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COUNTABLY ENLARGING WEAK BARRELLEDNESS[1]

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Abstract. *If* (E,ξ) *is a locally convex space with dual* E' *and* η *is the coarsest topology finer than* ξ *such that the dual of* (E,η) *is* $E'+M$ *for a given* $\aleph_0$*-dimensional subspace* $M \subset E^*$ *transverse to* E'*, then* η *is a* countable enlargement *(CE) of* ξ*. Here most barrelled CE (BCE) results are optimally extended within the fourteen properties introduced in the 1960s, '70s, '80s, '90s and recently studied in "Reinventing weak barrelledness", et al. If a CE exists, one exists with none of the fourteen properties. Yet CEs that preserve precise subsets of these properties essentially double the stock of distinguishing examples. If a CE exists, must one exist that preserves a given property enjoyed by* ξ*? Under metrizability, the fourteen cases become two: the metrizable BCE question we answered earlier, and the metrizable inductive CE (ICE) question we answer here (both positively). Without metrizability we are as yet unable to answer Robertson, Tweddle and Yeomans' original BCE question (1979), the ICE question and four others. We give negative answers for the eight remaining general cases, those between* $\aleph_0$*-barrelled and dual locally complete, inclusive, under the ZFC-consistent assumption that* $\aleph_1 < \mathfrak{b}$.

1 The fourteen properties

A decade ago Pérez Carreras and Bonet [8] segregated into Chapters 4 and 8 what they and others such as Robertson, Tweddle, Yeomans, Tsirulnikov, Husain, Webb, De Wilde, Ruess, Valdivia, Dierolf, Saxon, Levin had learned about BCEs and weak barrelledness (WB). Late breakthroughs in both areas cultivate (i) a promising partnership between BCEs and small cardinals [15, 18, 19, 20, 27], and (ii) a clearer, more comprehensive WB view [21, 22, 23, 24, 25, 26] whose Mackey aspect alone answers questions from 1971, 1982, 1991. Surveys [12, 30] supply additional definitions/motivations. After the initial BCE/WB encounter [29] comes now the full-scale union.

Countably enlarging just three WB properties [29] repaid BCEs with, e.g., this theorem: a CE preserves barrelledness when it retains the weaker property (S). For the best possible version within a much larger WB context, we substitute the yet weaker property of being dual locally complete (dlc). Here, in fact, most BCE results find optimal expression as they borrow from, and then add distinguishing examples to, all the WB wealth depicted in [23]:

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barrelled; (p_α)b	$\Longleftrightarrow$	barrelled; (f_α)b
$\Downarrow$		$\Downarrow$
$\aleph_0$-barrelled; (p_n)b	$\Longrightarrow$	ℓ^∞-barrelled; (f_n)b
1) separable		
$\Downarrow$		$\Downarrow$
C-barrelled; $(p_n)\,0$	$\Longrightarrow$	c_0-barrelled; $(f_n)\,0$
$\Downarrow$		$\Downarrow$
property (L); $p\|_{A_n}$	$\Longrightarrow$	dlc; $f\|_{A_n}$
2) Mackey		
3) metrizable		
$\Downarrow$		$\Downarrow$
inductive; $p\|_{E_n}$	$\Longrightarrow$	primitive; $f\|_{E_n}$
4) Mackey		

The mnemonic symbols after the semicolons recall seminorm p vs. linear form f definitions/characterizations for left vs. right sides; e.g., a locally convex space is $\langle \aleph_0\text{-}\rangle$ [C-] barrelled if and only if every pointwise $\langle$bounded$\rangle$ [null] sequence of continuous seminorms is equicontinuous, while the definition of $\langle \ell^\infty\text{-}\rangle$ [c_0-] barrelled spaces replaces "seminorms" with "linear forms". Again, a locally convex space E $\langle$has property $(L)\rangle$ [is inductive] if and only if, given a seminorm p and an absorbing sequence $\langle \{A_n\}\rangle$ $[\{E_n\}_n$ consisting of linear subspaces] with each $\langle p|_{A_n}\rangle$ $[p|_{E_n}]$ continuous, it must happen that p itself is continuous; while E is $\langle$dlc$\rangle$ [primitive] if and only if the preceding holds with linear form f in place of seminorm p. (Recall that a *$\langle$closed$\rangle$ absorbing sequence* is an increasing sequence of $\langle$closed$\rangle$ balanced convex sets whose union is absorbing.)

Beside the nine in the table, [7] defines two more weak barrelledness properties (C) and (S), with ℓ^∞-barrelled $\Rightarrow$ property (C) $\Rightarrow$ property (S) $\Rightarrow$ dlc. And Baire-like (BL) $\Rightarrow$ quasi-Baire (QB) $\Rightarrow$ barrelled [13, 16] suggests two *strong* barrelledness properties that would also fit inside the table's box 3), where concepts coincide under metrizability (cf. [5]). Finally, a locally convex space is S_σ if it is the union of an increasing sequence of proper closed subspaces, and quasi-Baire $\Rightarrow$ non-S_σ $\Rightarrow$ inductive [23]. The property of being non-S_σ is neither stronger nor weaker than barrelledness, but completes the total of fourteen properties we are concerned with here and in other recent papers on weak barrelledness. [A space (locally convex, Hausdorff, with real or complex scalar field) is *QB* if it is barrelled and non-S_σ, is *BL* if some member of every closed absorbing sequence is a neighborhood of zero.] Five of the fourteen are duality invariant properties, comprising properties (C), (S), dlc, primitive and non-S_σ. However, we will see here, as in other recent papers, that the property of being non-S_σ and the nine non-duality invariant properties exhibit similar behavior opposite to that of the four duality invariant properties between property (C) and primitivity. Another behavioral dichotomy separates those spaces that are non-S_σ, inductive or primitive from those having one of the eleven other properties. Exceptional behavior is often exhibited by C- and c_0-barrelledness.

2 Criteria for property-preserving enlargements

Let (E,ξ) be a space with dual E' and let M be a subspace of E^* transverse to E'. The topology $\eta = \sup(\xi, \sigma(E, E'+M))$ is the coarsest locally convex topology finer than ξ which makes each element of M continuous, and a basic η-neighborhood of 0 is formed by intersecting a ξ-neighborhood of 0 with the polar of a finite subset of M. The dual of (E,η) is $E'+M$.

Theorem 1 *If M is finite-dimensional, then, since all fourteen properties are preserved by quotients and by products with finite-dimensional spaces, (E,η) has a given one of the fourteen properties if and only if $(M^{\perp},\eta) = (M^{\perp},\xi)$ has the given property.*

When M is finite-dimensional, η is a *finite enlargement* of ξ and the finite-codimensional $M^{\perp}$ must preserve precisely twelve of the fourteen properties [23].

Corollary 2 *Finite enlargements preserve all fourteen properties except C- and c_0-barrelledness.*

If M is $\aleph_0$-dimensional, we say that η is a *countable enlargement* (CE) of ξ. Though at a slight variance with Tweddle and Yeomans' usage [31], the present is more generally suitable, as suggested by Tweddle and Catalán [29]. For ξ barrelled, the CE $\eta = \sup(\xi, \sigma(E, E'+M))$ is barrelled (a BCE) if and only if the Mackey topology $\tau(E, E'+M)$ is. Indeed, [31, Theorem 2] (in [30], Theorem 1) shows that if the latter topology is barrelled, it must coincide with the former, and every barrelled space is Mackey. Thus these definitions are interchangeable when discussing the long-standing BCE problem, which asks if every barrelled space E with $E' \neq E^*$ (i.e., with E *non-trivial* [30]) must admit a BCE. More generally, if a space E with a given one of the fourteen properties admits a CE, must it admit a CE that preserves the given property? Beyond the BCE question, three of these fourteen CE questions were considered by Tweddle and Catalán [29].

The CE answers are emphatically not of the simple positive sort found in the Corollary, as proves the following theorem, the most general form possible of [10, Theorem 3] and [29, Proposition 5] within context.

The sequel preserves notation and assumes η is the CE $\sup(\xi, \sigma(E, E'+M))$.

Theorem 3 *If (E,ξ) admits a CE, it admits one that is not primitive.*

Proof. If ξ is not primitive and η is, then there exist an increasing covering sequence $\{E_n\}_n$ of subspaces, $f \in E'$ and $g \in M \setminus \{0\}$ such that each $(f+g)|_{E_n} \in (E_n,\xi)'$. If $\check{M}$ is a 1-codimensional subspace of M with $g \notin \check{M}$, then $f+g \notin E'+\check{M}$ and the corresponding CE $\check{\eta}$ is not primitive.

If (E,ξ) is primitive and admits a CE, it has a dense $\aleph_0$-codimensional subspace H [23] with cobasis $\{x_n\}_n$. Let M be the span of $\{f_n\}_n \subset E^*$, where each f_n vanishes both on H and at x_j $(j > n)$ and has value 1 at x_i $(1 \le i \le n)$. By density M is transverse to E'. The corresponding CE η is not primitive, since the linear form that vanishes on H and is 1 at each x_n is not in $E'+M$. □

We say that a subset of a vector space has a certain *dimension*, or *codimension*, or is *transverse* to some subspace, if its linear span has the corresponding property. Several times we will use the following elementary fact:

(*) A barrel is a neighborhood of 0 if it intersects a finite-codimensional subspace in a relative neighborhood of 0.

Tweddle-Yeomans [31] and Tweddle-Catalán [29] showed that if ξ is barrelled, is $\aleph_0$-barrelled, is ℓ^∞-barrelled, or has property (S), then η has the same property if and only if η satisfies the Tweddle-Yeomans Criterion (given below). Thus η preserves the stronger properties if it preserves the weaker property (S). We shall replace property (S) with dual local completeness to obtain the best possible result in this context.

Definition 4 *The topology* η *satisfies the* Tweddle-Yeomans Criterion *if there is no infinite-dimensional* $\sigma(E'+M,E)$*-bounded set transverse to* E'.

Theorem 5 *Suppose* (E,ξ) *has a given one of the eleven properties between BL and dlc, inclusive.* (E,η) *has the same property if and only if*

1. η *satisfies the Tweddle-Yeomans Criterion and*

2. $(L^\perp,\xi)$ *has the given property for each finite-dimensional subspace* $L\subset M$.

Note: Part (2) is superfluous except for C- and c_0-barrelledness [23]. Consequently, if (E,ξ) has one of the properties stronger than dual local completeness, excepting the C- and c_0-barrelled cases, then (E,η) has the same stronger property if [and only if] (E,η) is dual locally complete.

Proof. [Necessity]. If (E,ξ) and (E,η) are both dlc and N is a $\sigma(E'+M,E)$-bounded set transverse to E', its balanced convex $\sigma(E'+M,E)$-closed hull B is the unit ball of a Banach space X and $B\cap E'$ is likewise a Banach disk. Therefore $Y=E'\cap X$ is a closed subspace of X. Since $\dim(X/Y)\leq\dim((E'+M)/E')=\aleph_0$, we have $\dim(X/Y)<\aleph_0$ (no Banach space has dimension $\aleph_0$), and thus N has finite dimension. That is, the Tweddle-Yeomans Criterion holds.

Part (2) needs proof in the C- and c_0-barrelled cases only. Since a dense c_0-barrelled subspace of a C-barrelled space is also C-barrelled [23], the proof reduces to showing that if (E,ξ) and (E,η) are both c_0-barrelled, so is $(L^\perp,\xi)$ for each finite-dimensional $L\subset M$. By Theorem 1, this is equivalent to showing (E,τ) is c_0-barrelled, where τ is the finite enlargement of ξ corresponding to L. If $\{f_n\}_n$ is a $\sigma(E'+L,E)$-null sequence, its polar B is a τ-barrel. Now B is an η-neighborhood of 0, and thus, by definition of η, meets a finite-codimensional subspace of E in a relative ξ-, hence a relative τ-neighborhood of 0. From (*), then, B is a neighborhood of 0 in (E,τ), as desired.

[Sufficiency]. For the remainder of the proof, assume η satisfies the Tweddle-Yeomans Criterion. Consider a $\sigma(E'+M,E)$-bounded sequence $(f_n)_n$, necessarily contained in $E'+L$ for some finite-dimensional subspace L of M. If ξ has property (C); has property (S) and $\{f_n\}_n$ is $\sigma(E'+M,E)$-Cauchy; or is dual locally complete and $\{a_n\}_n\in\ell^1$, then, since the finite enlargement $\sup(\xi,\sigma(E'+L))$ has the same property as ξ, in the $\sigma(E'+L,E)$ topology the sequence $\{f_n\}_n$ is such that, respectively, it has an adherence point; it has a limit; or the series $\sum_n a_nf_n$ converges, using [21, Theorem 2.3]. The same is thus respectively true in the $\sigma(E'+M,E)$ topology, which proves η preserves properties (C), (S) and dual local completeness.

Let $\{A_n\}_n$ be a closed absorbing sequence in (E,η). Taking polars in the duality $\langle E, E'+M\rangle$, there must be some q for which $A_q^\circ \subset E'+L$ with $L \subset M$ finite-dimensional. Otherwise, we could inductively find $\{f_n\}_n \subset E'+M$ with each $f_{n+1} \in A_{n+1}^\circ \setminus [E'+\mathrm{sp}(\{f_1 \ldots f_n\})]$, contradicting the Criterion. Now $\{A_n\}_{n\geq q}$ is a closed absorbing sequence in (E,τ), where τ is the finite enlargement of ξ corresponding to L. If ξ is BL; is QB and each A_n is a subspace; or is barrelled and each $A_n = A_1$, then, since τ has the same property as ξ, some A_n $(n \geq q)$ is a τ-neighborhood, hence an η-neighborhood of 0 in E. This shows that η preserves BL, QB and barrelled spaces.

Finally, let $\{U_n\}_n$ be a sequence of absolutely convex closed 0-neighborhoods in (E,η) whose intersection U is absorbing. The Criterion provides a finite-dimensional $L \subset M$ such that $U^\circ \subset E'+L$, so each $U_n^\circ \subset E'+L$. Thus each U_n is closed for the corresponding finite enlargement τ. From the definition of η, each U_n meets a finite-codimensional subspace of E in a relative ξ-, hence a relative τ-neighborhood of 0. From (*), then, each U_n is a (closed) neighborhood of 0 in (E,τ). (i) If ξ is $\aleph_0$-barrelled, so is τ (Corollary 2). Therefore U is a τ-, hence an η-neighborhood of 0; i.e., η is also $\aleph_0$-barrelled. (ii) Similarly, replacing U by $\cap_n nU_n$ preserves property (L) as characterized in [23]. (iii) If ξ is ℓ^∞-barrelled, proceed as in (i) but with each $U_n = \{f_n\}^\circ$ for some $f_n \in E'+M$. (iv) If ξ is C-barrelled or (v) c_0-barrelled, we assume that each $x \in E$ is in U_n for almost all n and, in the latter case, that each $U_n = \{f_n\}^\circ$ for some $f_n \in E'+M$. In both cases, (2) and Theorem 1 imply that τ inherits the given property of ξ, so that U is a τ-, thus an η-neighborhood of 0, which proves that η also has the property. □

The following proposition motivates the next theorem.

Proposition 6 *The topology η satisfies the Tweddle-Yeomans Criterion if and only if there is no infinite-dimensional subset N of $E'+M$ transverse to E' such that N° is countable-codimensional in E.*

Proof. If N is $\sigma(E'+M,E)$-bounded, then N° is 0-codimensional, so one direction is clear. For the other, suppose N° is countable-codimensional in E for some infinite-dimensional $N \subset E'+M$ with N transverse to E'. Let $\{f_n\}_n$ be a linearly independent sequence in N and let $\{x_n\}_n$ be a sequence in E whose span is complementary to $\mathrm{sp}(N^\circ)$ in E. For each n, choose $1 \geq \varepsilon_n > 0$ such that $|\varepsilon_n f_n(x_k)| \leq 1$ for $1 \leq k \leq n$. Then $\{\varepsilon_n f_n(x_k)\}_n$ is bounded for each k, as is $\{\varepsilon_n f_n(y)\}_n$ for each $y \in N^\circ$. Hence $\{\varepsilon_n f_n\}_n$ denies the Tweddle-Yeomans Criterion. □

Lemma 7 *When ξ is inductive, a $\sigma(E'+M,E)$-bounded set R is η-equicontinuous if and only if P is ξ-equicontinuous and Q is finite-dimensional, where $\langle P\rangle$ $[Q]$ is the projection of R into $\langle E'$ along $M\rangle$ $[M$ along $E']$.*

Proof. $(\Leftarrow)$. Since P is ξ-equicontinuous, it is $\sigma(E'+M,E)$-bounded, as is R, which implies the same, then, for Q. Thus P° and Q° are η-neighborhoods of zero, as is, therefore, $R^\circ \supset (P+Q)^\circ \supset \frac{1}{2}(P^\circ \cap Q^\circ)$.

$(\Rightarrow)$. If R° is an η-neighborhood of zero, then there is a finite subset T of M and a ξ-neighborhood U of zero such that $U \cap T^\circ \subset R^\circ$. Thus $R \subset R^{\circ\circ} \subset (U \cap T^\circ)^\circ \subset \left(U \cap T^\perp\right)^\circ$ implies the set of restrictions $R|_{T^\perp}$ is equicontinuous on $\left(T^\perp, \xi\right)$. Density of $T^\perp$ yields a unique ξ-equicontinuous set $(R|_{T^\perp})^-$ of extensions to E and, because T is finite-dimensional, $R \subset (R|_{T^\perp})^- + \mathrm{sp}(T)$, which implies $P \subset (R|_{T^\perp})^-$ and $Q \subset \mathrm{sp}(T)$. □

The necessity part of the following theorem generalizes [11, Theorem 4].

Theorem 8 *If (E,ξ) is non-S_σ, inductive or primitive, then (E,η) has the same respective property if and only if there is no infinite-dimensional subset N of $E' + M$ transverse to E' such that $N^\perp$ is countable-codimensional in E.*

Note: Consequently, if (E,ξ) is non-S_σ or inductive, then (E,η) is also non-S_σ or inductive, respectively, if (and only if) it is primitive.

Proof. [Necessity]. Suppose E is primitive under both ξ and η and N is a subset of $E' + M$ transverse to E' with $N^\perp$ countable-codimensional in E. Let G be an algebraic complement in E to the ξ-closure $\overline{N^\perp}$ of $N^\perp$. The space Q of linear forms which vanish on $N^\perp + G$ is contained in $E' + M$, since these linear forms vanish on $N^\perp$ and (E,η) is primitive. Similarly, since (E,ξ) is primitive, the space P of linear forms which vanish on $\overline{N^\perp}$ is contained in E'. Now $N \subset N^{\perp\perp} = P + Q$ and N is transverse to $P \subset E'$, so the dimension of N cannot exceed that of Q: we need only show that Q has finite dimension. Clearly, the dimension of Q is either finite or $\geq \mathfrak{c}$, according to whether the dimension of $\overline{N^\perp}/N^\perp$ is finite or infinite. The latter is impossible, for $Q \subset E' + M$ is transverse to E' by density, and therefore $\dim(Q) \leq \dim(M) = \aleph_0$.

[Sufficiency]. For the remainder of the proof we assume the non-existence condition on N.

First, suppose (E,ξ) is non-S_σ. If there is a strictly increasing sequence $\{E_n\}_n$ of η-closed subspaces covering E, we may choose $f_n \in E' + M$ such that each f_n vanishes on E_n but not on E_{n+1}. Some E_q is ξ-dense in E, so that $N = \{f_n\}_{n \geq q}$ is transverse to E', while $N^\perp$ is $\aleph_0$-codimensional in E by a simple linear algebra argument (cf. [16, Theorem 1]), contradicting our assumption. Thus (E,η) is also non-S_σ, and the condition suffices in the first case.

Next, suppose (E,ξ) is inductive. Lemma 3.1 of [24] says that (E,η) is inductive if and only if $R = \cup_n R_n$ is η-equicontinuous whenever $\{R_n\}_n$ is a sequence of η-equicontinuous subsets of $E' + M$ such that each $x \in E$ is in all but finitely many $R_n^\perp$. Denote the projections of R and R_n into [E' along M] [[M along E']] by [P and P_n] [[Q and Q_n]], respectively $(n = 1, 2, \ldots)$. By Lemma 7, each P_n is ξ-equicontinuous and each Q_n is finite-dimensional, and we must show that P is ξ-equicontinuous and Q is finite-dimensional. If $Q = \cup_n Q_n$ is infinite-dimensional, then there exist a subsequence $\{n_k\}_k$ of $\{1, 2, \ldots\}$ and $f_k \in R_{n_k}$ such that $N = \{f_k\}_k$ is a linearly independent subset of $E' + M$ transverse to E'. The hypothesis on $\{R_n\}_n$ ensures that for each $x \in E$, the equation $f_k(x) = 0$ holds for almost all k, so that $N^\perp$ is again $\aleph_0$-codimensional in E (cf. [16, Theorem 1]), a contradiction. Therefore Q is finite-dimensional and the finite-codimensional subspace $Q^\perp$ of (E,ξ) is inductive [6], which implies by [24, Lemma 3.1] that the set of restrictions $\cup_n R_n|_{Q^\perp} = \cup_n P_n|_{Q^\perp} = P|_{Q^\perp}$ is equicontinuous on $(Q^\perp, \xi)$. Now $Q^\perp$ dense in (E,ξ) implies P is equicontinuous on (E,ξ).

For the last case, when (E,ξ) is primitive, the Mackey topology $\widehat{\xi} = \tau(E,E')$ is inductive [22], and so is its CE $\widehat{\eta} = \sup\left(\widehat{\xi}, \sigma(E, E' + M)\right)$, by duality invariance and the previous case. Hence $\widehat{\eta}$ is primitive, as is η, again, by duality invariance. □

The dlc and primitive properties govern not only CE preservation of most properties (cf. Notes of Theorems 5, 8), but similarly govern dense subspace inheritance [23]. In both instances, C- and c_0-barrelled spaces rebel. A variant in [23] avoids confrontation: *If E has any given property in the left column of the table and if every subspace containing a fixed dense subspace F has the weaker property immediately to the right, then F also enjoys the stronger given property.* An equally tactful parallel variant of (the Notes to) Theorems 5, 8 follows.

Theorem 9 *If (E,ξ) has a given property in the left column of the table in Section 1, then any CE η with the weaker property immediately to the right also enjoys the stronger given property.*

Proof. Theorems 5 and 8 leave us to prove only that if ξ is C-barrelled and η is c_0-barrelled, then $\left(L^{\perp},\xi\right)$ is C-barrelled for each finite-dimensional subspace L of M. Since L is finite-dimensional and transverse to E', we have $F = L^{\perp}$ is dense, and by part (2) of Theorem 5, every subspace between F and E is c_0-barrelled. The above-quoted result from [23] thus shows that F is, indeed, C-barrelled. □

A space F *dominates* a space E if the two spaces are algebraically the same and F has a finer topology than does E.

Theorem 10 *Suppose (E,ξ) has a given one of the fourteen properties and is dominated by a space (F,π) that has the property. If F admits a CE that preserves the property, then so does E. Indeed, if M is an $\aleph_0$-dimensional subspace of E^* transverse to F' such that the CE* $\sup(\pi,\sigma(F'+M))$ *preserves the property, then M is also transverse to E' $(\subset F')$, and the CE* $\eta = \sup(\xi,\sigma(E'+M))$ *likewise preserves the property.*

Proof. If the CE corresponding to $F'+M$ satisfies the Tweddle-Yeomans Criterion or the condition of Theorem 8, so does η. This proves the Theorem for all but the C- and c_0-barrelled cases, where we only need to show that if ξ is C- or c_0-barrelled and L is a finite-dimensional subspace of M, then $L^{\perp}$ is also C- or c_0-barrelled, respectively. For this it is enough to show that $\left(L^{\perp},\xi\right)$ is c_0-barrelled under assumption that (E,ξ) is, using [23] as in the previous proof. Let $\{f_n\}_n$ be a $\sigma\left(\left(L^{\perp},\xi\right)',L^{\perp}\right)$-null sequence with unique ξ-continuous extension f_n^- of each f_n to E. It is also a $\sigma\left(\left(L^{\perp},\pi\right)',L^{\perp}\right)$-null sequence, and so by (2) of Theorem 5 it is π-equicontinuous on $L^{\perp}$, and thus so is $\{f_n^-\}_n$ on F, since $L^{\perp}$ is dense in F. Now $\{f_n^-\}_n$ is $\sigma(F',F)$-null, for if it were not, it would have a non-zero $\sigma(F',F)$-adherence point that would necessarily vanish on the dense finite-codimensional subspace $L^{\perp}$, an impossibility. Hence $\{f_n^-\}_n$ is $\sigma(E',E)$-null, and therefore ξ-equicontinuous, as is, then, the set $\{f_n\}_n$ of restrictions to $L^{\perp}$. □

Example 11 *In precisely the C- and c_0-barrelled cases, one cannot omit part (2) from Theorem 5.* Let E denote the space c of convergent scalar sequences endowed with the Mackey topology $\xi = \tau\left(c,\ell^1\right)$. Obviously dual ℓ^1-complete, E is dlc [21], hence C-barrelled [22]. [Properties in Box 2) coincide under the Mackey topology.] If F denotes c with its usual sup norm topology, then F dominates E and admits a BCE corresponding to some M transverse to

F', by any one of [15, 20, 31]. The preceding theorem concludes that $\eta = \sup(\xi, \sigma(E, E'+M))$ is also C-barrelled. Let f be a linear form such that $f^{\perp} = c_0$, a dense hyperplane of E which, as observed in [23], is not c_0-barrelled. Since $f \in F' \backslash E'$, we see that $\hat{M} = \mathrm{sp}(M \cup \{f\})$ is transverse to E' and the CE $\hat{\eta} = \sup\left(\xi, \sigma\left(E, E'+\hat{M}\right)\right)$ is not c_0-barrelled via part (2) of Theorem 5, even though $\hat{\eta}$, as a finite enlargement of η, is dlc (Corollary 2) and hence satisfies the Tweddle-Yeomans Criterion.

Theorem 12 *The CE η satisfies the Tweddle-Yeomans Criterion (resp., the condition of Theorem 8) if and only if, for each linearly independent sequence $\{f_n\}_n \subset E'+M$ transverse to E', the mapping $\theta : x \mapsto \{f_n(x)\}_n$ from E into ω is onto a barrelled (resp., inductive) subspace of ω.*

Proof. Given scalars $a_1, \ldots, a_k$, the linear independence of $f_1, \ldots, f_k$ yields $x \in E$ such that $f_i(x) = a_i \quad (1 \leq i \leq k)$, so θ is always onto a dense subspace of ω, and is obviously linear. If the image under each such θ is barrelled, then, since the coefficient functionals are not equicontinuous, they are not weakly bounded, which is equivalent to saying that $\{f_n(x)\}_n$ is an unbounded scalar sequence for some $x \in E$; i.e., $\{f_n\}_n$ is not $\sigma(E'+M, E)$-bounded, and η satisfies the Criterion. If some such image is not barrelled, then there exists a linearly independent sequence in the dual of ω that is bounded at each point in the image of θ; equivalently, there exists a linearly independent sequence in $\mathrm{sp}(\{f_n\}_n)$ that is $\sigma(E'+M, E)$-bounded, which means η fails the Criterion.

If $\theta(E)$ is always inductive, then, being metrizable, it is always of uncountable dimension, as is, then, $E/\theta^{-1}(\{0\}) = E/N^{\perp}$, where $N = \{f_n\}_n$. Hence the condition of Theorem 8 is satisfied. If for some θ as above, $\theta(E)$ is not inductive, then it is an S_σ space and contains a closed $\aleph_0$-codimensional subspace H [23]. Thus there is a linearly independent sequence S in the span of the coordinate functionals on $\theta(E)$ such that $S^{\perp} = H$. There is a corresponding linearly independent sequence $N \subset \mathrm{sp}(\{f_n\}_n)$ such that $N^{\perp}$ is $\aleph_0$-codimensional in E, so that η fails the condition of Theorem 8. □

3 Maximal extensions of standard BCE results

The previous section noted that, within context, Theorem 3 maximally extends certain results in [10, 29]. The following lemma and theorem prove that Theorem 5, which generalizes results in [31, 29], does not extend to non-S_σ, inductive or primitive spaces, nor does Theorem 8 extend to any of the remaining eleven properties; i.e., neither theorem can be further extended within our domain of fourteen properties.

If F is a closed countable-codimensional subspace of a primitive space E, so is every subspace between F and E. ("Between" includes F and E.) And if F is a dense subspace of E which is either non-S_σ or has one of the nine non-duality invariant properties, then every subspace between F and E has the same property, but the statement fails in the four remaining duality invariant cases [23].

Lemma 13 *Suppose F is a dense subspace of E and every subspace between F and E is primitive. If G is a closed subspace of E such that $G \cap F$ is countable-codimensional in F, then so is G in E.*

Proof. Let $x \in E$. Since $G \cap (F + \mathrm{sp}(\{x\}))$ is closed and countable-codimensional in the primitive space $F + \mathrm{sp}(\{x\})$, so is the intermediate subspace $(F+G) \cap (F + \mathrm{sp}(\{x\}))$, and since it contains the dense F, it is all of $F + \mathrm{sp}(\{x\})$. Indeed, then, $F+G$ contains all $x \in E$, and the codimension of G in $E = F+G$ is that of $G \cap F$ in F, and so is countable. □

Each infinite-dimensional Banach space E enjoys all fourteen properties and is barrelledly fit [17], thus satisfies the hypothesis of our next theorem (cf. [14, 27]), thus is an example showing maximality of Theorems 5, 8.

Theorem 14 *Let (E, ξ) be a space with a dense $\aleph_1$-codimensional subspace F such that every subspace between F and E is primitive. There is a countable enlargement η of ξ that preserves primitivity but denies dual local completeness.*

Proof. Let B denote a cobasis for F. Choose a sequence $\{f_n\}_n \subset E^*$ such that $(\{f_n\}_n)^\perp = F$ and $\{(f_n(x))_n : x \in B\}$ is a Hamel basis for χ, an $\aleph_1$-dimensional dense non-barrelled inductive subspace of ω [25]. By density, $M = \mathrm{sp}(\{f_n\}_n)$ is transverse to E', producing a CE η. From Theorem 12 it follows that η does not satisfy the Tweddle-Yeomans Criterion, hence could not inherit dual local completeness from ξ. But (E, η) cannot be dual locally complete in any case, since its quotient E/F is isomorphic to the metrizable non-barrelled, hence non-dual locally complete space χ, and quotients preserve all fourteen properties. (There is only one compatible topology for χ.)

On the other hand, suppose $N \subset E' + M$ with $N^\perp$ countable-codimensional in E. Let P be the projection of N into E' along M, and let Q be the projection of N into M along E'. We identify $\mathrm{sp}(B)$ with χ via the map $x \longmapsto (f_n(x))_n$. By Theorem 8, η preserves primitivity if we can prove that $\dim(Q)$ is finite. Since $N^\perp \cap F = P^\perp \cap F$ is countable-codimensional in F, so is $P^\perp$ in E (Lemma 13). Therefore $P^\perp \cap \chi$ is countable-codimensional in χ, as is $Q^\perp \cap P^\perp \cap \chi = N^\perp \cap (P^\perp \cap \chi)$ in $P^\perp \cap \chi$, so that $Q^\perp \cap P^\perp \cap \chi$ is countable-codimensional in χ. Thus so is the larger $Q^\perp \cap \chi$ in χ, and it is also closed in the non-S_σ space χ. We conclude that $Q^\perp \cap \chi$ is finite-codimensional in χ, and then, of necessity, Q is finite-dimensional. □

A related theorem maximally extends the classic [10, Theorem 5] and its codimensional refinement [19, Theorem 5]. The proof suggests S. Dierolf's three-space technique [2, 2.2], [3]; see [8, 4.5.5(i)].

Theorem 15 *If (E, ξ) has a given one of the fourteen properties and contains a dense subspace F of codimension at least $\mathfrak{b}$ such that all the subspaces between F and E have the same property, then there is a CE of ξ that preserves the property. If the given property is primitive, inductive or non-S_σ, one may replace $\mathfrak{b}$ with $\aleph_1$.*

Proof. First, suppose $\dim(E/F) = \mathfrak{b}$. Let G be an algebraic complement to F in E and choose a subspace $M \subset E^*$ such that $M^\perp \supset F$ with $(G, \sigma(G, M))$ isomorphic to the (metrizable) barrelled subspace $\psi_\mathfrak{b}$ of ω [18]. By density $M \cap E' = \{0\}$ so that $\eta = \sup(\xi, \sigma(E, E' + M))$ is a CE of ξ. Unchanged, F retains its given property under η and E/F is isomorphic to the Baire-like space $\psi_\mathfrak{b}$ because, by density, $F^\perp = M$ and the metrizable space $(G, \sigma(G, M))$ has only one compatible topology (cf. [8, 4.5.5(i)]). We claim that η preserves the property in all fourteen cases. Preservation is immediate for the seven cases [24] in which the given property is a three-space property.

In all cases, part (2) of Theorem 5 holds by hypothesis, and we only need to show that η preserves either dual local completeness or primitivity, which are duality invariant. Thus it suffices to show that the CE $\hat{\eta} = \sup(\tau(E,E'), \sigma(E,E'+M))$ for E with its Mackey topology is either dual locally complete or primitive, respectively. If the given property for ξ implies either dual locally complete or primitive, then $\tau(E,E')$ either has property (L) or is inductive [22], and induces on all the subspaces between F and E a dual locally complete or primitive topology, so that F actually either acquires property (L) or becomes inductive, respectively [23], while E/F remains Baire-like under $\hat{\eta}$. Since property (L) is a three-space property and inductivity "nearly" is [24], it follows that $(E,\hat{\eta})$ either has property (L) or is inductive, and thus is either dual locally complete or primitive, respectively. The proof is complete for $\dim(E/F) \geq \mathfrak{b}$. [If $\dim(E/F) > \mathfrak{b}$, replace F with a larger subspace whose codimension in E is exactly $\mathfrak{b}$.]

Now suppose the given property is either primitive, inductive or non-S_σ with $\dim(E/F) = \aleph_1$. From Theorem 14, there is a CE that preserves primitivity, thus either of the other two properties ξ may enjoy (Theorem 8). □

Other maximal extensions derive from [8, 31, 19].

Theorem 16 *If (E,ξ) has a given one of the fourteen properties, excepting the C- and c_0-barrelled cases, and contains a bounded set of dimension at least $\mathfrak{b}$, then there is a CE of ξ that preserves the property.*

Proof. We proceed precisely as in the proof of [19, Theorem 6] to find a CE that satisfies the Criterion, and thus also the condition of Theorem 8 (see Proposition 6), preserving the property. □

Question: For the above theorem, is it possible in the non-S_σ, inductive and primitive cases to replace $\mathfrak{b}$ with $\aleph_1$?

Theorem 17 *If (E,ξ) and its subspace F have a given one of the fourteen properties, excepting the C- and c_0-barrelled cases, and there is a CE for (E,ξ) that preserves the property, then there is a CE for (F,ξ) that also preserves the property, provided the codimension of F is, in the non-S_σ, inductive and primitive cases, less than $\aleph_1$; in the remaining cases, less than $\mathfrak{b}$.*

Proof. Let P be an $\aleph_0$-dimensional subspace of E^* transverse to E' whose corresponding CE preserves the property. Let F' denote the dual of (F,ξ), let M be a maximal subspace of $P|_F$ transverse to F', where $P|_F$ denotes the restrictions to F of members of P, and let η be the corresponding enlargement for (F,ξ). The Hahn-Banach Theorem provides a subspace $Q \subset E^*$ with $Q|_F = M$ such that $E' + Q = E' + P$.

In the non-S_σ, inductive and primitive cases we assume F has codimension less than $\aleph_1$. Now if M were finite-dimensional, then $Q^\perp$ would be of codimension less than $\aleph_1$ in E, contradicting Theorem 8. Hence η is a CE. Let $\{f_n\}_n$ be a linearly independent sequence in $F' + M$ transverse to F'. For each n choose g_n in $E' + P$ whose restriction to F is f_n. The subspace $\{\{f_n(x)\}_n : x \in F\} = \{\{g_n(x)\}_n : x \in F\}$ of ω has codimension less than $\aleph_1$ in $\{\{g_n(x)\}_n : x \in E\}$, a non-$S_\sigma$ space by Theorem 12. Hence the countable-codimensional subspace is also non-S_σ, and η preserves the property via Theorem 12.

In the remaining cases, one proves precisely as in [19, Theorem 4] that η is a CE and satisfies the Tweddle-Yeomans Criterion, so that the given property is preserved. $\square$

4 Dimension and the fourteen CE questions

If, in some ZFC-consistent model of set theory, there exists a non-trivial barrelled space E with $\dim(E) < \mathfrak{b}$, this would provide a negative answer to the still-open BCE question [19]. However, we were able to determine $\mathfrak{b}$ as the least infinite-dimensionality for metrizable barrelled spaces [19] and answer positively the metrizable BCE question in every ZFC-consistent model [15] (cf. [20]), although we as yet have no familiar identification of the least infinite-dimensionality for normable barrelled spaces. In [25] we gave four characterizations of $\aleph_1$ as the least infinite-dimensionality for: non-S_σ spaces, metrizable inductive spaces, normable inductive spaces, and metrizable non-normable inductive spaces. In this section we show that $\aleph_1$ is the least dimensionality for non-trivial spaces E having any given one of the properties strictly weaker than barrelled. We give negative answers to the eight CE questions between $\aleph_0$-barrelled and dual locally complete, inclusive, under assumption that $\aleph_1 < \mathfrak{b}$, which is ZFC-consistent. In the second half of the section, we complement the positive metrizable BCE answer with a positive CE answer for metrizable inductive spaces, so that the fourteen metrizable CE questions, which collapse to just two distinct ones, all have positive answers in every ZFC-consistent system.

4.1 Eight negative CE answers

The least infinite-dimensionality for non-trivial spaces having one of the strictly weak barrelledness properties is always $\aleph_1$. Certainly, it is at least as big as $\aleph_1$, since each $\aleph_0$-dimensional primitive space is the union of an increasing sequence of finite-dimensional subspaces, making every linear form continuous.

Example 18 *Let E be a vector space with Hamel basis B of size $\kappa \geq \aleph_1$. We give E the topology ξ generated by the seminorms on E that vanish on all but countably many members of B. Clearly, ξ is $\aleph_0$-barrelled, but is not barrelled, since the balanced convex hull of B is a barrel. Moreover, there exists a dense $\aleph_0$-barrelled subspace G with* $\dim(E/G) = \dim(E)$.

Note: Since $|f|$ is a seminorm for each linear form f, the dual of E is just the $\aleph_0$-dual with respect to the basis B [1]. By [7, p.101], (E, ξ) is not Mackey, and when given its Mackey topology, the space is ℓ^∞-barrelled but not $\aleph_0$-barrelled (cf.[23, 26]). **Proof.**

Let $\{B_\alpha\}_{\alpha \in I}$ be a partition of B into $\kappa = |I|$ pairwise disjoint sets each of size κ. Each $E_\alpha = \mathrm{sp}(B_\alpha)$ is isomorphic to E, thus is $\aleph_0$-barrelled and contains a dense hyperplane H_α, also necessarily $\aleph_0$-barrelled. Now $G = \mathrm{sp}(\cup_\alpha H_\alpha)$ is clearly dense and κ-codimensional in E; we show it is $\aleph_0$-barrelled. Let $\{U_n\}_n$ be a sequence of closed absolutely convex 0-neighborhoods in G whose intersection U is absorbing in G. By density, each closure $\overline{U}_n$ is an absolutely convex 0-neighborhood in E. Each $U \cap H_\alpha$ is a 0-neighborhood in the $\aleph_0$-barrelled space H_α, and thus so is each $\overline{U} \cap E_\alpha$ in E_α, which proves $\overline{U}$ is absorbing in E. Thus so is the larger $\cap_n \overline{U}_n$, which is therefore a 0-neighborhood in the $\aleph_0$-barrelled E. Thus $U = \cap_n U_n = \cap_n \left(\overline{U}_n \cap G\right) = \left(\cap_n \overline{U}_n\right) \cap G$ is a 0-neighborhood in G. $\square$

Tweddle and Catalán [29] gave negative answers to the CE problem for the $\aleph_0$-barrelled, ℓ^∞-barrelled and property (S) cases, but under the assumption that measurable cardinals exist. No one knows if there is any model of set theory consistent with the usual ZFC system that warrants such an assumption, and thus it is possible that their examples are invalid. Example 18 provides negative answers for all eight cases between $\aleph_0$-barrelled and dual locally complete, inclusive, under assumption that $\aleph_1 < \mathfrak{b}$, which is known to be consistent with ZFC [4], as is the assumption that $\aleph_1 = \mathfrak{d}$, needed in [27], which implies $\aleph_1 = \mathfrak{b}$.

Theorem 19 *Let (E,ξ) be the space of Example 18. There exists a CE that preserves inductivity. However, there exists a CE that preserves dual local completeness (and thus also the stronger properties up to and including $\aleph_0$-barrelledness) if and only if $\kappa \geq \mathfrak{b}$.*

Proof. Theorem 14 preserves primitivity and, by Theorem 8, inductivity.

Suppose $\kappa < \mathfrak{b}$. If $\{f_n\}_n$ is any linearly independent sequence in E^* then, since $\mathfrak{b}$ is the least infinite-dimensionality for metrizable barrelled spaces [19], $\{(f_n(x))_n : x \in E\}$ is a non-barrelled subspace of ω, and Theorem 12 implies that no CE of ξ satisfies the Tweddle-Yeomans Criterion; i.e., none preserves dual local completeness.

If $\kappa \geq \mathfrak{b}$, then Theorem 15 permits preservation of $\aleph_0$-barrelledness (a three-space property [8, 8.2.32] requiring only the first paragraph of the proof of Theorem 15). □

4.2 The metrizable CE answers are positive

Six CE questions as yet lack a general answer. For metrizable spaces, however, precisely eleven of the fourteen properties are equivalent to barrelledness, and the three remaining properties, non-S_σ-ness, inductivity and primitivity, are each equivalent to the other [8, 25]. Thus the metrizable CE questions all reduce to the recently answered metrizable BCE question and the metrizable ICE question, whose positive answer we discover below.

A space E is *[(non-S_σ)-ly] fit* (cf. [17, 27]) if it contains a dense [non-S_σ] subspace whose codimension is $\dim(E)$. Clearly, every infinite-dimensional non-S_σ space is fit.

Theorem 20 *If a non-S_σ space E has infinite dimension less than $\mathfrak{b}$, then E is (non-S_σ)-ly fit. In fact, given any increasing sequence $\{E_k\}_k$ of subspaces covering E, some E_k is both dense and non-S_σ.*

Proof. Since E is infinite-dimensional, it is the union of an increasing sequence of subspaces, each having codimension in its successor equal to $\dim(E)$. Thus it suffices to prove the second statement. By way of contradiction, if each E_k is either non-dense or S_σ, then for each k there exists an increasing sequence $\{E_{kj}\}_j$ of proper closed subspaces of E covering E_k. Let B be a Hamel basis for E. For each $x \in B$, define $f_x \in \mathbb{N}^{\mathbb{N}}$ by writing

$$f_x(k) = \begin{cases} 1 & \text{if } x \notin E_k \\ \min\{j : x \in E_{kj}\} & \text{if } x \in E_k \end{cases}$$

for each $k \in \mathbb{N}$. As $|B| < \mathfrak{b}$, there exists $g \in \mathbb{N}^{\mathbb{N}}$ with $f_x \leq^* g$ for each $x \in B$. For each $l \in \mathbb{N}$, define

$$F_l = \cap_{k \geq l} E_{k,g(k)}.$$

Given $x \in B$, there exists $m_0 \in \mathbb{N}$ such that $x \in E_{m_0}$, and thus $x \in E_k$ for all $k \geq m_0$. Also, there exists n_0 such that $g(k) \geq f_x(k)$ for all $k \geq n_0$. Setting $l = \max(m_0, n_0)$, if $k \geq l$ we have $x \in E_k$, hence $x \in E_{k,f_x(k)} \subset E_{k,g(k)}$, and thus $x \in F_l$. It follows that $\{F_l\}_{l \in \mathbb{N}}$ is an increasing sequence of proper closed subspaces covering $\mathrm{sp}(B) = E$. Thus a properly increasing subsequence may be extracted, contradicting the fact that E is non-S_σ. □

Theorem 21 *Every infinite-dimensional metrizable inductive space E has an inductive countable enlargement.*

Proof. If $\dim(E) \geq \mathfrak{b}$, (the proof of) [15] provides a CE satisfying the Tweddle-Yeomans Criterion, and thus the weaker criterion for inductivity. If $\dim(E) < \mathfrak{b}$, the previous theorem yields a dense inductive subspace whose codimension is $\dim(E) \geq \aleph_1$ and Theorem 15 completes the proof. □

5 Preservation with precision

A CE may have certain of the properties not enjoyed by the original topology. These consist of the duality invariant ones between property (C) and primitivity.

Example 22 *If E is an $\aleph_0$-dimensional vector space, then there exists, clearly, a $\sigma(E^*, E)$-dense $\aleph_0$-codimensional subspace E' of E^*, and we give E any topology ξ compatible with the dual pairing $\langle E, E' \rangle$. Now $E' \neq E^*$ implies ξ is not primitive, but the CE $\sup(\xi, \sigma(E, E^*))$ obviously has property (C).*

Theorem 23 *If a CE η is non-S_σ or has a given one of the non-duality invariant properties, then so is or has the original topology ξ.*

Proof. The non-S_σ case is obvious. In all the non-duality invariant cases the argument is essentially the same and quite easy. For example, if η has property (L) and $\{U_n\}_n$ is a sequence of closed balanced convex neighborhoods of 0 in (E, ξ) with absorbing intersection, then, since each U_n is also η-closed, a characterization [23] of the property (L) of η is that $B = \cap_n n U_n$ is an η-neighborhood of 0. Thus (*) implies B is a ξ-neighborhood of 0, and ξ has property (L). Or, if η is Baire-like and $\{A_n\}_n$ is a closed absorbing sequence in (E, ξ), then some A_n is an η-neighborhood of 0. Thus (*) implies A_n is a ξ-neighborhood of 0, and ξ is Baire-like. Or, again, if η is inductive and A is a balanced convex subset of E such that each $A \cap E_n$ is a 0-neighborhood in (E_n, ξ), where $\{E_n\}_n$ is a given increasing sequence of subspaces covering E, then the closure B of $\frac{1}{2}A$ is a barrel in (E, ξ) with $B \subset A$, and B is clearly a 0-neighborhood in the inductive (E, η), and, by (*), is likewise in (E, ξ). Thus so is A, and (E, ξ) is inductive. □

Example 24 *There is a non-S_σ space E that is not dlc, yet has a CE with property (C).*

Proof. As in [28, P_4] (cf.[6, 23, 25]), let E be a non-complete normable space dominated by a Banach space F, with respective duals E' and F'. The Closed Graph Theorem implies E is not barrelled, hence, by metrizability, is not dlc. Now F'/E' is infinite-dimensional, for otherwise E and F would induce the same metrizable (Mackey) topology on a certain dense finite-codimensional subspace of E that is a closed (complete) subspace of the Banach space F, forcing the contradiction $E = F$. By [21, Theorem 2.3], there exist a $\sigma(E',E)$-bounded sequence $\{f_n\}_n$ and $\{a_n\}_n \in \ell^1$ such that the pointwise limit f of $\sum_n a_n f_n$ is not in E'. Let H be an $\aleph_0$-codimensional subspace of F' such that $H \supset E'$ with $f \notin H$. Then $(E,\sigma(E,H))$ is dominated by the non-S_σ space F, and hence is non-S_σ, is not dlc, and yet the countable enlargement $\sigma(E,F')$ has the duality invariant property (C). □

Theorem 25 *Given a CE η of ξ, some CE $\check{\eta}$ of ξ preserves precisely the properties common to η and ξ.*

Proof. If there are no common properties, the conclusion is that of Theorem 3.

By Theorems 5, 8 and 23, only three cases remain to be proved. We must find a CE $\check{\eta}$ of ξ such that: (i) $\check{\eta}$ is primitive and not dlc when ξ is primitive and not dlc and η is dlc; (ii) $\check{\eta}$ is dlc and without property (S) when ξ is dlc and without property (S) and η has property (S); (iii) $\check{\eta}$ has property (S) and not property (C) when ξ has property (S) and not property (C) and η has property (C).

In case (i), some pointwise limit f of an ℓ^1-sum of a $\sigma(E',E)$-bounded sequence is in $E'+M$ but not in E' [21, Theorem 2.3]. Let $\check{M}$ be a 1-codimensional subspace of M such that $f \notin E'+\check{M}$, so that the corresponding CE $\check{\eta}$ of ξ is not dlc. Since η had to satisfy the condition of Theorem 8, it is obvious that $\check{\eta}$ must also, and thus is primitive.

Case (ii) is entirely analogous: f becomes the pointwise limit of a sequence from E' with $f \in (E'+M)\setminus E'$ and Theorem 5 replaces Theorem 8.

Case (iii) admits in $(E',\sigma(E',E))$ a bounded sequence A with no adherence point. Since η satisfies the Tweddle-Yeomans Criterion, the $\sigma(E'+M,E)$ closure $\overline{A}$ has a finite-dimensional projection B into M along E'. Let $\check{M}$ be a finite-codimensional subspace of M transverse to B. The corresponding CE $\check{\eta}$ satisfies the Tweddle-Yeomans Criterion since η does, and thus has property (S) since ξ does. But A clearly has no $\sigma\left(E'+\check{M},E\right)$-adherence point, so $\check{\eta}$ does not have property (C). □

Application of CEs essentially doubles the list of distinguishing examples in [23]. Most of the examples are dominated by non-trivial Banach or (LB)-spaces, which always have BCEs [31, 20, 15, 17]. Each such example admits, by the last part of Theorem 10, a single CE that preserves all its properties, and thus Theorem 25 yields a CE that has precisely those properties. These CEs cannot be dominated by a Banach topology, since CEs so dominated obviously fail the Tweddle-Yeomans Criterion. Therefore most differ significantly from the original topologies.

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