v_{Nk}n_k - \lambda)d(\sqrt{2\alpha}\rho_N W_i) + n_i(\rho S dT + \rho_N d\alpha + \\
&\qquad + \rho_N\sqrt{2\alpha} W_j dv_{Sj}) + n_k\sqrt{2\alpha}\rho_N W_i d(\sqrt{2\alpha} W_k + v_{Sk}) = 0,\\
&(v_{Nk}n_k - \lambda)\rho dS + S n_k d(\sqrt{2\alpha}\rho_S W_k) = 0,
\end{aligned} \tag{32}$$

where $(31)_3$ is the sum of $(29)_3$, $(29)_1$ multiplied by $-v_{Si}$, $(29)_2$ multiplied by $-\rho$; similarly, $(31)_4$ is the sum of $(29)_1$ multiplied by $-S$ and of $(29)_4$.

Case 2: $\underline{v}_S \neq \underline{v}_N$, $(\underline{v}_S - \underline{v}_N)^2 - [n_i(v_{Si} - v_{Ni})]^2 = 0$.

In this case we have $\underline{W} = W\underline{n}$ with $W = \pm 1$. We see that $\lambda = v_{Nk}n_k$ is an eigenvalue with multiplicity 4 for the system (31). The corresponding eigenvectors are $dP = dT = d\alpha = 0$,

$$dv_{Si} = X_1 e_{1i} + X_2 e_{2i}, dW_i = X_3 e_{1i} + X_4 e_{2i},$$

where $\underline{e}_1, \underline{e}_2$ are two *l. i.* vectors orthogonal to $\underline{n}$ and X_1, X_2, X_3, X_4 are arbitrary coefficients; therefore, we have 4 *l. i.* eigenvectors.

If $\lambda \neq v_{Nk}n_k$, eq. $(31)_3$ gives $dW_i = Yn_i$; but $W_iW_i = 1$ implies $W_i dW_i = 0$, from which $Y = 0$ follows. Therefore $(31)_{2,3}$ give $dv_{Si} = Y_1\ Wn_i$; $dW_i = 0$ which allow the system (32) to be written as

$$\begin{aligned}
&(v_{Nk}n_k - \lambda)d\rho - Wd(\sqrt{2\alpha}\rho_S) + \rho W(Y_1 + d\sqrt{2\alpha}) = 0,\\
&(v_{Nk}n_k - \lambda)Y_1 W + d\mu - \sqrt{2\alpha}Y_1 = 0,\\
&(v_{Nk}n_k - \lambda)[\rho Y_1 W + \rho W d\sqrt{2\alpha} - Wd(\sqrt{2\alpha}\rho_S)] + dP + \rho_S d\alpha +\\
&\qquad - 2\sqrt{2\alpha}\rho_S(Y_1 + d\sqrt{2\alpha}) + \sqrt{2\alpha}d(\sqrt{2\alpha}\rho_S) = 0,\\
&(v_{Nk}n_k - \lambda)d(\rho S) + \rho S W(Y_1 + d\sqrt{2\alpha}) = 0,
\end{aligned} \tag{33}$$

where $(33)_3$ is the sum of $(32)_2$ multiplied by ρ, of $(32)_1$ multiplied by $-W\sqrt{2\alpha}$, and of $(32)_3$; moreover $(33)_4$ is the sum of $(32)_1$ multiplied by S, and of $(32)_4$.

Let us consider the change of variables

$$\begin{aligned}
dP &= \rho\left(1 + 2\alpha K^{-1}\frac{\partial \rho_S}{\partial P}\right)X_1 + W\sqrt{2\alpha}K^{-1}X_2 + 2\alpha K^{-1}\left(\frac{\partial \rho_S}{\partial T} + \rho S\frac{\partial \rho_S}{\partial P} + K\frac{\rho S}{2\alpha}\right)X_4,\\
dT &= X_4,\\
d\alpha &= -2\alpha K^{-1}\frac{\rho}{\rho_S}\frac{\partial \rho_S}{\partial P}X_1 - W\sqrt{2\alpha}K^{-1}\frac{1}{\rho_S}X_2 +\\
&\quad - 2\alpha K^{-1}\frac{1}{\rho_S}\left(\frac{\partial \rho_S}{\partial T} + \rho S\frac{\partial \rho_S}{\partial P}\right)X_4,\\
Y_1 &+ d\sqrt{2\alpha} = WX_3,
\end{aligned} \tag{34}$$

where $K = 1 + 2\alpha\left(\frac{1}{\rho_S}\frac{\partial \rho_S}{\partial \alpha} - \frac{\partial \rho_S}{\partial P}\right)$.

The system (33) becomes

$$\sum_{j=1}^{4}[b_{ij} + (v_{Nk}n_k - \lambda)a_{ij}]X_j = 0 \tag{35}$$

where

$$
\begin{aligned}
a_{11} &= -\rho^3 \frac{\partial^2 \mu}{\partial P^2}\left(1 + 2\alpha K^{-1}\frac{\partial \rho_S}{\partial P}\right) + \rho^2 \frac{\partial^2 \mu}{\partial P \partial \alpha} 2\alpha K^{-1} \frac{\rho}{\rho_S}\frac{\partial \rho_S}{\partial P},\\
a_{12} &= a_{21} = W\sqrt{2\alpha}K^{-1}\frac{\rho}{\rho_S}\frac{\partial \rho_S}{\partial P}, a_{13} = a_{31} = 0\\
a_{14} &= a_{41} = \frac{\partial(\rho S)}{\partial P}\rho\left(1 + 2\alpha K^{-1}\frac{\partial \rho_S}{\partial P}\right) +\\
&\quad - \frac{\partial(\rho S)}{\partial \alpha} 2\alpha K^{-1}\frac{\rho}{\rho_S}\frac{\partial \rho_S}{\partial P},\\
a_{22} &= \frac{1}{K\rho_S}, a_{23} = a_{32} = 1,\\
a_{24} &= a_{42} = \left(\frac{\partial \rho_S}{\partial T} + \rho S\frac{\partial \rho_S}{\partial P}\right)\sqrt{2\alpha}K^{-1}\frac{1}{\rho_S}W,\\
a_{33} &= \rho, a_{34} = a_{43} = 0,\\
a_{44} &= \frac{\partial(\rho S)}{\partial P} 2\alpha K^{-1}\left(\frac{\partial \rho_S}{\partial T} + \rho S\frac{\partial \rho_S}{\partial P} + K\frac{\rho S}{2\alpha}\right) + \frac{\partial(\rho S)}{\partial T} +\\
&\quad - \frac{\partial(\rho S)}{\partial \alpha} 2\alpha K^{-1}\frac{1}{\rho_S}\left(\frac{\partial \rho_S}{\partial T} + \rho S\frac{\partial \rho_S}{\partial P}\right),\\
b_{11} &= 0, b_{12} = b_{21} = 1, b_{13} = b_{31} = \rho, b_{14} = b_{41} = 0,\\
b_{22} &= 0, b_{23} = b_{32} = -\sqrt{2\alpha}W, b_{24} = b_{42} = 0,\\
b_{33} &= -2\rho_S\sqrt{2\alpha}W, b_{34} = b_{43} = \rho S, b_{44} = 0.
\end{aligned}
\tag{36}
$$

Moreover, $\sum_{i,j=1}^{4} a_{ij}X_iX_j = d\rho d(\mu + \alpha) + dTd(\rho S) + d(\rho_S \sqrt{2\alpha})\, d\sqrt{2\alpha} + \rho\, (W_i dv_{Ni})^2 - 2\, d(\rho_S \sqrt{2\alpha})\, W_i dv_{Ni}$ which is equal to the quadratic form Q in eq. (21), multiplied by T and calculated for $dv_S^i = W_j\, dv_{Sj}\, W^i$, $dv_N^i = dv_S^i + W^i\, d\sqrt{2\alpha}$ (from which $dW^i = 0$). Therefore, a_{ij}, b_{ij} are symmetric and a_{ij} is positive definite; consequently, the problem (37) has 4 real eigenvalues and 4 *l. i.* eigenvectors. These eigenvalues are distinct from $\lambda = v_{Nk}\, n_k$, because $\det(b_{ij}) = \rho^2 S^2 \neq 0$.

Case 3: $\underline{v}_S \neq \underline{v}_N$, $(\underline{v}_S - \underline{v}_N)^2 - [n_i(v_{Si} - v_{Ni})]^2 \neq 0$.

In this case $\underline{W}$ and $\underline{n}$ are *l. i.*. We see that $\lambda = v_{Nk} n_k$ is an eigenvalue with multiplicity 4 for the system (32). The corresponding eigenvectors are given by

$$
\begin{aligned}
dW_k &= X_1[n_k - W_i n_i W_k] + X_2 \varepsilon_{kij} n_i W_j,\\
dv_{Sk} &= X_3 n_k + X_4 W_k + X_5 \varepsilon_{kij} n_i W_j,\\
dP &= -\frac{\rho}{\rho_N}\rho_S S dT - \rho_S d\alpha,
\end{aligned}
$$

where (X_1, X_3, X_4) is the solution of the system

$$\rho_S\sqrt{2\alpha}X_1[1-(W_in_i)^2]+W_in_id(\rho_S\sqrt{2\alpha})=0,$$
$$X_3+X_4W_in_i=\frac{1}{\rho_S}W_in_i\sqrt{2\alpha}d\rho_S,$$
$$X_3W_in_i+X_4=-\frac{\rho S}{\rho_N\sqrt{2\alpha}}dT-d\sqrt{2\alpha},$$

while $X_2, X_5, dT, d\alpha$ are free unknowns; consequently there are 4 *l. i.* eigenvectors corresponding to $\lambda = v_{Nk}n_k$.

If $\lambda \neq v_{Nk}n_k$, from $(32)_2$ we see that $dv_{Si} = X_1n_i$, $dW_i = X_2\,(n_i - W_jn_jW_i)$, where $W_idW_i = 0$ has been also considered. By using these relations, the system (33) becomes

$$(v_{Nk}n_k-\lambda)d\rho-\sqrt{2\alpha}n_kW_kd\rho_S+n_kW_k\rho_Nd\sqrt{2\alpha}+\rho X_1+\sqrt{2\alpha}\rho_N[1-(n_kW_k)^2]X_2=0, \quad (38)$$
$$d\mu+(v_{sk}n_k-\lambda)X_1=0,$$
$$dP-\rho d\mu+n_kW_k\rho_N\sqrt{2\alpha}X_1+(v_{nk}n_k-\lambda)\rho_N\sqrt{2\alpha}X_2=0,$$
$$\sqrt{2\alpha}(v_{Nk}n_k-\lambda)d\rho_N+\rho_N(v_{Nk}n_k-\lambda+\sqrt{2\alpha}W_kn_k)d\sqrt{2\alpha}+$$
$$+\left\{\sqrt{2\alpha}[1-(n_kW_k)^2]-n_iW_i(v_{Nk}n_k-\lambda)\right\}\rho_N\sqrt{2\alpha}X_2+$$
$$\sqrt{2\alpha}\rho_NX_1=0,$$
$$\rho dS+Sd\rho-\frac{\rho S}{\rho_N}d\rho_N-\frac{\rho S}{2\alpha}d\alpha+\rho Sn_iW_iX_2=0,$$

where the sum of $(38)_4$ multiplied by $-\frac{\rho S}{\rho_N\sqrt{2\alpha}}$, of $(38)_1$ multiplied by S, and of $(32)_4$ has been taken; the result, divided by $(v_{Nk}\,n_k-\lambda)$, gives $(38)_5$.

Obviously, the system (38) can be put in the symmetric form; the resulting expression is very long to write. Consequently, I just notice how the subcase $W_kn_k = 0$ is enough simple and elegant. In fact, in this case $(38)_{2,3}$ give

$$X_1=-(v_{Nk}n_k-\lambda)^{-1}d\mu, X_2=[\sqrt{2\alpha}\rho_N(v_{Nk}n_k-\lambda)]^{-1}(\rho d\mu-dP).$$

By using these expressions, the eqs. $(38)_{1,4}$ become

$$-(v_{Nk}n_k-\lambda)^2d\rho+dP=0,$$
$$(v_{Nk}n_k-\lambda)^2\rho d(-\rho_N/\rho)-(v_{Nk}n_k-\lambda)^2\frac{\rho_N}{2\alpha}d\alpha+\frac{\rho_S}{\rho}dP-\rho_Sd\mu=0, \quad (39)$$

where $(39)_2$ is the sum of $(38)_4$ multiplied by $(v_{Nk}n_k-\lambda)$, and of $(38)_1$ multiplied by $-\frac{\rho_N}{\rho}\sqrt{2\alpha}\,(v_{Nk}n_k-\lambda)$.

Let us consider the change of variables $Y_1 = dP$, $Y_2 = \rho S\,\sqrt{\rho_S/\rho_N}\,(dT+\frac{\rho_N}{\rho S}d\alpha)$, $Y_3 = d\alpha$. Eq. $(38)_5$ can be written as

$$Y_3=q^{-1}\left[\left(-\rho\frac{\partial^2\mu}{\partial P\partial T}+\frac{\rho^2S}{\rho_N}\frac{\partial^2\mu}{\partial P\partial\alpha}\right)Y_1+\left(-\rho\frac{\partial^2\mu}{\partial T^2}+\frac{\rho^2S}{\rho_N}\frac{\partial^2\mu}{\partial\alpha\partial T}\right)\frac{1}{\rho S}\sqrt{\rho_N/\rho_S}Y_2\right], \quad (40)$$

with

$$q = -\frac{\rho_N}{S}\left[\frac{\partial^2 \mu}{\partial T^2} - 2\frac{\rho S}{\rho_N}\frac{\partial^2 \mu}{\partial \alpha \partial T} + \left(\frac{\rho S}{\rho_N}\right)^2 \frac{\partial^2 \mu}{\partial \alpha^2} - \rho\frac{S^2}{2\alpha\rho_N}\right] > 0.$$

By substituting this expression in eqs. (39), after having multiplied $(39)_2$ by $\rho / \sqrt{\rho_S \rho_N}$, this system becomes

$$\sum_{j=1}^{2}[(v_{Nk}n_k - \lambda)^2 a_{ij} + \delta_{ij}]Y_j = 0 \tag{41}$$

for $i = 1, 2$ and with

$$a_{11} = \rho^2\frac{\partial^2 \mu}{\partial T^2} + q^{-1}\rho^4\frac{S}{\rho_N}\left(\frac{\partial^2 \mu}{\partial P \partial \alpha} - \frac{\rho_N}{\rho S}\frac{\partial^2 \mu}{\partial P \partial T}\right)^2,$$

$$a_{22} = \frac{\rho}{\rho_S S^2}\frac{\partial^2 \mu}{\partial T \partial \alpha} +$$
$$+ \frac{1}{\rho_S S q}\left(\rho\frac{\partial^2 \mu}{\partial \alpha^2} - \frac{\rho_N}{2\alpha} - \frac{\rho_N}{S}\frac{\partial^2 \mu}{\partial T \partial \alpha}\right)\left(-\rho\frac{\partial^2 \mu}{\partial T^2} + \frac{\rho^2 S}{\rho_N}\frac{\partial^2 \mu}{\partial \alpha \partial T}\right),$$

$$a_{12} = a_{21} = \frac{\rho^2}{\sqrt{\rho_S\rho_N}}\frac{\partial^2 \mu}{\partial P \partial \alpha} +$$
$$+ \frac{1}{q\sqrt{\rho_S\rho_N}}\left(\rho^2\frac{\partial^2 \mu}{\partial \alpha^2} - \frac{\rho_N\rho}{2\alpha} - \frac{\rho_N\rho}{S}\frac{\partial^2 \mu}{\partial T \partial \alpha}\right)\left(-\rho\frac{\partial^2 \mu}{\partial P \partial T} + \frac{\rho^2 S}{\rho_N}\frac{\partial^2 \mu}{\partial \alpha \partial P}\right).$$

Now the quadratic form

$$Q = \frac{\partial^2 \mu}{\partial P^2}(dP)^2 + 2\frac{\partial^2 \mu}{\partial P \partial T}dPdT + 2\frac{\partial^2 \mu}{\partial P \partial \alpha}dPd\alpha + \frac{\partial^2 \mu}{\partial T^2}(dT)^2 +$$
$$+ 2\frac{\partial^2 \mu}{\partial T \partial \alpha}dTd\alpha + \left(\frac{\partial^2 \mu}{\partial \alpha^2} - \frac{\rho_N}{2\alpha\rho}\right)(d\alpha)^2$$

is negative definite; moreover

$$\rho^2 Q = a_{11}(Y_1)^2 + 2a_{12}Y_1Y_2 + a_{22}(Y_2)^2 - \frac{\rho_n}{S}q\left\{Y_3 +\right.$$
$$- q^{-1}\left[\left(-\rho\frac{\partial^2 \mu}{\partial P \partial T} + \frac{\rho^2 S}{\rho_N}\frac{\partial^2 \mu}{\partial \alpha \partial P}\right)Y_1 + \left(-\rho\frac{\partial^2 \mu}{\partial T^2} +\right.\right.$$
$$\left.\left.\left. + \frac{\rho^2 S}{\rho_N}\frac{\partial^2 \mu}{\partial \alpha \partial T}\right)\frac{1}{\rho S}\sqrt{\rho_N / \rho_S}Y_2\right]\right\}^2;$$

by using eq. (40), we see that the matrix $A = (a_{ij})$ is negative definite. Let $\omega_1 > 0$, $\omega_2 > 0$ be the real eigenvalues of the matrix $-A$; we have $\det(-A - \omega_i I) = 0$, from which $\det(\omega_i^{-1} A + I) = 0$. Therefore, for the system (41) we have 4 real eigenvalues, *i.e.*, $\lambda = \pm\omega_1^{-1/2} + v_{Nk}\, n_k$ and $\lambda = \pm\omega_2^{-1/2} + v_{Nk}n_k$.

In this way, the eigenvalues and eigenvectors problem for the system (22), has been completely investigated.

Obviously, the present methodology can be applied to many other physical problem and this will be the object of future works.

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