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NATURAL NORMS ON SYMMETRIC TENSOR PRODUCTS OF NORMED SPACES[1]

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Abstract. *The basics of the theory of symmetric tensor products of normed spaces and some applications are presented.*

0 Introduction

0.1. Though known for quite a while to algebraists (at least since Chevalley's monograph [C] in 1956), it was only in 1980 that R. Ryan in his doctoral thesis [R] introduced symmetric tensor products for the study of polynomials on Banach spaces; before Gupta [Gu] had discovered in 1968 that the space of nuclear n-homogeneous polynomials on a Banach space E is a natura1 predual (via trace duality) of the space of continuous n-homogeneous polynomials on E' (if E' has the approximation property). Unfortunately, Ryan's thesis was not published and 1 have the impression that many researchers do not feel attracted by symmetric tensor products and prefer to use other methods. I think, however. that a consequent (but not exclusive!) use of tensor products will give good and new insights into the theory – exactly as Grothendieck did it successfully in his "résumé" ([Gro], see also [DF]) for the theory of linear operators. Moreover, there are already various "metri? results and so it seems to be adequate to develop a "metric theory" of n-th symmetric tensor products in the spirit of Grothendieck. Therefore the purpose of this paper is two-fold: presenting a thorough introduction of the algebraic basics of symmetric tensor products and the two extreme natura1 norms (the symmetric projective norm π_s and symmetric injective norm ε_s) in order to facilitate the use of symmetric tensor products *and* to prepare a theory of so-called s-tensor norms the beginning of which will appear in [F2].

0.2. This paper starts with a study of the algebraic aspects, the norms π_s and ε_s, continues with the duality between ε_s and π_s and applications to the polarization constants and finishes with extensions of polynomials to the bidual. Though many of the results are explicitly or implicitly known, the thorough construction of the theory gives various simplified proofs and also new information – not only for the theory of symmetric tensor products but also for the study of polynomials.

0.3. If $E_1, \ldots, E_n$ and F are vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ the space of n-linear mappings $\varphi : E_1 \times \cdots \times E_n \longrightarrow F$ is denoted by $L(E_1, \ldots, E_n; F)$. If all $E_1 = \cdots = E_n = E$ Nachbin's notation $L({}^nE; F) := L(\underbrace{E, \ldots, E}_{n}; F)$ will be used; ${}^nE$ should be read as n-times

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E. The subspace $L_s({}^nE;F) \subset L({}^nE;F)$ is the space of those φ which are *symmetric*, i.e. $\varphi(x_1,\dots,x_n)=\varphi(x_{\eta(1)},\dots,x_{\eta(n)})$ for all permutations $\eta\in S_n$ (the group of permutations of $\{1,\dots,n\}$). If the E_j and F are normed spaces the subspaces of continuous n-linear maps will be denoted by $\mathcal{L}(E_1,\dots,E_n;F)$, $\mathcal{L}({}^nE;F)$ and $\mathcal{L}_s({}^nE;F)$ respectively; $E' := \mathcal{L}(E;\mathbb{K})$. The closed unit ball of E is B_E. If $G \subset E$ is a subspace $I_G^E : G \hookrightarrow E$ and $Q_G^E : E \longrightarrow E/G$ denote the natura1 injection and quotient mapping. $E \cong F$ means topologically isomorphic, $E \overset{1}{=} F$ isometrically isomorphic, $E \overset{1}{\hookrightarrow} F$ denotes an (iso-)metric injection and $T : E \overset{1}{\twoheadrightarrow} F$ a metric surjection (i.e. $T\overset{\circ}{B}_E = \overset{\circ}{B}_F$). The set $C^n_{\mathbb{K}}$ is $:= \{-1, 1\}$ if $\mathbb{K} = \mathbb{R}$ and n is even and $:= \{1\}$ otherwise.

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1 The algebraic theory of symmetric tensor products

1.1. If $n \in \mathbb{N}$ and $El,\dots, E_n$ are $\mathbb{K}$-vectors spaces, then an n-fold tensor product (H_0, ψ_0) (where H_0 is a $\mathbb{K}$-vector space and $\psi_0 \in L(E_1,\dots,E_n;H_0)$) is defined by the following universa1 property: for every $\mathbb{K}$-vector space F and every $\varphi \in L(E_1,\dots,E_n;F)$ there is a unique $T \in L(H_0;F)$ with $\varphi = T \circ \psi_0$. The pair (H_0,ψ_0) is unique up to isomorphisms and exists. The following notation will be used: $\otimes(E_1,.,E,)$, $\otimes_{j=1}^n E_j$, $E_1 \otimes \dots \otimes E_n$,

$$\otimes^n E := \otimes(E,\dots,E) \quad \textit{and} \quad \otimes(x_1,\dots,x_n) = x_1 \otimes \dots \otimes x_n,$$

as well as $\otimes^n x := x \otimes \dots \otimes x$. To distinguish it from the symmetric tensor product (which will be defined and constructed in a moment) it is reasonable to call $\otimes^n E$ the *full* n-fold tensor product of E. The isomorphism

$$L(E_1,\dots,E_n;F) = L(\otimes_{j=1}^n E_j;F)$$

will be denoted by $\varphi \rightsquigarrow \varphi^L$. Clearly, $\otimes^1 E = E$.

1.2. The symmetric tensor product will linearize only *symmetric* n-linear mappings.

Definition. *Let E, H be $\mathbb{K}$-vector spaces and $\psi_0 \in L_s({}^nE;H)$. The pair (H,ψ_0) is called an n-th symmetric tensor product of E if for every $\mathbb{K}$-vector space F and every $\varphi \in L_s({}^nE;F)$ there is a unique $T \in L(H;F)$ with $\varphi = T \circ \psi_0$.*

The algebraists call (H,ψ) also an n-th symmetric (tensor) power of E. If it is clear which $n \in \mathbb{N}$ is used, the adjective "n-th" will be omitted. The universa1 property immediately implies the following aftirmations for fixed n:

(1) If (H_0, ψ_0) is a symmetric tensor product of E, then $span\,\psi_0(E^n) = H_0$ and, for all F and $T_1, T_2 \in L(H_0; F)$, one has: $T_1 = T_2$ if and only if $T_1 \circ \psi_0 = T_2 \circ \psi_0$.

(2) If (H_j, ψ_j) are two symmetric tensor products of E, then there exists a unique isomorphism (onto) $S \in L(H_1; H_2)$ with $\psi_2 = S \circ \psi_1$ and $\psi_1 = S^{-1} \circ \psi_2$.

(3) Let (H_0, ψ_0) be a symmetric tensor product of E and $S \in L(H_0; H_1)$ and $T \in L(F;E)$ isomorphisms. Then $(H_1, S \circ \psi_0 \circ (T, \ldots, T))$ is a symmetric tensor product of F.

The statement (2) gives the same kind of uniqueness as for the full tensor product. Once one has existence, it is therefore reasonable to speak about the (n-th) symmetric tensor product of E.

1.3. To prove the existence, the following operation will be helpful: if $\eta \in S_n$, then the n-linear map $E^n \longrightarrow \otimes^n E$ defined by

$$(x_1, \ldots, x_n) \rightsquigarrow x_{\eta^{-1}(1)} \otimes \cdots \otimes x_{\eta^{-1}(n)}$$

has a linearization $\otimes^n E \longrightarrow \otimes^n E$ which will be denoted by $z \rightsquigarrow z^\eta$. It is easy to see that $(z^\eta)^\sigma = z^{\sigma \circ \eta}$. The use of η^{-1} in the definition instead of η is sometimes practical (see also [Gre]). For $x_1, \ldots, x_n \in E$ define

$$x_1 \vee \ldots \vee x_n := \frac{1}{n!} \sum_{\eta \in S_n} x_{\eta^{-1}(1)} \otimes \cdots \otimes x_{\eta^{-1}(n)} \in \otimes^n E$$

and for $z \in \otimes^n E$

$$\sigma_E^n(z) := \frac{1}{n!} \sum_{\eta \in S_n} z^\eta \in \otimes^n E$$

which clearly is the linearization $\otimes^n E \longrightarrow \otimes^n E$ of the n-linear (even symmetric) map $\vee : E^n \longrightarrow \otimes^n E$. We shall show that $(\mathrm{im}\sigma_E^n, \vee)$ is an n-th symmetric tensor product of E. Note that $\sigma_E^n(\otimes^n x) = \otimes^n x$.

1.4. Before doing this, let us state the

Polarization formula. *Let E be a $\mathbb{K}$-vector space, (Ω, P) a probability space, $\varepsilon_1, \ldots, \varepsilon_n : \Omega \longrightarrow \mathbb{K}$ functions in $\mathcal{L}_2(P)$ which are stochastically independent, centered (i.e. $\int_\Omega \varepsilon_k \, dP = 0$) and normalized (i.e. $\int_\Omega |\varepsilon_k|^2 dP = 1$). Then, for every $x_0, x_1, \ldots, x_n \in E$*

$$x_1 \vee \cdots \vee x_n = \frac{1}{n!} \int_\Omega \varepsilon_1(w) \cdots \varepsilon_n(w) \otimes^n \left[x_0 + \sum_{k=1}^n \overline{\varepsilon_k(w)} x_k \right] P(dw).$$

It is clear that the Bochner-integral exists in the finite dimensional subspace $\otimes^n [span\,\{x_0, \ldots, x_n\}]$ of $\otimes^n E$.

Proof. The proof is straightforward: if μ_k is the distribution measure on $\mathbb{K}$ of ε_k, then using $t_0 := 1$

$$\begin{aligned}&\int_\Omega \cdot \cdot P(dw) = \\ &= \sum_{k_1,\dots,k_n=0}^{n} \int_{\mathbb{K}} \cdots \int_{\mathbb{K}} t_1 \cdots t_n \cdot \overline{t_{k_1}} \cdots \overline{t_{k_n}} \mu_1(dt_1) \cdots \mu_n(dt_n) x_{k_1} \otimes \cdots \otimes x_{k_n} = \\ &= \sum_{\eta \in S_n} x_{\eta^{-1}(1)} \otimes \cdots \otimes x_{\eta^{-1}(n)} = n!\, x_1 \vee \cdots \vee x_n\end{aligned}$$

since the iterated integra1 is 1 if $\{k_1, \dots, k_n\} = \{1, \dots, n\}$ and 0 otherwise. ✍

Many special situations are of interest:

(a) P the countable product measure on $\Omega := \mathbb{K}^{\mathbb{N}}$ of the normalized Gauß-measure on $\mathbb{K}$ and ε_k being the k-th projection;

(b) the Rademacher functions: $\Omega := \{-1, +1\}^{\mathbb{N}}$ with P the product measure of $\frac{1}{2}$ (6-t + δ_{+1}) and $\varepsilon_k := r_k$ the k-th projection;

(c) more general – the n-Rademacher functions which were first used by Aron and Globevnik [AG] for the study of polynomials: Substitute $\{-1, 1\}$ by the n-th unit roots $\lambda_k := \exp\left(\frac{2\pi i k}{n}\right)$, hence Ω is the set $\{\lambda_0, \dots, \lambda_{n-1}\}^{\mathbb{N}}$ and $\varepsilon_k := s_k^n$ is the k-th projection. These n-Rademacher functions are often useful in the complex theory of n-linear mappings and polynomials since they are n-orthonormal, i.e.

$$\int_\Omega s_{k_1}^n \cdot \cdot s_{k_n}^n \, dP = \begin{cases} 1 & if\, k_1 = \dots = k_n \\ 0 & otherwise \end{cases}$$

(P being the product measure of $\frac{1}{n}(\delta_{\lambda_1} + + \delta_{\lambda_n})$), hence

$$\sum_{k=1}^{m} x_k(1) \otimes \cdots \otimes x_k(n) = \int_\Omega \left[\sum_{k=1}^{m} s_k^n(t) x_k(1)\right] \otimes \cdots \otimes \left[\sum_{k=1}^{m} s_k^n(t) x_k(n)\right] P(dt),$$

and satisfy a Khintchine inequality (see [ALRT], [FM] and [MeT]).

1.5. Using the Rademacher functions one obtains the classica1 polarization formula

$$x_1 \vee \cdots \vee x_n = \frac{1}{n! 2^n} \sum_{\delta_1, \dots, \delta_n \in \{-1,1\}} \delta_1 \dots \delta_n \otimes^n \left[x_0 + \sum_{k=1}^{n} \delta_k x_k\right]$$

and in particular the

Corollary.

$$\begin{aligned}\sigma_E^n(\otimes^n E) &= span\{x_1 \vee \cdots \vee x_n \mid x_j \in E\} = span\{\otimes^n x \mid x \in E\} = \\ &= \left\{\sum_{j=1}^{m} \alpha_j \otimes^n x_j \;\middle|\; m \in \mathbb{N},\, x_j \in E,\, \alpha_j \in C_{\mathbb{K}}^n\right\}\end{aligned}$$

where $C_{\mathbb{K}}^n := \{-1, 1\}$ *if* $\mathbb{K} = \mathbb{R}$ *and* n *is even and* $C_{\mathbb{K}}^n := \{1\}$ *otherwise.*

Now everything is prepared for the

Theorem. (imσ^n_E, ∨) *is an n-th symmetric tensor product of E.*

The embedding $\mathrm{im}\sigma^n_E \longrightarrow \otimes^n E$ will be denoted, if necessary, by ι^n_E.

Proof. The unique factorization of every $\varphi \in L_s({}^nE; F)$ through a linear $T \in L(\mathrm{im}\sigma^n_E; F)$ has to be verified. Clearly $T := \varphi^L \circ \iota^n_E$ satisfies

$$T \circ \vee(x_1, \ldots, x_n) = T(x_1 \vee \cdots \vee x_n) = \left(L \tfrac{1}{n!} \textstyle\sum_\eta (x_1 \otimes \cdots \otimes x_n)^\eta\right) =$$
$$= \tfrac{1}{n!} \textstyle\sum_\eta \varphi^L((x_1 \otimes \cdots \otimes x_n)^\eta) = \varphi(x_1, \ldots, x_n)$$

since φ is symmetric. If the mappings $T_1, T_2 \in L(\mathrm{im}\sigma^n_E; F)$ satisfy $T_1 \circ \vee = T_2 \circ \vee$, then $T_1(\otimes^n x) = T_2(\otimes^n x)$ for all x and hence, by the corollary, $T_1 = T_2$. □

Though there are clearly other "realizations" of the n-th symmetric tensor product we shall – if not otherwise stated (and if the full tensor product $\otimes^n E$ is fixed) – consider the subspace $\mathrm{im}\sigma^n_E \subset \otimes^n E$ together with ∨ as *the* n-th symmetric tensor product: $\otimes^{n,s}E := \mathrm{im}\sigma^n_E$. It is obvious that $\otimes^{1,s}E = \otimes^1 E = E$. Note that $\sigma^n_E : \otimes^n E \longrightarrow \otimes^{n,s}E$ is a projection and $\sigma^n_E \circ \otimes = \vee$, but clearly $\otimes \neq \iota^n_E \circ \vee$ if $n > 1$. It is easy to see that if (H_0, ψ_0) is an n-th symmetric tensor product and (H_1, ψ_1) a full n-fold tensor product of E, then

$$J(\psi_0(x_1, \ldots, x_n)) := \frac{1}{n!} \sum_{\eta \in S_n} \psi_1(x_{\eta^{-1}(1)}, \ldots, x_{\eta^{-1}(n)}) \in H_1$$

defines the natura1 injection $J : H_0 \longrightarrow H_1$

1.6. The universa1 property of the n-th symmetric tensor product gives an isomorphism

$$L_s({}^nE; F) \xrightarrow{=} L(\otimes^{n,s}E; F), \quad \varphi \rightsquigarrow \varphi^{L,s} := \varphi^L \circ \iota^n_E;$$

its inverse is $T \rightsquigarrow T \circ \vee$. Since σ^n_E is the linearization of ∨ it follows that the embedding

$$L(\otimes^{n,s}E; F) = L_s({}^nE; F) \hookrightarrow L({}^nE; F) = L(\otimes^n E; F)$$

is the mapping $T \rightsquigarrow T \circ \sigma^n_E$. For $F = \mathbb{K}$ one obtains

$$(\sigma^n_E)^* : (\otimes^{n,s}E)^* = L_s({}^nE) \hookrightarrow L({}^nE) = (\otimes^n E)^*.$$

1.7. If $T \in L(E; F)$, then there is a unique $S \in L(\otimes^{n,s}E; \otimes^{n,s}F)$ with $S(\otimes^n x) = \otimes^n Tx$ for all $x \in E$: just take $S := \otimes^n T \circ \iota^n_E$ and note that $\mathrm{im}S \subset \otimes^{n,s}F$; uniqueness comes from the fact that the elements $\otimes^n x$ span the space $\otimes^{n,s}E$. Notation: $\otimes^{n,s}T : \otimes^{n,s}E \longrightarrow \otimes^{n,s}F$. It is easy to see that $\otimes^{n,s}T(x_1 \vee \cdots \vee x_n) = Tx_1 \vee \cdots \vee Tx_n$ and $\ker \otimes^{n,s}T = \sigma^n_E(\ker \otimes^n T)$.

1.8. It is worthwhile to note, see [C], that $\dim_{\mathbb{K}}(\otimes^{n,s}\mathbb{K}^k) = \binom{n+k-1}{k-1}$.

1.9. The elements in $\otimes^{n,s}E \subset \otimes^n E$ are called *symmetric.*

Remark.

(1) $z \in \otimes^n E$ *is symmetric if and only if* $z = z^\eta$ *for all* $\eta \in S_n$.

(2) *Let* $x_1, \ldots, x_n \in E \backslash \{0\}$. *Then* $x_1 \otimes \cdots \otimes x_n$ *is symmetric if and only if span* $\{x_1, \ldots, x_n\}$ *is one-dimensional.*

Proof. (1) is immediate; for (2) assume that $x_2 \notin span\,\{x_1\}$ *and* take $x_k^* \in E^*$ with $\langle x_k^*, x_k \rangle = 1$ for all k and $\langle x_1^*, x_2 \rangle = 0$. For $\varphi := x_1^* \otimes \cdots \otimes x_n^* \in (\otimes^n E)^*$ one gets the contradiction $1 = \langle \varphi, x_1 \otimes x_2 \otimes \cdots \otimes x_n \rangle = \langle \varphi, x_2 \otimes x_1 \otimes x_3 \cdots \otimes x_n \rangle = 0$ since $x_1 \otimes x_2 \otimes x_3 \cdots \otimes x_n = x_2 \otimes x_1 \otimes x_3 \otimes \cdots \otimes x_n$ by (1). □

1.10. The symmetric tensor product $\otimes^{n,s} E$ is a complemented subspace of the full one $\otimes^n E$ with the projection σ_E^n. Vice versa, take $F := \prod_{i=1}^n E$, and $I_i : E_i \longrightarrow F$ and $P_i : F \longrightarrow E_i$ the natura1 injections and projections, then

$$\mathrm{id}_{\otimes_{i=1}^n E_i} : \otimes_{i=1}^n E_i \xrightarrow{I_1 \otimes \cdots \otimes I_n} \otimes^n F \xrightarrow{\sigma_F^n} \otimes^{n,s} F \hookrightarrow \otimes^n F \xrightarrow{n! P_1 \otimes \cdots \otimes P_n} \otimes_{i=1}^n E_i$$

(this construction is, for $n = 2$, due to Bonet-Peris [BP] and was successively extended to the present form by Defant-Maestre [DM], [AlF1] and Blasco [Bi]). It follows that $\otimes_{i=1}^n E_i$ is isomorphic to a complemented subspace of $\otimes^{n,s} F$ (with "natura]" mappings, which is important in view of the topologica1 situation). In particular: $\otimes^n E$ is isomorphic to a complemented subspace of $\otimes^{n,s} E^n$. If $E \cong E^n$, then $\otimes^n E$ and $\otimes^{n,s} E$ are complemented in each other. One can even show more [DD]: if $E \cong E^2$, then $\otimes^n E \cong \otimes^{n,s} E$; this result is also a consequence of the formula ([AnF])

$$\otimes^{n,s}(F \oplus G) = \bigoplus_{k=0}^{n} \left[\otimes^{k,s} F\right] \otimes \left[\otimes^{n-k,s} G\right]$$

(again with natura1 mappings and the convention $\otimes^{0,s} E := \mathbb{K}$).

1.11. Blasco [B2] showed that $\otimes^{n,s} E$ is isomorphic to a complemented subspace of $\otimes_s^{n+1} E$ – also with natura1 mappings; in particular: $E = \otimes^{1,s} E$ is complemented in all $\otimes^{n,s} E$. The dual result (i.e. for n-homogeneous polynomials) was proved in 1976 by Aron-Schottenloher [AS].

1.12. Reca11 that $q : E \longrightarrow F$ is an *n-homogeneous polynomial* (notation: $q \in P^n(E; F)$) if there is a $\varphi \in L(^n E; F)$ with $q(x) = \varphi(x, \ldots, x)$ for all $x \in E$; notation: $q := \varphi^\Delta := \varphi \circ A$ where $A(x) := (x, \ldots, x)$. It is clear that also the *symmetrization* $\varphi_s \in L_s(^n E; F)$ of φ defined by

$$\varphi_s(x_1, \ldots, x_n) = \frac{1}{n!} \sum_{\eta \in S_n} \varphi(x_{\eta^{-1}(1)}, \ldots, x_{\eta^{-1}(n)})$$

satisfies $q(x) = \varphi_s(x, \ldots, x)$. Note the relation

$$\varphi_s = \varphi^L \circ \iota_E^n \circ \vee \,. \qquad (*)$$

The polarization formula implies that there is a *unique* $\check{q} \in L_s({}^nE;F)$ with $q = \check{q}^{\Delta}$. The following notation will be used:

$$\begin{array}{ccccc} P^n(E;\,F\,) & = & L_s({}^nE;F) & = & L(\otimes^{n,s}E;\,F\,) \\ 4 & \rightsquigarrow & 4 & \rightsquigarrow & q^L := (\check{q})^{L,s} \\ & & \varphi & \rightsquigarrow & \varphi^{L,s} = \varphi^L \circ \iota_E^n . \end{array}$$

Applying $\check{q} = q^L \circ \vee$ to the polarization formula gives

$$\begin{aligned} \check{q}(x_1,\ldots,x_n) &= \tfrac{1}{n!}\int_\Omega \varepsilon_1(w)\cdots\varepsilon_n(w)\, q\left(x_0 + \textstyle\sum_{k=1}^n \overline{\varepsilon_k(w)}x_k\right) P(dw) = \\ &= \tfrac{1}{n!2^n}\textstyle\sum_{\delta_1,\ldots,\delta_n\in\{-1,1\}} \delta_1\cdots\delta_n\, q\left(x_0 + \sum_{k=1}^n \delta_k x_k\right) \end{aligned}$$

for all $q \in P^n(E;F)$ and $x_0,\ldots,x_n \in E$.

1.13. If $\langle E_j,\, F_j\rangle$ are dual systems of vector spaces, then $\langle \otimes_{j=1}^n E_j, \otimes_{j=1}^n F_j\rangle$ forms a dual system with the duality bracket

$$\Big\langle \sum_I x_1^i \otimes\cdots\otimes x_n^i,\ \sum_J y_1^j \otimes\cdots\otimes y_n^j \Big\rangle = \sum_{i,j}\prod_{k=1}^n \langle x_k^i, y_k^j\rangle$$

If all $\langle E_j, F_j\rangle$ are separating, then $\langle \otimes_{j=1}^n E,\, , \otimes_{j=1}^n F_j\rangle$ is also separating (for a proof, by induction, reduce to $n = 2$). In particular: $\langle \otimes^n E, \otimes^n F\rangle$ is separating if (E, *F)* is. Clearly, the restriction (via ι_E^n x ι_F^n) to $\otimes^{n,s}E$ x $\otimes^{n,s}F$ gives a duality bracket. It is clear that $\langle u^\eta, v\rangle = \langle u, v^{\eta^{-1}}\rangle$ for all $(u, \mathrm{v}) \in \otimes^n E$ x $\otimes^n F$ and $\eta \in S_n$, whence

$$\langle \sigma_E^n(u), v\rangle = \langle u, \sigma_F^n(v)\rangle . \qquad (*)$$

Proposition. *If (E. F) is a separating dual system, then $\langle \otimes^{n,s}E, \otimes^{n,s}F\rangle$ is also a separuting dual system with the duality bracket*

$$\Big\langle \sum_i \delta_i \otimes^n x_i,\ \sum_j \eta_j \otimes^n y_j\Big\rangle = \sum_{i,j}\delta_i\eta_j(\langle x_i, y_j\rangle)^n .$$

Proof. For $0 \neq u \in \otimes^{n,s}E \subset \otimes^n E$ there is a $v \in \otimes^n F$ with $1 = \langle u, v\rangle = \langle \sigma_E^n(u), v\rangle = \langle u, \sigma_F^n(v)\rangle$ whence $\sigma_F^n(v) \in \otimes^{n,s}F \subset \otimes^n F$ separates u from 0. □

In particular: the natura1 map $J_E : \otimes^{n,s}E \hookrightarrow (\otimes^{n,s}F)^* = P^n(F)$ is injective and

$$\begin{aligned} J_E(x_1 \vee\cdots\vee x_n)(y) &= \textstyle\prod_{j=1}^n \langle x_j, y\rangle \\ J_E(\otimes^n x)(y) &= \langle x, y\rangle^n . \end{aligned}$$

Following the notation of [DF], the polynomial $J_E(\otimes^n x)$ will be denoted by $\underline{\otimes}^n x$, hence (@x)(y) = $\langle x,y\rangle^n$; this notation is helpful since the extension of J_E to the completion of $\otimes^{n,s}E$ (with respect to π_s, see chap. 2) may fai1 to be injective (see 4.3.). If E is normed, then

$$J_{E'}: \otimes^{n,s}E' \hookrightarrow \mathcal{P}^n(E) \; c \; P"(E)$$

($\mathcal{P}^n(E)$ are the continuous n-homogeneous polynomials) is injective. In particular: $\langle \otimes^{n,s}E,$ *T"(E))* is a separating dual system with the duality bracket

$$\langle z,q\rangle := \langle q^L, z\rangle .$$

Having in mind the tensor product description of the trace and the trace duality for linear operators (see e.g. [DF, 2.5. and 2.6.]) one may call this last duality and the duality in the proposition *trace duality* as well.

A polynomial $q \in P^n(E;F)$ is called of *finite* type (notation: $q \in P_f^n(E;F)$) if there are $(x_m^*, y_m) \in E^*$ x F with

$$q(x) = \sum_{m=1}^{n} \langle x_m^*, x\rangle^n y_m \qquad forall x \in E.$$

It follows that $P_f^n(E;F) = (\otimes^{n,s}E^*) \otimes F$ and for normed spaces E. F

$$\mathcal{P}_f^n(E;F) := P_f^n(E;F) \; rl \; \mathcal{P}^n(E;F) = (\otimes^{n,s}E') \otimes F;$$

for a proof use $(\otimes^n E^*) \cap \mathcal{L}(^nE) = \otimes^n E'$ (which can be proved by induction) and the polarization formula. The relation $\mathcal{L}(^n(E, \sigma(E,F))) \cap \otimes^n E^* = \otimes^n F$ for a separating dual system $(E, F\rangle$ and the weak topology $\sigma(E,F)$ implies

$$\mathcal{P}^n(E, \sigma(E,F)) \cap \otimes^{n,s}E^* = \otimes^{n,s}F .$$

In particular: the weak-*-continuous n-homogeneous polynomials on E' of finite type are $\otimes^{n,s}E$. These formulas were first observed by Ryan [R].

1.14. It is worthwhile to note that 1.6. and the formulas $(*)$ in 1.13. and 1.12. give that the following diagrams commute for each dual system *(E, F):*

$$\begin{array}{ccccc}
\otimes^{n,s}E & \xrightarrow{J_s} & (\otimes^{n,s}F)^* & = & L_s(^nF) \;\ni\; \varphi \\
\downarrow \iota_E^n & & \downarrow (\sigma_F^n)^* & & \downarrow \\
\otimes^n E & \xrightarrow{J} & (\otimes^n F)^* & = & L(^nF) \;\ni\; \varphi
\end{array}$$

$$\begin{array}{ccccccc} \otimes^{n,s}E & \xrightarrow{J_s} & (\otimes^{n,s}F)^* & = & L_s({}^nF) & \ni & \varphi_s \\ {\scriptstyle \sigma_E^n}\uparrow & & \uparrow{\scriptstyle (\iota_F^n)^*} & & \wr & & \\ \otimes^{n}E & \xrightarrow{J} & (\otimes^{n}F)^* & = & L({}^nF) & 3 & \varphi \end{array}$$

where J_s and J are just the mappings coming from the respective duality brackets; they are injective if (E, $F\rangle$ is separating.

1.15. Just for the sake of a certain completeness of this introduction to the algebraic theory of symmetric tensor products: the addition formula

$$(x_1+y_1)\otimes\cdots\otimes(x_n+y_n) \quad = \sum_{D\subset\{1,\dots,n\}} (z_1^D\otimes\cdots\otimes z_n^D)$$

(where $z_\ell^D := x_\ell$ if $\ell\in D$ and $:= y_\ell$ otherwise) gives

$$\otimes^n(x+y) \;=\; \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$$

with the definition $x^k y^{n-k} := \underbrace{x\vee\cdots\vee x}_{k-times}\vee\underbrace{y\vee\cdots\vee y}_{n-k-times}$. It follows that

$$q(x+y)=\sum_{k=0}^{n}\binom{n}{k}q^L(x^k y^{n-k})=\sum_{k=0}^{n}\binom{n}{k}\check{q}(\underbrace{x,\dots,x}_{k},\underbrace{y,\dots,y}_{n-h})$$

for all $q\in P^n(E;F)$ and $x,y\in E$.

1.16. As a consequence of these formulas every $\varphi\in L_{\mathbb{R}}(E_1,\dots,E_n;F)$ and every $q\in P^n_{\mathbb{R}}(E;F)$ (where E_j, E and F are rea1 vector spaces) has a unique extension $\varphi^{\mathbb{C}}\in L_{\mathbb{C}}(E_1^{\mathbb{C}},\dots,E_n^{\mathbb{C}};F^{\mathbb{C}})$ and $q^{\mathbb{C}}\in P^n_{\mathbb{C}}({}^nE^{\mathbb{C}};F")$ to the complexification ($G^{\mathbb{C}} := G\oplus iG$) given by ($z_\ell^D$ is as in 1.15.)

$$\begin{aligned}\varphi^{\mathbb{C}}(x_1+iy_1.\dots)\,x_n+iy_n) \quad &:= \textstyle\sum_{D\subset\{1,\dots,n\}} i^{|D|}\varphi(z_1^D,\dots,z_n^D)\\ q^{\mathbb{C}}(x+iy) \;&:= \textstyle\sum_{k=0}^{n}\binom{n}{k}i^k q^L(x^k y^{n-k}).\end{aligned}$$

Kirwan [Ki] and Muñoz-Sarantopoulos-Tonge [MST] studied the behaviour of the norms $\|\varphi^{\mathbb{C}}\|$ and $\|q^{\mathbb{C}}\|$ if the complexifications $E_j^{\mathbb{C}}$ and $F^{\mathbb{C}}$ are equipped with (possibly different) "complexification norms".

2 The projective s-tensor norm

2.1. Let E and F be normed spaces. Then the projective norm $\pi(\cdot;\otimes^n E)$ on the full tensor product satisfies

$$\mathcal{L}({}^nE;F)\overset{1}{=}\mathcal{L}(\otimes^n_\pi E;F)$$

($\stackrel{1}{=}$ means isometrically equal). $\mathcal{L}_s({}^nE;F) \subset \mathcal{L}({}^nE;F)$ has the induced norm. For the natural norm

$$\|q\|_{\mathcal{P}^n(E,F)} := \sup\{\|q(x)\|_F \quad x \in B_E\}$$

of a continuous n-homogeneous polynomial $q \in \mathcal{P}^n(E;F)$ one has

$$\|q\|_{\mathcal{P}^n(E;F)} \leq \sup\{\|\check{q}(x_1,\ldots,x_n)\|_F \mid x_1,\ldots,x_n \in B_E\} = \\ = \|\check{q}\|_{\mathcal{L}_s({}^nE;F)} \leq \frac{n^n}{n!}\|q\|_{\mathcal{P}^n(E;F)}$$

by the polarization formula. It follows that the n-th *polarization constant* of E defined by

$$c(n,E) := \sup\left\{\|\check{q}\|_{\mathcal{L}_s({}^nE;F)} \ F normed, q \in B_{\mathcal{P}^n(E;F)}\right\} = \\ = \sup\left\{\|\check{q}\|_{\mathcal{L}_s({}^nE)} \mid q \in B_{\mathcal{P}^n(E)}\right\}$$

is $\leq \frac{n^n}{n!}$. It is well-known that $c(n,\ell_1) = \frac{n^n}{n!}$ and $c(n,\ell_2) = 1$ (Harris [Ha] comments in the Scottish book, that this can easily be deduced from results of Kellogg 1928 and from van der Corput-Schaake 1935, but that also Banach [B] proved it). In particular: $c(n,H) = 1$ for all Hilbert spaces; conversely, Benitez and Sarantopoulos [BeS] showed that each *real* normed space with $c(2,E) = 1$ is pre-Hilbert. For other examples see [S], [D2], [D4].

It follows that $\|q\| \neq \|\check{q}\|$ in general. If $\pi|_s$ denotes the restriction to $\otimes^{n,s}E$ of the projective norm on $\otimes^n E$, then

$$\|\sigma_E^n : \otimes_\pi^n E \longrightarrow \otimes_{\pi|_s}^{n,s} E\| = 1$$

(if $E \neq \{0\}$). Since $q^L(\sigma_E^n(z)) = (\check{q})^L(z) \in F$ for all $q \in \mathcal{P}^n(E;F)$ one obtains

$$\|\check{q}\|_{\mathcal{L}_s({}^nE;F)} = \|q^L\|_{\mathcal{L}(\otimes_{\pi|_s}^{n,s}E;F)}$$

This shows that $\pi|_s$ is not an appropriate norm fora *metric* theory of continuous n-homogeneous polynomials.

2.2. For this one needs a norm π_s on $\otimes^{n,s}E$ such that

$$\mathcal{P}^n(E;F) \stackrel{1}{=} \mathcal{L}(\otimes_{\pi_s}^{n,s}E;F).$$

The key calculation is the following:

$$\|q\|_{\mathcal{P}^n(E;F)} = \sup\left\{\|q(x)\|_F \quad x \in \overset{\circ}{B}_E\right\} = \sup\left\{\|q^L(\otimes^n x)\|_F \quad x \in \overset{\circ}{B}_E\right\} = \\ = \sup\left\{\|q^L(z)\|_F \quad z \in \Gamma(\Delta^n \overset{\circ}{B}_E)\right\}$$

(where Δ^n is the "diagonal" map $E \longrightarrow \otimes^{n,s}E$ defined by $x \rightsquigarrow \otimes^n x$). For the absolute convex hull $C := \Gamma(\Delta^n \overset{\circ}{B}_E)$ one has $span C = \otimes^{n,s}E$ (by Corollary 1.5.). The Minkowski-gauge functional of C on $\otimes^{n,s}E$ will be denoted by $\pi_s(\cdot;\otimes^{n,s}E)$ (or shortly π_s; notation: $\otimes_{\pi_s}^{n,s}E$)

$$\pi_s(z;\otimes^{n,s}E) := \inf\left\{\lambda \geq 0 \;\middle|\; z \in \lambda\Gamma(\Delta^n \overset{\circ}{B}_E)\right\}.$$

It is clear that

$$\|q\|_{\mathcal{P}^n(E;F)} = \sup\left\{\|q^L(z)\|_F \quad \pi_s(z;\otimes^{n,s}E) \leq 1\right\}$$

Since $\langle \otimes^{n,s}E, \mathcal{P}^n(E)\rangle$ is a separating dual system (see 1.13.) this implies that π_s is even a norm. π_s is called the *projective s-tensor norm* (or shortly: the projective s-norm). The completion of $\otimes^{n,s}_{\pi_s}E$ will be denoted by $\widetilde{\otimes}^{n,s}_{\pi_s}E$.

The following properties of π_s can be proved in rather the same way as the analogous ones for π on $\otimes^2 E$ (see e.g. [DF, §3 and §5]) for all normed spaces *E:*

Proposition.

(1) For all normed spaces F one has

$$\mathcal{P}^n(E;F) \overset{1}{=} \mathcal{L}\left(\otimes^{n,s}_{\pi_s}E;F\right);$$

in particular: $\mathcal{P}^n(E;F)$ is complete if F is and

$$\mathcal{P}^n(E;F) \overset{1}{=} \mathcal{L}\left(\widetilde{\otimes}^{n,s}_{\pi_s}E;F\right)$$

in this case.

This "universa1 property" of π_s can also be formulated as follows: $\|q(x)\|_F \le c\|x\|^n_E$ for all $x \in E$ if and only if $\|q^L : \otimes^{n,s}_{\pi_s}E \longrightarrow F\| \le c$.

(2) π_s is the unique seminorm α on $\otimes^{n,s}E$ which satisfies

$$(\otimes^{n,s}E, \alpha)' \overset{1}{=} \mathcal{P}^n(E)$$

(3) $\pi_s(\otimes^n x; \otimes^{n,s}E) = \|x\|^n$ for all $x \in E$.

(4)

$$\pi_s(z;\otimes^{n,s}E) = \inf\left\{ \sum_{j=1}^m |\lambda_j|\|x_j\|^n \;\middle|\; m\in\mathbb{N}, z = \sum_{j=1}^m \lambda_j \otimes^n x_j \right\} =$$
$$= \inf\left\{ \sum_{j=1}^\infty |\lambda_j|\|x_j\|^n \;\middle|\; z = \sum_{j=1}^\infty \lambda_j \otimes^n x_j \,(\pi_s\text{-convergence}) \right\}$$

Note that $\lambda_j \otimes^n x_j$ can be written as $\delta_j \otimes^n \mu_j x_j$ with $\delta_j \in C^n_{\mathbb{K}}$ (see 1.5.) and $|\lambda_j|\|x_j\|^n = \|\mu_j x_j\|^n$.

(5) The open unit ball with respect to π_s is $\Gamma(\Delta^n \overset{\circ}{B}_E)$. In particular: if $Q : E \longrightarrow F$ is a metric surjection (notation: $Q : E \overset{1}{\twoheadrightarrow} F$), then

$$\otimes^{n,s}Q : \otimes^{n,s}_{\pi_s}E \overset{1}{\twoheadrightarrow} \otimes^{n,s}_{\pi_s}F$$

("π_s respects metric surjections").

This justifies the name "projective". But it does not respect metric injections, see 2.9.

(6) If $T \in \mathcal{L}(E;F)$, *then*

$$\| \otimes^{n,s} T : \otimes^{n,s}_{\pi_s} E \longrightarrow \otimes^{n,s}_{\pi_s} F \| = \|T\|^n .$$

("π_s *satisfies the metric mapping property").*

(7) π_s *is finitely generated in the following sense:*

$$\pi_s(z; \otimes^{n,s} E) = \inf \{ \pi_s(z; \otimes^{n,s} M) \quad M \in \mathrm{FIN}(E), z \in \otimes^{n,s} M \}$$

(reca11 from [DF] the notation FIN(E) *for the set of finite-dimensional subspaces of* E*).*

See 2.5. for the somehow dual situation. The norm in the completion $\widetilde{\otimes}^{n,s}_{\pi_s} E$ will also be denoted by π_s.

(8) If $K \subset \widetilde{\otimes}^{n,s}_{\pi_s} E$ *is compact and* $\varepsilon > 0$, *then there are a zero-sequence* (x_j) *in* B_E *and a compact set* $D \subset \ell_1$ *with* $\sup \|D\| \leq (1+\varepsilon) \sup \pi_s(K)$ *such thatfor every* $z \in K$ *there is a* $(\lambda_j) \in D$ *with*

$$z = \sum_{j=1} \lambda_j \otimes^n x_j .$$

(9) In particular: every $z \in \widetilde{\otimes}^{n,s}_{\pi_s} E$ *has a representation*

$$z = \sum_{j=1}^{\infty} \lambda_j \otimes^n x_j$$

with $\sum |\lambda_j| \|x_j\|^n < \infty$ *and*

$$\pi_s(z) = \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| \|x_j\|^n \;\middle|\; z = \sum_{j=1}^{\infty} \lambda_j \otimes^n x_j \right\} .$$

(10) For every compact set $K \subset \widetilde{\otimes}^{n,s}_{\pi_s} E$ *and* $\varepsilon > 0$ *there is a compact set* $C \subset E$ *with* $\sup \|C\| \leq (1+\varepsilon)(\sup \pi_s(K))^{1/n}$ *and* $\overline{\Gamma(\Delta^n C)} \supset K$.

2.3. The "full" projective norm on $\otimes^n E$ can be calculated by

$$\pi(z; \otimes^n E) = \inf \left\{ \sum_{k=1}^{\ell} \prod_{m=1}^{n} \|x_{k,m}\| \quad z = \sum_{k=1}^{\ell} x_{k,1} \otimes \cdots \otimes x_{k,n} \right\}$$

and therefore n(z) $\leq$ n,(z) for all $z \in \otimes^{n,s} E$. On the other hand it follows from the definition and 2.2.(3) that

$$\pi_s(x_1 \vee \cdots \vee x_n; \otimes^n E) \leq \frac{n^n}{n!} \|x_1\| \cdots \|x_n\|$$

hence $\vee : E^n \longrightarrow \otimes^{n,s}_{\pi_s} E$ is continuous and therefore also its linearization $\sigma^n_E : \otimes^n_\pi E \longrightarrow \otimes^{n,s}_{\pi_s} E$. From 1.10. one obtains

$$\mathcal{P}^n(E) \stackrel{1}{=} (\otimes^{n,s}_{\pi_s} E)' \stackrel{(\sigma^n_E)'}{\longrightarrow} (\otimes^n_\pi E)' \stackrel{1}{=} \mathcal{L}({}^nE)$$

and this mapping is just $q \rightsquigarrow \check{q}$. Altogether:

Proposition. *Let E be a normed space. Then*

(1) $\|\iota^n_E : \otimes^{n,s}_{\pi_s} E \longrightarrow \otimes^n_\pi E\| = 1$ *(if* $E \neq \{0\}$*)*

(2) $\|\sigma^n_E : \otimes^n_\pi E \longrightarrow \otimes^{n,s}_{\pi_s} E\| = c(n,E)$

(3) $\otimes^{n,s}_{\pi_s} E$ *is a topologically complemented subspace of* $\otimes^n_\pi E$.

In particular: $\pi|_s \leq \pi_s \leq c(n,E)\pi|_s$ and $\pi_s \neq \pi|_s$ in general but $\pi_s = \pi|_s$ for Hilbert spaces. For $z \in \otimes^{n,s} E$ it is not difficult to see that

$$\pi(z) = \inf\left\{ \sum_{k=1}^{\ell} \|x^1_k\| \cdots \|x^n_k\| \;\middle|\; \ell \in \mathbb{N};\ z = \sum_{k=1}^{\ell} x^1_k \vee \cdots \vee x^n_k \right\}$$

and that $\pi|_s$ is the quotient norm of $\sigma^n_E : \otimes^n_\pi E \longrightarrow \otimes^{n,s} E$.

2.4. It is well-known (see e.g. [DF, 5.8.] for $n = 2$) that $\widetilde{\otimes}^n T : \widetilde{\otimes}^n_\pi E \longrightarrow \widetilde{\otimes}^n_\pi F$ is injective if $T \in \mathcal{L}(E;F)$ is injective and E is a Banach space with the approximation property. Hence 2.3.(3) implies the

Corollary. *If E is a Banach space with the approximation property and $T \in \mathcal{L}(E;F)$ is injective, then $\widetilde{\otimes}^n_s T : \widetilde{\otimes}^{n,s}_{\pi_s} E \longrightarrow \widetilde{\otimes}^{n,s}_{\pi_s} F$ is also injective.*

2.5. Denote by $\mathrm{COFIN}(E)$ the set of closed finite-codimensional subspaces of a normed space. It is clear that $\pi_s\big(z; \otimes^{n,s} E\big) \geq \sup\big\{\pi_s\big(\big(\otimes^{n,s} Q^E_F\big)(z); \otimes^{n,s} E/F\big)\big\}$ (where the supremum is taken over all $F \in \mathrm{COFIN}(E)$), but – in general – there is no equality: as in Example 2 in 4.3. below this can be deduced from the respective fact for full tensor products (see e.g. [DF, 16.2.]).

Proposition. *If the normed space E has the metric approximation property, then*

$$\pi_s\big(z; \otimes^{n,s} E\big) = \sup\big\{\pi_s\big(\big(\otimes^{n,s} Q^E_F\big)(z); \otimes^{n,s} E/F\big) \quad F \in \mathrm{COFIN}(E)\big\}.$$

Proof. For $\varepsilon > 0$ take a representation $z = \sum_{j=1}^m \lambda_j \otimes^n x_j$ with $\sum |\lambda_j| \|x_j\|^n \leq (1+\varepsilon)\pi_s(z)$ and – using the m.a.p. – a $T \in \mathcal{L}(E;E)$ of finite rank, $\|T\| \leq 1 + \varepsilon$ and $Tx_j = x_j$ (see e.g. [DF, 16.9.]). Then $T = \widehat{T} \circ Q^E_{\ker T}$ and

$$\pi_s\big(z; \otimes^{n,s} E\big) = \pi_s\big(\otimes^{n,s} \widehat{T} \circ Q^E_{\ker T}(z)\big) \leq \|\widehat{T}\|^n \pi_s\big(\otimes^{n,s} Q^E_{\ker T}(z); \otimes^{n,s} E/\ker T\big)$$

which implies the result. cl

2.6. A polynomial $q \in \mathcal{P}^n(E)$ is called *nuclear* (notation: $q \in \mathcal{P}^n_{\mathrm{nuc}}(E)$) if there are $\lambda_m \in \mathbb{K}$ and $x'_m \in E'$ such that

$$q(x) = \sum_{m=1} \lambda_m \langle x'_m, x\rangle^n \qquad \text{for all } x \in E$$

with $\sum_{m=1}^{\infty} |\lambda_m| \|x'_m\|^n < \infty$, i.e. $q = \sum_{m=1}^{\infty} \lambda_m \underline{\otimes}^n x'_m$ (see 1.13. for $\underline{\otimes}^n$). It is well-known and easy that

$$\|q\|_{\mathrm{nuc}} = \inf\left\{ \sum_{m=1}^{\infty} |\lambda_m| \|x'_m\|^n \;\; q = \sum_{m=1}^{\infty} \lambda_m \underline{\otimes}^n x'_m \right\}$$

is a norm and $(\mathcal{P}^n_{\mathrm{nuc}}(E), \| \|_{\mathrm{nuc}})$ is a Banach space. The description 2.2.(9) of $\widetilde{\otimes}^{n,s}_{\pi_s} E'$ shows that the map $J^n_{E'}$ (see 1.13 .) extends to a metric surjection

$$\widetilde{J}^n_{E'} : \widetilde{\otimes}^{n,s}_{\pi_s} E' \overset{1}{\twoheadrightarrow} \mathcal{P}^n_{\mathrm{nuc}}(E).$$

Reca11 $\otimes^{n,s}E' = \mathcal{P}^n_f(E)$ from 1.13.. In 4.3. the injectivity of this map will be investigated.

2.7. For a set $D \subset \mathcal{P}^n(E) = (\otimes^{n,s}_{\pi_s} E)' = (\widetilde{\otimes}^{n,s}_{\pi_s} E)'$ the following are equivalent if E is a Banach space *(Mackey theorem for polynomials):*

(a) D is norm-bounded.

(b) D is $\sigma(\mathcal{P}^n(E), \widetilde{\otimes}^{n,s}_{\pi_s} E)$-bounded.

(c) D is $\sigma(\mathcal{P}^n(E), \otimes^{n,s} E)$-bounded.

(d) $\{q(x) \;\; q \in D\}$ is bounded for all $x \in E$.

The proof is immediate from this kind of theorem in $\mathcal{L}({}^nE)$ and the polarization formula.

2.8. The construction in 1 .10. shows that the full projective tensor product $\otimes^n_\pi E$ is isomorphic to a complemented subspace of $\otimes^{n,s}_{\pi_s} E^n$. The formula for $\otimes^{n,s}(F \oplus G)$ at the end of 1.10. holds topologically (see [AnF]) which implies that $\otimes^{n,s}_{\pi_s} E \cong \otimes^n_\pi E$ if $E \cong E^2$ (a result which is due to Diaz-Dineen [DD]).

2.9. The construction 1.10. is also quite useful to transfer counterexamples from π to π_s; example: if $G \subset E$ is a subspace, but the norm $\otimes^n_\pi G$ is not equivalent to the induced norm from $\otimes^n_\pi E$, then the same holds for $\otimes^{n,s}_{\pi_s} G^n$ and $\otimes^{n,s}_{\pi_s} E^n$: *the projective s-tensor norm does not respect subspaces topologically*. However, $\otimes^{n,s}_{\pi_s} E \hookrightarrow \otimes^{n,s}_{\pi_s} E''$ is always an isometry (see 6.7. below), in particular dense subspaces are respected (this can also easily be deduced from 2.2.(2)).

2.10. Blasco's construction [B2] mentioned in 1.11. gives that $\otimes^{n,s}_{\pi_s} E$ *is topologically isomorphic to a complemented subspace of* $\otimes^{n+1,s}_{\pi_s} E$. In particular, E is isomorphic to a complemented subspace of $\otimes^{n,s}_{\pi_s} E$ for all $n \in N$.

3 The injective s-tensor norm

3.1. The metric theory of full tensor products of normed spaces, due to Grothendieck and Schatten treats "reasonable" norms α on $\otimes^n(E_1,\dots,E_n)$ with $\varepsilon \le \alpha \le \pi$ and allows, for example, to treat interesting subclasses of multilinear forms or operators via duality. To follow such strategies for polynomials, π was substituted by π_s since the latter is more appropriate for polynomials. In this sense, the *injective s-tensor norm* ε_s on $\otimes^{n,s}E$ is defined to be the induced norm from

$$J : \otimes^{n,s}E \hookrightarrow \mathcal{P}^n(E') \overset{1}{=} (\otimes^{n,s}_{\pi_s} E')'$$

hence

$$\varepsilon_s(z;\otimes^{n,s}E) := \|J(z)\|_{\mathcal{P}^n(E')} = \sup\{|\langle z, \otimes^n x'\rangle| \;\; x' \in B_{E'}\} =$$
$$= \sup\left\{\left|\textstyle\sum_{k=1}^m \lambda_k \langle x', x_k\rangle^n\right| \;\; x' \in B_{E'}\right\}$$

if $z = \sum_{k=1}^m \lambda_k \otimes^n x_k$. Notation: $\otimes^{n,s}_{\varepsilon_s}E$ and $\widetilde{\otimes}^{n,s}_{\varepsilon_s}E$ for the completion. From the commutative diagrams (see 1.14.)

$$\begin{array}{ccc} \otimes^{n,s}_{\varepsilon_s}E & \overset{1}{\hookrightarrow} & (\otimes^{n,s}_{\pi_s}E')' \\ \downarrow{\scriptstyle \iota^n_E} & & \downarrow{\scriptstyle (\sigma^n_{E'})'} \\ \otimes^n_{\varepsilon}E & \overset{1}{\hookrightarrow} & (\otimes^n_{\pi}E')' \end{array} \qquad \begin{array}{ccc} \otimes^{n,s}_{\varepsilon_s}E & \overset{1}{\hookrightarrow} & (\otimes^{n,s}_{\pi_s}E')' \\ \uparrow{\scriptstyle \sigma^n_E} & & \uparrow{\scriptstyle (\iota^n_{E'})'} \\ \otimes^n_{\varepsilon}E & \overset{1}{\hookrightarrow} & (\otimes^n_{\pi}E')' \end{array}$$

and the same with the rôles of E and E' interchanged one obtains from 2.3. the

Proposition.

(1) $\|\iota^n_E : \otimes^{n,s}_{\varepsilon_s}E \longrightarrow \otimes^n_{\varepsilon}E\| \le c(n,E')$ *and* $\|\iota^n_{E'} : \otimes^{n,s}_{\varepsilon_s}E' \longrightarrow \otimes^n_{\varepsilon}E'\| \le c(n,E)$

(2) $\|\sigma^n_E : \otimes^n_{\varepsilon}E \longrightarrow \otimes^{n,s}_{\varepsilon_s}E\| = 1$ *if* $E \neq \{0\}$

(3) $\otimes^{n,s}_{\varepsilon_s}E$ *is a topologically complemented subspace of* $\otimes^n_{\varepsilon}E$.

If $\varepsilon|_s$ denotes the restriction of the injective norm ε of the full tensor product to the symmetric one, one has in particular

$$\varepsilon_s \le \varepsilon|_s \le c(n,E')\varepsilon_s ;$$

in particular: $\varepsilon_s = \varepsilon|_s$ for Hilbert spaces. For equality in (1) see 5.3..

3.2. More properties of ε_s are collected in the

Proposition.

(1) $\varepsilon_s(\otimes^n x;\otimes^{n,s}E) = \|x\|^n$ *for all* $x \in E$; *in particular:* $\varepsilon_s \leq \pi_s$.

(2) ε_s *satisfies the rnetric mapping property, i.e.*

$$\| \otimes^{n,s} T : \otimes^{n,s}_{\varepsilon_s} E \longrightarrow \otimes^{n,s}_{\varepsilon_s} F\| = \|T : E \longrightarrow F\|^n .$$

(3) If E *is a Banach space and* $T \in \mathcal{L}(E;F)$ *is injective, then*

$$\widetilde{\otimes}^n_s T : \widetilde{\otimes}^{n,s}_{\varepsilon_s} E \longrightarrow \widetilde{\otimes}^{n,s}_{\varepsilon_s} F$$

is injective as well.

(4) If $I : G \overset{1}{\hookrightarrow} E$ *is a metric injection, then* $\otimes^{n,s} I : \otimes^{n,s}_{\varepsilon_s} G \overset{1}{\hookrightarrow} \otimes^{n,s}_{\varepsilon_s} E$
("ε_s *respects metric injections").*

(5) $\varepsilon_s(z;\otimes^{n,s}E) = \inf\{\varepsilon_s(z;\otimes^{n,s}M) \;\; M \in \mathrm{FIN}(E), z \in \otimes^{n,s}M\}$ *(i.e.* ε_s *is finitely generated). The infimum is attained.*

(6) If $C \subset B_{E'}$ *is* $\sigma(E', E)$*-dense, then*

$$\varepsilon_s(z;\otimes^{n,s}E) = \sup\{\, |\langle z, \otimes^n x'\rangle| \,\big|\, x' \in c\}$$

for all $z \in \otimes^{n,s}E$. *In particular:*

$$\varepsilon_s(z';\otimes^{n,s}E') = \sup\{\, |\langle z', \otimes^n x\rangle| \;\; x \in B_E\}$$

for all $z' \in \otimes^{n,s}E'$ *in other words:*

$$\mathcal{P}^n_f(E) \overset{1}{=} \otimes^{n,s}_{\varepsilon_s} E' \overset{1}{\hookrightarrow} \mathcal{P}^n(E) \overset{1}{=} (\otimes^{n,s}_{\pi_s} E)'$$

The proofs of these statements are straightforward; for (6) one uses that for $z = \sum_{k=1}^m \lambda_k \otimes^n x_k$ the function $E' \ni x' \rightsquigarrow \langle z, \otimes^n x'\rangle = \sum_{k=1}^m \lambda_k \langle x_k, x'\rangle^n$ is o(E',E)-continuous. Reca11 that $D \subset B_{E'}$ is called *norming* if $\|x\| = \sup\{\, |\langle x', x\rangle| \;\; x' \in D\}$ which is equivalent to ΓD being $\sigma(E', \mathrm{E})$-dense in $B_{E'}$.

However, (6) does *not* hold for norming C $\subset B_{E'}$: take $E = C[0,1]$ and C := $\{\delta_t \mid 0 \leq t \leq 1\}$, since

$$\otimes^{2,s} C[0,1] \subset C[0,1]^2$$

one obtains $\sup\{|\langle h, \otimes^2 \delta_t\rangle| \mid 0 \leq t \leq 1\} = \sup\{|h(t,t)| \mid 0 \leq t \leq 1\}$ but there are $0 \neq h \in {}^{2,s}C[0,1]$ which are 0 on the diagonal.

It can be seen as in 2.9. that ε_s *does not respect quotient mappings topologically.*

3.3. A neat application of the basic properties of ε_s (in particular its injectivity 3.2.14)) is the following: it is straightforward from the definition that (for $n \geq 2$)

$$\varepsilon_s\left(\sum_{k=1}^n \lambda_k \otimes^n e_k; \otimes^{n,s}\ell_2^n\right) = \max\{|\lambda_k| \mid k = 1, \ldots, n\}$$

hence $\ell_\infty^n \overset{1}{\hookrightarrow} \otimes_{\varepsilon_s}^{n,s} \ell_2^n$. Dvoretzky's theorem ($\ell_2^n$ is $(1+\varepsilon)$-isomorphic to a subspace of every infinite-dimensiona1 normed space E) implies that $\ell_\infty^n \overset{1}{\hookrightarrow} \otimes_{\varepsilon_s}^{n,s} \ell_2^n \overset{1+\varepsilon}{\hookrightarrow} \otimes_{\varepsilon_s}^{n,s} E$. It follows that ℓ_∞ is finitely represented in $\otimes_{\varepsilon_s}^{n,s} E$ and in $\mathcal{P}^n(E) \overset{1}{\hookleftarrow} \otimes_{\varepsilon_s}^{n,s} E'$; this result is due to Dineen [D3]. In particular: none of these spaces have proper type or cotype.

3.4. As in the n-linear case the description of the dual will be crucial. Since $\varepsilon_s \leq \pi_s$ one has

$$(\otimes_{\varepsilon_s}^{n,s} E)' \; c \; (\otimes_{\pi_s}^{n,s} E)' \; = \; Z'\text{“}(E).$$

A polynomial $q \in \mathcal{P}^n(E)$ is called *integra1* if $q^L \in (\otimes_{\varepsilon_s}^{n,s} E)'$; notation $q \in \mathcal{P}_{\text{int}}^n(E)$. It is clear that with $\|.\|_{\text{int}}$ defined by

$$\|q\|_{\text{int}} := \|q^L\|_{(\otimes_{\varepsilon_s}^{n,s} E)'}$$

$\mathcal{P}_{\text{int}}^n(E)$ becomes a Banach space. Note that it is obvious from the Hahn-Banach theorem and the fact that ε_s respects subspaces (see 3.2.(3)) that every integra1 polynomial q on a subspace G *CE* has an integra1 extension $\tilde{q} \in \mathcal{P}_{\text{int}}^n(E)$ with $\|\tilde{q}\|_{\text{int}} = \|q\|_{\text{int}}$.

Theorem. [Dineen [Dl]]. *Let $q \in \mathcal{P}^n(E)$. Then q is integral if and only if there is a signed Borel-measure μ on $B_{E'}$ (with the $\sigma(E',E)$-topology) such that*

$$q(x) = \int_{B_{E'}} \langle x', x\rangle^n \mu(dx') \qquad (*)$$

for all $x \in E$. Moreover:

$$\|q\|_{\text{int}} = \min\{\|\mu\| \;\; \mu \text{ as in } (*)\}.$$

If $\mathbb{K} = \mathbb{C}$ or: $\mathbb{K} = \mathbb{R}$ and n odd, then a best measure μ can be chosen positive, but otherwise, in general, not.

Proof. If q (and hence also q^L) has such an representation, it is immediate that $\|q\|_{\text{int}} \leq \|\mu\|$. Vice versa $I(z)(x') := \langle \otimes^n x', z\rangle$ defines an isometry $I : \otimes_{\varepsilon_s}^{n,s} E \hookrightarrow C(B_{E'})$ and the Hahn-Banach theorem gives a signed (regular) Borel-measure $\mu \in C(B_{E'})'$ which extends q^L (i.e.: $I'(\mu) = q^L$) and $\|\mu\| = \|q^L\|_{(\otimes_{\varepsilon_s}^{n,s} E)'}$. If $\mathbb{K} = \mathbb{R}$ and n is even, then positive measures represent (via $(*)$) only non-negative q. In the remaining cases Defant's proof for the $\otimes_\varepsilon$-situation ([DF, 4.6.]) can be adopted: denote by D the Dirac-measures, by M_1^+ the probability measures and by $M := C(B_{E'})'$ all signed Borel-measures on $B_{E'}$. Then, by definition, *Z'(D)* is norming for ε_s on $\otimes^{n,s} E$, equivalently:

$$I'(D)^0 = B_{\otimes_{\varepsilon_s}^{n,s} E}.$$

For $\lambda \in B_{\mathbb{K}}$ there is $\alpha \in B_{\mathbb{K}}$ with $\alpha^n = \lambda$ (this were not possible if $\mathbb{K} = \mathbb{R}$ and n even!) hence $\lambda I'(\delta_{x'}) = I'(\delta_{\alpha x'})$ for all $x' \in B_{E'}$. Since $0 \in Z'(D)$ if follows that $conv(I'(D)) = \Gamma(I'(D))$. The bipolar theorem and the $\sigma(M, C(B_{E'}))$-compacity of M_1^+ give for $\sigma_* := \sigma((\otimes_{\varepsilon_s}^{n,s} E)', \otimes_{\varepsilon_s}^{n,s} E)$

$$B_{(\otimes_{\varepsilon_s}^{n,s} E)'} \; = \; I'(D)^{00} \; = \; \overline{\Gamma I'(D)}^{\sigma_*} \; = \; \overline{conv\, I'(D)}^{\sigma_*} \; c \; I'(M_1^+)$$

□

For $x' \in E'$ it is clear that $\otimes^n x' \in \otimes^{n,s} E' \subset \mathcal{P}^n(E)$ is integra1 and $\|\check{\otimes}^n x'\|_{\mathrm{int}} = \|x'\|^n$ (by 3.2.(6)) hence (by the universa1 property 2.2.(1))

$$\|\otimes_{\pi_s}^{n,s} E' \longrightarrow \mathcal{P}_{\mathrm{int}}^n(E)\| \leq 1$$

and also

$$\tilde{J}_{E'}^n : \tilde{\otimes}_{\pi_s}^{n,s} E' \overset{1}{\twoheadrightarrow} \mathcal{P}_{\mathrm{nuc}}^n(E) \hookrightarrow \mathcal{P}_{\mathrm{int}}^n(E)$$

has norm ≤ 1. It follows that $\|q\| \leq \|q\|_{\mathrm{int}} \leq \|q\|_{\mathrm{nuc}}$. In section 4 it will be investigated under which circumstances $\tilde{J}_{E'}^n$ is injective, an isomorphism (in) and onto.

3.5. The following example will turn out to be typical: let $\mu \neq 0$ be a signed measure such that $|\mu|$ is strictly localizable (e.g. if μ is o-finite), then $\varphi_M \in L_s({}^n L_\infty(\Omega, |\mu|); L_\infty(\Omega, |\mu|))$ is defined to be the multiplication

$$\varphi_M(\tilde{f}_1, \ldots, \tilde{f}_n) := \left(\prod_{k=1}^n f_k\right)^{\sim}.$$

Remark.

(1) $(\varphi_M)^{L,s} \in \mathcal{L}(\otimes_{\varepsilon_s}^{n,s} L_\infty(\Omega, |\mu|); L_\infty(\Omega, |\mu|))$ *with norm 1.*

(2) If μ is finite, then the n-homogeneous "integrating" polynomial q_n dejìned by

$$q_n(\tilde{f}) = \int_\Omega f^n d\mu$$

is integral and $\|q_n\|_{\mathrm{int}} = \|q_n\| \leq |\mu|(\Omega)$.

Proof. If $\lambda : L_\infty(\Omega, |\mu|) \longrightarrow \mathcal{L}_\infty(\Omega, |\mu|)$ is a lifting (see e.g. [F1, 16.9.]), then $\delta_w(\tilde{f}) := \mathrm{h(f)(w)}$ defines a functional in $B_{L'_\infty}$, hence one obtains for $\tilde{g} = \sum \alpha_m \otimes^n \tilde{f}_m$

$$\begin{aligned} \|\varphi_M^{L,s}(\tilde{g})\|_{L_\infty} &= \|\sum \alpha_m \widetilde{f_m^n}\|_{L_\infty} \leq \sup_{w \in \Omega} |\langle \delta_w, \sum \alpha_m \widetilde{f_m^n} \rangle| = \\ &= \sup_{w \in \Omega} \langle \otimes^n \delta_w, \sum \alpha_m \otimes^n \widetilde{f_m} \rangle \leq \varepsilon_s(\tilde{g}; \otimes^{n,s} L_\infty). \end{aligned}$$

If μ is finite and $\psi_I(\tilde{f}) := \int f \, d\mu$, then $q_n^L := \psi_I \circ \varphi_M^{L,s}$ which gives that q_n is integral; the rest is easy. □

Corollary. *For $q \in l?^{\alpha}(E)$ the following statements are equivalent:*

(1) *q is integral.*

(2) There exists a signed finite measure μ on some Ω and $T \in \mathcal{L}(E; L_\infty(|\mu|)$ with

$$q(x) = \int_\Omega [(Tx)(w)]^n \mu(dw)$$

for all $x \in E$.

(3) As in (2), *hut with a signed Borel measure on a compact set.*

In this case:

$$\begin{aligned} ||q||_{\text{int}} &= \min\{||\mu||\,||T||^n \;\; \mu, T \text{ as in (2)}\} = \\ &= \min\{\cdots \;.\; \cdots (3)\} \end{aligned}$$

Proof. (2) implies (1) and $||q||_{\text{int}} \leq ||\mu||\,||\otimes_{\varepsilon_s}^{n,s} T|| = ||\mu||\,||T||^n$. If q is integral, then define $T : E \overset{1}{\hookrightarrow} C(B_{E'})$ and Theorem 3.4. gives a representation (3) with $||q||_{\text{int}} = ||\mu||\,||T||^n$. □

In other words: q factors through the integrating polynomial

$$q : E \xrightarrow{T} L_\infty(|\mu|) \xrightarrow{q_n} \mathbb{K}.$$

Conceming positivity of μ the same statements as in 3.4. apply; in particular: if $\mathbb{K} = \mathbb{C}$ the measures in the corollary can be chosen positive.

3.6. Define for a normed space and the canonica1 mappings J_0 and J_1 the mapping Φ by

$$\Phi : \otimes_{\pi_s}^{n,s} E' \xrightarrow{J_0} \otimes_{\varepsilon_s}^{n,s} E' \xrightarrow{J_1} (\otimes_{\pi_s}^{n,s} E)' \overset{1}{=} \mathcal{P}^n(E);$$

note that J_0 is onto and J_1 an isometry (in). It follows that the *Borel transform* $B = \Phi'$ factors

$$B : \mathcal{P}^n(E)' \xrightarrow{J_1'} (\otimes_{\varepsilon_s}^{n,s} E')' = \mathcal{P}_{\text{int}}^n(E') \xrightarrow{J_0'} (\otimes_{\pi_s}^{n,s} E')' \overset{1}{=} \mathcal{P}^n(E')$$

and J_1' is a metric surjection and J_0' injective. In other words: $\text{im}B = \mathcal{P}_{\text{int}}^n(E')$ (an observation from [CZ]). Note that J_0' and J_1' are both norm-norm and weak$*$-weak$*$ continuous. For $\varphi \in \mathcal{P}^n(E)'$ the integra1 polynomial $B(\varphi) \in \mathcal{P}^n(E')$ can be calculated as follows:

$$B(\varphi)(x') = \langle J_0' \circ J_1'(\varphi), \otimes^n x' \rangle = \langle \varphi, \underline{\otimes}^n x' \rangle$$

(see 1.13. for the notation $\otimes^n x'$).

3.7. The statements of 2.8. and 2.10. hold also for the injective norms: *$\otimes_\varepsilon^n E$ is isomorphic to a complemented subspace of $\otimes_{\varepsilon_s}^{n,s} E^n$ und these two spaces are topologically isomorphic if $E \cong E^2$* (see [AnF]). It was observed in [AnF] that a careful check of Blasco's construction [B2] gives also that *$\otimes_{\varepsilon_s}^{n,s} E$ is topologically isomorphic to a complemented subspace of $\otimes_{\varepsilon_s}^{n+1,s} E$*; in particular: E is isomorphic to a complemented subspace of $\otimes_{\varepsilon_s}^{n,s} E$ for all $n \in \mathbb{N}$.

4 Duality and the approximation property

4.1. If E is a normed space, the definition of ε_s (and 3.2.(6)) give that the natura1 mappings

$$\begin{aligned} \widetilde{\otimes}_{\varepsilon_s}^{n,s} E &\longrightarrow (\otimes_{\pi_s}^{n,s} E')' \overset{1}{=} \mathcal{P}^n(E') \\ \widetilde{\otimes}_{\varepsilon_s}^{n,s} E' &\longrightarrow (\otimes_{\pi_s}^{n,s} E)' \overset{1}{=} \mathcal{P}^n(E) \end{aligned}$$

are metric injections. The polynomials in $\widetilde{\otimes}^{n,s}_{\varepsilon_s} E' \subset \mathrm{T}^n(E)$ are usually called *upproximable.* How is the dual situation? When are the mappings

$$J^d_E : \widetilde{\otimes}^{n,s}_{\pi_s} E \longrightarrow (\otimes^{n,s}_{\varepsilon_s} E')' = \mathcal{P}^n_{\mathrm{int}}(E') \subset \mathcal{P}^n(E')$$
$$J_{E'} : \widetilde{\otimes}^{n,s}_{\pi_s} E' \longrightarrow (\otimes^{n,s}_{\varepsilon_s} E)' = \mathcal{P}^n_{\mathrm{int}}(E) \subset \mathcal{P}^n(E)$$

injective or even metric injections, when are they surjective?

4.2. The injectivity and surjectivity can easily be deduced from the analogous properties of the full tensor product. Here the result is as follows:

Theorem. *Let $E_1, \ldots, E_{n-1}$ be Banach spaces $\neq \{0\}$.*

(1) *$E_1, \ldots, E_{n-1}$ have the approximation property if and only if for all Banach spaces E_n (or only separable reflexive E_n) the canonical map*

$$\widetilde{\otimes}^n_{\pi, j=1} E_j \longrightarrow \widetilde{\otimes}^n_{\varepsilon, j=1} E_j$$

is injective.

(2) *$E'_1, \ldots, E'_{n-1}$ have the Radon-Nikodým property (=: RNP) if and only if for all Banach spnces E_n the canonical map*

$$\widetilde{\otimes}^n_{\pi, j=1} E'_j \longrightarrow (\widetilde{\otimes}^n_{\varepsilon, j=1} E_j)'$$

is surjective. In this case it is even a metric surjection.

Proof. These results are known. Proofs for $n = 2$ can be found e.g. in [DF, 5.6., 21.9., 16.5.] – and the general case can be deduced from this: For (1) use [DF, 4.3.(2)] and

$$E_1 \widetilde{\otimes}_\pi (E_2 \widetilde{\otimes}_\pi \ldots) \longrightarrow E_1 \widetilde{\otimes}_\varepsilon (E_2 \widetilde{\otimes}_\pi \ldots) \longrightarrow E_1 \widetilde{\otimes}_\varepsilon (E_2 \widetilde{\otimes}_\varepsilon (E_3 \widetilde{\otimes}_\pi \ldots)) \longrightarrow$$
$$\longrightarrow E_1 \widetilde{\otimes}_\varepsilon \cdots \widetilde{\otimes}_\varepsilon E_n$$

and, for the other direction, that the condition implies that $\widetilde{\otimes}^{n-1}_{\pi, j=1} E_j$ has a.p.. For (2) look at

$$\widetilde{\otimes}_\pi (E'_1, \ldots, E'_n) \overset{1}{\twoheadrightarrow} \widetilde{\otimes}_\pi (E'_1, \ldots, E'_{n-2}, (E_{n-1} \otimes_\varepsilon E_n)') \overset{1}{\twoheadrightarrow} \cdots \overset{1}{\twoheadrightarrow} (\otimes^n_{\varepsilon, j=1} E_j)' . \qquad (*)$$

For the converse note that the condition implies that

$$(\otimes^{n-1}_{\varepsilon, j=1} E_j)' \widetilde{\otimes}_\pi E'_n \longrightarrow (\otimes^n_{\varepsilon, j=1} E_j)' \qquad (**)$$

is always onto, hence $(\otimes^{n-1}_{\varepsilon, j=1} E_j)'$ has RNP and so do all E'_j (for j = 1,. . , n – 1) since they are complemented in $(\otimes^{n-1}_{\varepsilon, j=1} E_j)'$. □

In particular: $\widetilde{\otimes}^n_\pi E$ has the approximation property if E has it and $(\widetilde{\otimes}^n_\varepsilon E)'$ has the RNP if E' has RNP (use $(**)$ of the foregoing proof for this). Complementation of the symmetric in the full projective tensor product and Blasco's results cited in 2.10. and 3.7. give the

Corollary. *The Banach space E has the approximation property (resp. E' has RNP) if and only if $\widetilde{\otimes}^{n,s}_{\pi_s} E$ has the approximation property (resp. $\mathcal{P}^n_{\mathrm{int}}(E) = (\widetilde{\otimes}^{n,s}_{\varepsilon_s} E)'$ has RNP).*

The result about the a.p. is from [Mu2]. Since $(\otimes^n_\pi \ell_2)'$ does not have the a.p. (for $n = 2$ this is the famous result of Szankowski, the case $n > 2$ easily follows from this) 2.8. implies that also $(\otimes^{n,s}_{\pi_s} \ell_2)' = \mathcal{P}^n(\ell_2)$ *does not have the approximation property*. Note that $\widetilde{\otimes}^{n,s}_{\varepsilon_s} E$ also has the a.p. if E has it since this result is true for $\widetilde{\otimes}^n_\varepsilon E$ (see e.g. [Ko, §44.5.(7)]).

4.3. The natural maps $\widetilde{\otimes}^{n,s}_{\pi_s} E \longrightarrow \mathcal{P}^n(E')$ and $\widetilde{\otimes}^{n,s}_{\pi_s} E' \longrightarrow \mathcal{P}^n(E)$ have ranges in $\widetilde{\otimes}^{n,s}_{\varepsilon_s} E \cap (\otimes^{n,s}_{\varepsilon_s} E')'$ and $\widetilde{\otimes}^{n,s}_{\varepsilon_s} E' \cap (\otimes^{n,s}_{\varepsilon_s} E)'$ respectively, hence the injectivity of J^d_E and $J_{E'}$ (from 4.1.) is equivalent to the injectivity of $\widetilde{\otimes}^{n,s}_{\pi_s} \cdot \longrightarrow \widetilde{\otimes}^{n,s}_{\varepsilon_s} \cdot$ respectively. Since certainly the diagram

$$\begin{array}{ccc} \widetilde{\otimes}^{n,s}_{\pi_s} E & \longrightarrow & \widetilde{\otimes}^{n,s}_{\varepsilon_s} E \\ {\scriptstyle \iota^{n,\pi}_E}\downarrow & & \downarrow{\scriptstyle \iota^{n,\varepsilon}_E} \\ \widetilde{\otimes}^n_\pi E & \longrightarrow & \widetilde{\otimes}^n_\varepsilon E \end{array}$$

(natural mappings) is commutative, 4.2.(1) gives the

Proposition. *If E is a Banach space with the approximation property, then, for all $n \in \mathbb{N}$, the natural map*

$$\widetilde{\otimes}^{n,s}_{\pi_s} E \longrightarrow \widetilde{\otimes}^{n,s}_{\varepsilon_s} E$$

is injective.

Note that $\widetilde{\otimes}^{n,s}_{\varepsilon_s} E \overset{1}{\hookrightarrow} \mathcal{P}^n(E')$ and $\widetilde{\otimes}^{n,s}_{\varepsilon_s} E' \overset{1}{\hookrightarrow} \mathcal{P}^n(E)$. For the nuclear polynomials (see 2.6.) it follows that $\widetilde{\otimes}^{n,s}_{\pi_s} E' \overset{1}{=} \mathcal{P}^n_{\mathrm{nuc}}(E)$ holds if E' has the approximation property; in particular: $\mathcal{P}^n_{\mathrm{nuc}}(E)' \overset{1}{=} \mathcal{P}^n(E')$ in this case-a result which is due to Gupta [Gu] in 1968 (see 0.1.).

Is the condition of E having a.p. in the proposition necessary?

Example. *Let P be a Banach space without a.p. such that $P \otimes_\pi P = P \otimes_\varepsilon P$ holds topologically (Pisier [Pi] has constructed such spaces). It follows that $\widetilde{\otimes}^{2,s}_{\pi_s} P \longrightarrow \widetilde{\otimes}^{2,s}_{\varepsilon_s} P$ is injective, hence the converse **of** the proposition is false **for** $n = 2$. However, an example like Pisier's spaces P does not exist for $n \geq 3$: John [J] has shown that a Banach space E with $\otimes^n_\pi E \cong \otimes^n_\varepsilon E$ **for** some $n \geq 3$ is finite dimensional.*

Example. *Let $E_1, \ldots, E_n$ be Banach spaces such that $\widetilde{\otimes}^n_{\pi, j=1} E_j \longrightarrow \widetilde{\otimes}^n_{\varepsilon, j=1} E_j$ is* not *injective and take $F_n := \prod^n_{j=1} E_j$. It follows from the construction in 1.10., the metric mapping property **of** ε_s and π_s and the continuity **of** $\sigma^n_{F_n}$ that*

$$\widetilde{\otimes}^{n,s}_{\pi_s} F_n \longrightarrow \widetilde{\otimes}^{n,s}_{\varepsilon_s} F_n$$

is not *injective. Take now* $(E_1, \dots, E_n) = (\mathbb{M}, \dots, \mathbb{K}, E, E')$ *with an* E *without a.p. such that* $E \cong E \times \mathbb{K}$ *(for example* $E = G \oplus \ell_2$ *and* G *without a.p.) one obtains that* $F_n \cong \mathbb{K}^{n-2} \times E \times E' \cong E \times E'$ *hence* $G := E \times E'$ *has the property that*

$$\widetilde{\otimes}^{n,s}_{\pi_s} G \longrightarrow \widetilde{\otimes}^{n,s}_{\varepsilon_s} G$$

is not injective for all $n \geq 2$.

It is not known whether the injectivity for some $n \geq 3$ *(or all* n*) implies the a.p..*

4.4. If $\langle F, \mathrm{G}\rangle$ is a separating dual system of normed spaces and the natura1 map

$$\widetilde{\otimes}^{n,s}_{\pi_s} F \longrightarrow \mathcal{P}^n(G) = (\otimes^{n,s}_{\pi_s} G)'$$

is injective, then (using 1.13.) $\langle \widetilde{\otimes}^{n,s}_{\pi_s} F, \otimes^{n,s} G\rangle$ is a separating dual system. In particular

Corollary. *Let* E *be a Banach space. If* E *(resp.* E'*) has the approximation property, then* $\langle \widetilde{\otimes}^{n,s}_{\pi_s} E, \otimes^{n,s} E'\rangle$ *(resp.* $\langle \widetilde{\otimes}^{n,s}_{\pi_s} E', \otimes^{n,s} E\rangle$*) is a separating dual system.*

Just one application of this result:

Proposition. [Ryan [R]]. *Let* E *be a Banach space with the upproximution property such that* $\mathcal{P}^n(E)$ *is reflexive. Then every* $q \in \mathcal{P}^n(E)$ *is* $\sigma(E, E')$*- continuous on bounded subsets of* E.

Proof. The fact that $\langle D := \otimes^{n,s} E', \widetilde{\otimes}^{n,s}_{\pi_s} E =: \mathrm{G}\rangle$ is a separating dual system implies that $\sigma(G, D)$ is a Hausdorff topology on G which (by the reflexity of G) coincides with $\sigma(G, \mathrm{G}')$ on bounded = $\sigma(G, \mathrm{G}')$-relatively compact sets.

Now take a bounded net (x,) in E with $\sigma(E, E')$-limit x. Then $\langle \otimes^n x_\alpha, z'\rangle \longrightarrow \langle \otimes^n x, z'\rangle$ for every $z' = \sum \lambda_m \otimes^n x'_m \in D = \otimes^{n,s} E' \subset \mathcal{P}^n(E) = \mathrm{G}'$; hence (by what was just said) $\otimes^n x_\alpha$ $\sigma(G, G')$-converges to $\otimes^n x$, i.e. for all $q \in \mathrm{G}' = P'^*(E)$

$$q(x_\alpha) = \langle q^L, \otimes^n x_\alpha\rangle \longrightarrow \langle q^L, \otimes^n x\rangle = q(x).$$

□

The space $P''(E)$ is reflexive for example in the following cases: $E = \ell_p$ (if $n < p < \infty$; see [AlF1] and [GoJ]) or $E = T$, the origina1 Tsirelson space (see [AAD]). Note that the claim of the proposition holds also for the non-reflexive space c_0 since all $q \in \mathcal{P}^n(c_0)$ are weakly sequentially continuous (due to Bogdanowicz [Bo] and Pełczyński [Pe], see also [AlF1]) and the bounded sets in c_0 are weakly metrizable.

4.5. The diagram of natura1 maps

$$\begin{array}{ccc} \widetilde{\otimes}^{n,s}_{\pi_s} E' & \xrightarrow{J^s} & (\widetilde{\otimes}^{n,s}_{\varepsilon_s} E)' \\ \uparrow{\scriptstyle \sigma^n_{E'}{}^{\pi}} & & \uparrow{\scriptstyle (\iota^n_E)'} \\ \widetilde{\otimes}^{n}_{\pi} E' & \xrightarrow{J} & (\widetilde{\otimes}^{n}_{\varepsilon} E)' \end{array}$$

is commutative (see 1.14.). Clearly, if J is surjective, J^s is as well. Therefore, if E' has RNP 4.2.(b) implies that J^s is onto and hence open, in particular (use 2.6.) $\mathcal{P}^n_{\mathrm{nuc}}(E) = \mathcal{P}^n_{\mathrm{int}}(E)$ (with equivalent norms); this result is due to Alencar [Al] who even treated the vector-valued case. To see the norm equality, Boyd and Ryan [BR] first used the metric surjection $I' : C(B_{E'})' \overset{1}{\twoheadrightarrow} (\otimes^{n,s}_{\varepsilon_s} E)' \overset{1}{=} \mathcal{P}^n_{\mathrm{int}}(E)$ (see the proof of 3.4.) to show that the extreme points of the unit ball of $\mathcal{P}^n_{\mathrm{int}}(E)$ (where E is an arbitrary normed space) are of the form $\pm \otimes^n x'$ with $x' \in B_{E'}$ hence

$$\mathrm{ext} B_{\mathcal{P}^n_{\mathrm{int}}(E)} \subset B_{\mathcal{P}^n_{\mathrm{nuc}}(E)} \subset B_{\mathcal{P}^n_{\mathrm{int}}(E)} \,. \qquad (*)$$

If E' has RNP, then (as shown above) $\mathcal{P}^n_{\mathrm{nuc}}(E) = \mathcal{P}^n_{\mathrm{int}}(E)$ and the norms are equivalent. Since $\mathcal{P}^n_{\mathrm{int}}(E)$ has also RNP (see Corollary 4.2.) a result of Lindenstrauss' (see [DU, p. 190]) implies that the unit ball of $\mathcal{P}^n_{\mathrm{int}}(E)$ is the *norm*-closed convex hull of its extreme points, hence $(*)$ and 2.6. give the

Proposition. [[Al], [BR], [CD]]. *If E' has the RNP, then the natural map $J^s : \widetilde{\otimes}^{n,s}_{\pi_s} E' \longrightarrow \mathcal{P}^n_{\mathrm{int}}(E)$ is a metric surjection, in particular: $\mathcal{P}^n_{\mathrm{nuc}}(E) \overset{1}{=} \mathcal{P}^n_{\mathrm{int}}(E)$.*

As a conscquence one obtains that

$$(\widetilde{\otimes}^{n,s}_{\pi_s} E')\widetilde{\otimes}_\pi F \overset{1}{\twoheadrightarrow} (\otimes^{n,s}_{\varepsilon_s} E)'\widetilde{\otimes}_\pi F \overset{1}{\twoheadrightarrow} \mathcal{N}(\widetilde{\otimes}^{n,s}_{\varepsilon_s} E; F) \overset{1}{=} \mathcal{P}I(\widetilde{\otimes}^{n,s}_{\varepsilon_s} E; F)$$

($\mathcal{N}$ for the nuclear and TI for the Pietsch-integral linear operators; see [DF, D8.] for the last isometry and note that $(\otimes^{n,s}_{\varepsilon_s} E)'$ has RNP) is a metric surjection:

Corollary. [[Al], [CD]]. *If E' has the RNP, then the natural map*

$$(\widetilde{\otimes}^{n,s}_{\pi_s} E')\widetilde{\otimes}_\pi F \longrightarrow \mathcal{P}I(\widetilde{\otimes}^{n,s}_{\varepsilon_s} E; F)$$

is a metric surjection for all Banach spaces F.

It follows (see Carando-Dimant [CD] for details) that – in this case – the latter space is the space of integral n-homogeneous polynomials $E \longrightarrow F$ in the sense of Alencar [Al].

Note that the mappings in the proposition (use 4.3.) and in the corollary (use the proposition, 4.2. and [DF, 5.7.]) are injective (hence isometric) if E' has the approximation property.

4.6. Concerning the isometric embeddings one has the

Duality theorem. *Let E be a normed space.*

(1) *If E has the metric approximation property, then*

$$\otimes^{n,s}_{\pi_s} E \hookrightarrow (\otimes^{n,s}_{\varepsilon_s} E')'$$

is a metric injection.

(2) *If E' has the metric approximation property, then*

$$\otimes^{n,s}_{\pi_s} E' \hookrightarrow (\otimes^{n,s}_{\varepsilon_s} E)'$$

is a metric injection.

The natura1 setting for the duality theorem and its proof is the theory of s-tensor norms which will be presented in [F2]. Therefore only the proof of (2) will be given; (1) can be shown along the same lines.

Proof.[Proof of (2)] Note first that for finite-dimensiona] G one has $\otimes_{\varepsilon_s}^{n,s} G \stackrel{1}{=} (\otimes_{\pi_s}^{n,s} G')'$ hence $\otimes_{\pi_s}^{n,s} G' \stackrel{1}{\hookrightarrow} (\otimes_{\varepsilon_s}^{n,s} G)'$. Now define for *dual* Banach spaces F' the norm $\gamma(\cdot; \otimes^{n,s} F')$ on $\otimes^{n,s} F'$ (notation: $\otimes_{\gamma}^{n,s} F'$) by

$$\otimes_{\gamma}^{n,s} F' \stackrel{1}{\hookrightarrow} (\otimes_{\varepsilon_s}^{n,s} F)' .$$

The following properties of γ are easily checked:

(a) $\gamma \leq \pi_s$ on all $\otimes^{n,s} F'$.

(b) $\gamma = \pi_s$ on $\otimes^{n,s} F'$ if $\dim F < \infty$.

(c) If $T \in \mathcal{L}(F_1; F_2)$, then $\| \otimes^{n,s} T' : \otimes_{\gamma}^{n,s} F_2' \longrightarrow \otimes_{\gamma}^{n,s} F_1' \| \leq \|T'\|^n = \|T\|^n$.

Statement (2) says that $\pi_s = \gamma$ on $\otimes^{n,s} E'$ if E' has the m.a.p.. Now suppose – a bit more general – that E' has the h-approximation property and take $z' \in \otimes^{n,s} E'$. Then there is a finite dimensiona1 subspace $F \subset E'$ with $z' \in \otimes^{n,s} F$. Thc quotient map $Q : E \longrightarrow E/_{F^0}$ (the dual of which is the embedding $I : F \hookrightarrow E'$) has finite rank, hence (see e.g. [DF, 16.9. Cor. (2)]) the λ-a.p. of E' implies the existence of a finite rank operator $S \in \mathcal{L}(E; E)$ with $\|S\| \leq \lambda(1 + \varepsilon)$ and $Q = Q \circ S$; it follows that $S' \circ I = I$ and therefore

$$z' = (\otimes^{n,s} S')(z') \in \otimes^{n,s} S'(E') .$$

The properties (b), (c) and the metric mapping property for π_s imply

$$\begin{aligned} \pi_s(z'; \otimes^{n,s} E') &= \pi_s((\otimes^{n,s} S')(z'); \otimes^{n,s} E') \leq \pi_s((\otimes^{n,s} S')(z'); \otimes^{n,s} S'(E')) = \\ &= \gamma((\otimes^{n,s} S')(z'); \otimes^{n,s} S'(E')) \leq \|S\|^n \gamma(z'; \otimes^{n,s} E') \leq \\ &\leq \lambda^n (1+\varepsilon)^n \gamma(z'; \otimes^{n,s} E') . \end{aligned}$$

It follows that $\gamma \leq \pi_s \leq \lambda^n \gamma$ on $\otimes^{n,s} E'$. □

4.7. In particular: $\mathcal{P}_{\mathrm{nuc}}^n(E) \stackrel{1}{\hookrightarrow} \mathcal{P}_{\mathrm{int}}^n(E)$ *if E' has the m.a.p.*. The proof even showed that, if E' has the Lapproximation property, then

$$\|q\|_{\mathrm{int}} \leq \|q\|_{\mathrm{nuc}} \leq \lambda^n \|q\|_{\mathrm{int}}$$

for all $q \in \mathcal{P}_{\mathrm{nuc}}^n(E)$.

5 Some consequences for the polarization constants

5.1. From 92 it is known that

$$c(n, E) = \sup\{ \|\check{q}\|_{\mathcal{L}(^n E)} \;\; \|q\|_{\mathcal{P}^n(E)} \leq 1 \} = \|\sigma_E^n : \otimes_{\pi}^n E \longrightarrow \otimes_{\pi_s}^{n,s} E\|$$

for every normed space E. Since $(\otimes^n T) z^{\eta} = [\otimes^n T(z)]^{\eta}$ the diagram

$$\begin{array}{ccc} \otimes^n E & \xrightarrow{\sigma_E^n} & \otimes^{n,s} E \\ \downarrow{\scriptstyle \otimes^n T} & & \downarrow{\scriptstyle \otimes^{n,s} T} \\ \otimes^n F & \xrightarrow{\sigma_F^n} & \otimes^{n,s} F \end{array} \qquad \text{if } T \in \mathcal{L}(E;F)$$

commutes and simple diagram chasing (or manipulation with polynomials) give the

Proposition.

(1) *If G c E is a closed subspace, then* $c(n, E/G) \le c(n, E)$.

(2) If F c E is complemented subspace with projection P, then

$$c(n,F) \le \|P\|^n c(n,E).$$

(3) If $\mathcal{M}$ is a jiltruting subset in FIN(E) (i.e for each $M, N \in \mathcal{M}$ exists an $L \in \mathcal{M}$ with M UN C L) such that $\cup\mathcal{M}$ is dense in E, then

$$c(n,E) \le \sup\{c(n,M) \quad M \in \mathcal{M}\}.$$

(4) $c(n, \ell_p) = \sup_k c(n, \ell_p^k)$.

(5) *If E is an $\mathcal{L}_{p,\lambda}^g$-space, then* $c(n, E) \le \lambda^n c(n, \ell_p)$.

Reca11 from [DF, §23] that E is an $\mathcal{L}_{p,\lambda}^g$-space if for all M ∈ FIN(E) and ε > 0 there is factorization $I_M^E = S \circ R$ with $\|R : M \longrightarrow \ell_p^k\| \|S : \ell_p^k \longrightarrow E\| \le \lambda + \varepsilon$.

Proof.[Proof of (5):] Take z ∈ $\otimes^n E$ and M ∈ FIN(E) with z ∈ $\otimes^n M$. For a factorization $I_M^E = S \circ R$ through ℓ_p^k one obtains

$$\begin{aligned} \pi_s\left(\sigma_E^n(z); \otimes^{n,s} E\right) &= \pi_s\left([\otimes^{n,s} S] \circ \sigma_{\ell_p^k}^n \circ [\otimes^n R](z); \otimes^{n,s} E\right) \le \\ &\le \|S\|^n \|R\|^n c\left(n, \ell_p^k\right) \pi(z; \otimes^n M) \end{aligned}$$

and the fact that π is finitely generated (i.e. $\pi(\cdot, \otimes^n E) = \inf\{\pi(\cdot; \otimes^n M) \quad M \in \text{FIN(E)}\}$) easily gives the result. □

Since ℓ_p is 1-complemented in $L_{,,}(p)$, properties (5) and (2) imply

Corollary. [[S]]. *If $L_p(\mu)$ is infinite-dimensiona4 then* $c(n, L_p(\mu)) = c(n, \ell_p)$.

Sarantopoulos [S] gives estimates and some precise values for $c(n, \ell_p)$.

5.2. However, in general there is no equality in (3) (take, in the complex case $E = \ell_\infty$ and $M = \ell_1^n$ and note that $c(n, \ell_\infty) < \frac{n^n}{n!}$; see e.g. [D2, 1.3.]). Therefore the polarization constant

is *not locally determined,* but it is somehow "co-1ocal" – at least under the presence of the m.a.p.:

Proposition. *Let E be a normed space with the metric approximation property. If $\mathcal{G}$ is a cojinal subset of* $\mathrm{COFIN}(E)$ *(i.e. for each* $F \in \mathrm{COFIN}(E)$ *exists* $G \in \mathcal{G}$ *with* $G \subset F$*), then*

$$c(n,E) = \sup\{c(n,E/G) \;\; G \in \mathcal{G}\}$$

Proof. Since $\mathcal{G}$ is cofinal in $\mathrm{COFIN}(E)$ 2.5. and the metric mapping property of π_s give

$$\pi_s(z;\otimes^{n,s}E) = \sup\{\pi_s((\otimes^{n,s}Q_G^E)(z);\otimes^{n,s}E/G) \;\; \mathrm{GE} \;\; \mathcal{G}\}.$$

The same statement holds also for the "full" projective norm π on $\otimes^n E$ (see e.g. [DF, 16.2.] for $n = 2$ or use the same type of arguments as in 2.5.). For $z \in \otimes^n E$ and $G \in \mathcal{G}$ one has $[\otimes^{n,s}Q_G^E] \circ \sigma_E^n = \sigma_{E/G}^n \circ [\otimes^n Q_G^E]$ hence

$$\pi_s([\otimes^{n,s}Q_G^E]\sigma_E^n(z);\otimes^{n,s}E/G) \le \|\sigma_{E/G}^n\|\pi(\otimes^n Q_G^E(z);\otimes^n E/G).$$

Taking sup's gives $c(n,E) \le \sup\ldots$ The other inequality was already stated in 5.1.(1). Cl

5.3. The duality results in §4 have also interesting consequences for the polarization constants. The upper arrows of the commutative diagrams (see 1.14.)

$$\begin{array}{ccc} \otimes_{\pi_s}^{n,s}E & \hookrightarrow & (\otimes_{\varepsilon_s}^{n,s}E')' \\ {\scriptstyle\sigma_E^n}\uparrow & & \uparrow{\scriptstyle(\iota_{E'}^n)'} \\ \otimes_{\pi}^{n}E & \hookrightarrow & (\otimes_{\varepsilon}^{n}E')' \end{array} \qquad \begin{array}{ccc} \otimes_{\pi_s}^{n,s}E' & \hookrightarrow & (\otimes_{\varepsilon_s}^{n,s}E)' \\ {\scriptstyle\sigma_{E'}^n}\uparrow & & \uparrow{\scriptstyle(\iota_E^n)'} \\ \otimes_{\pi}^{n}E' & \hookrightarrow & (\otimes_{\varepsilon}^{n}E)' \end{array}$$

are isometries if E or E' has the m.a.p. respectively; the lower are also isometries in these cases (for a proof generalize the approximation lemma [DF], 13.1. and the duality theorem [DF], 15.5. from 2 to *n)* hence (with an obvious notation)

$$\begin{aligned} c(n,E) &= \|\sigma_{E,\pi}^n\| \le \|\iota_{E',\varepsilon}^n\| \qquad \text{if } E \text{ has m.a.p.} \\ c(n,E') &= \|\sigma_{E',\pi}^n\| \le \|\iota_{E,\varepsilon}^n\| \qquad \text{if E'has m.a.p.} \end{aligned}$$

From 3.1.(1) one obtains the

Proposition.

(1) If E has the m.a.p., then

$$\|\iota_{E'}^n : \otimes_{\varepsilon_s}^{n,s}E' \longrightarrow \otimes_{\varepsilon}^{n}E'\| = c(n,E)$$

(2) If E' has the m.a.p., then

$$\|\iota_E^n : \otimes_{\varepsilon_s}^{n,s} E \longrightarrow \otimes_{\varepsilon}^n E\| = c(n, E').$$

The continuous polynomials of finite type are

$$\mathcal{P}_f^n(E) \overset{1}{=} \otimes_{\varepsilon_s}^{n,s} E' \hookrightarrow \mathcal{P}^n(E)$$

hence one obtains from $\otimes_{\varepsilon}^n E' \overset{1}{\hookrightarrow} \mathcal{L}({}^nE)$ that

$$c(n, E) = \sup\{\|\check{q}\|_{\mathcal{L}({}^nE)} \mid q \in \mathcal{P}_f^n(E), \|q\|_{\mathcal{P}^n} \leq 1\}$$

if E has m.a.p. – but this can also be shown directly.

5.4. Another immediate consequence of this proposition (look at $\|\iota_{E',\varepsilon}^n\|$) is the following result from [LR]:

Corollary. *If E'' has m.a.p., then*

$$c(n, E) = c(n, E'').$$

6 Extensions to the bidual and ultraproducts

6.1. Let $E_1, \ldots, E_n$ be normed spaces and $\varphi \in \mathcal{L}(E_1, \ldots, E_n)$ with associated $L_\varphi \in \mathcal{L}(E_1, \ldots, E_{n-1}; E_n')$; the n-linear map $\varphi^{\wedge(n)} \in L(E_1, \ldots, E_{n-1}, E_n'')$ is defined by

$$\begin{aligned}\varphi^{\wedge(n)}(x_1, \ldots, x_{n-1}, x_n'') &:= \langle L_\varphi(x_1, \ldots, x_{n-1}), x_n'' \rangle_{E_n', E_n''} = \\ &= \langle x_j, [L_\varphi(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_{n-1})]' x_n'' \rangle_{E_j, E_j'} = \\ &= \lim \varphi(x_1, \ldots, x_{n-1}, y^\alpha)\end{aligned}$$

if (y") in E $\sigma(E_n'', E_n')$-converges to x_n''. It is obvious that $\|\varphi^{\wedge(n)}\| = \|\varphi\|$ and that $\varphi^{\wedge(n)}$ is the unique separately $\sigma(E_1, E_1')$-$\cdots$-$\sigma(E_{n-1}, E_{n-1}')$-$\sigma(E_n'', E_n')$-continuous $\psi \in L(E_1, \ldots, E_{n-1}, E_n'')$ which extends cp. For other $j \in \{1, \ldots, n-1\}$ the extension $\varphi^{\wedge(j)}$ is defined in the analogous way. If $\lambda \in S_n$ one defines

$$\varphi^{\wedge(\lambda)} := (\ldots((\varphi^{\wedge(\lambda(1))})^{\wedge(\lambda(2))})\ldots)^{\wedge\lambda(n)} \in \mathcal{L}(E_1'', \ldots, E_n'').$$

Clearly, $\|\varphi^{\wedge(\lambda)}\| = \|\varphi\|$. It follows that

$$\varphi^{\wedge(\lambda)}(x_1'', \ldots, x_n'') = \lim_{\alpha_{\lambda(n)} \in A_{\lambda(n)}} \cdots \lim_{\alpha_{\lambda(1)} \in A_{\lambda(1)}} \varphi(x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}) \qquad (*)$$

if the net $(x_j^{\alpha_j})_{\alpha_j \in A_j}$ in $E_j \sigma(E_j'', E_j')$-converges to x_j''. These extensions were first studied by Arens [Ar] for $n = 2$. The special extension

$$\overline{\varphi} := \left(\cdots\left((\varphi^{\wedge(n)})^{\wedge(n-1)}\right)\cdots\right)^{\wedge(1)}$$

is called the *Arens-extension* of cp. It is the unique extension $\psi \in L(E_1'', \ldots, E_n'')$ of $\varphi \in \mathcal{L}(E_1, \ldots, E_n)$ such that for all j = 1, . . ,n, all $x_k \in E_k$ and $x_k'' \in E_k''$

$$\psi(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}'', \ldots, x_n'') \qquad (**)$$

is $\sigma(E_j'', E_j')$-continuous. Clearly, an analogous charaterization holds for $\varphi^{\wedge(\lambda)}$.

Just one simple example: if $T_j \in \mathcal{L}(E_j; F_j)$ and $\varphi \in \mathcal{L}(F_1, \ldots, F_n)$, then it is easy to check (e.g. with $(**)$) that $[\mathrm{cp} \circ (T_1, \ldots, T_n)]^{-} = \overline{\varphi} \circ (T_1'', , T_n'')$.

6.2. The characterizations $(*)$ and $(**)$ easily imply the

Proposition. *Let $\varphi \in \mathcal{L}(E_1, . , E_n)$. Then the Arens-extension $\overline{\varphi}$ is separately weak-*-continuous if and only if $\overline{\varphi} = \varphi^{\wedge(\lambda)}$ for all $\lambda \in S_n$.*

Now recall (e.g. from [DF, 1.6.]) that for $\varphi \in \mathcal{L}(E, F)$ the extension $\overline{\varphi}$ is separately weak-*-continuous if and only if $L_\varphi : E \longrightarrow F'$ is weakly compact. Since every permutation $\lambda \in S_n$ is a product of transpositions one obtains (b) $\curvearrowright$ (a) of the well-known (see e.g. [ACG, sect. 8])

Corollary. *Let E be a normed space and $n \geq 3$. Then the following are equivalent:*

(a) *For every $\varphi \in \mathcal{L}(^nE)$ the Arens-extension $\overline{\varphi}$ is separately weak-*-continuous.*

(b) *The same as* (a) *with $n = 2$*

(c) Every $T \in \mathcal{L}(E; E')$ is weakly compact.

Proof. For the remaining implication (a) $\curvearrowright$ (b) take $\varphi \in \mathcal{L}(^2E)$ and consider $\psi(x_1, , x_n) := \varphi(x_1, x_2)\langle x', x_3\rangle \cdot \langle x', x_n\rangle$ for x' $\neq 0$. □

6.3. Unfortunately, it is *not* true that $\overline{\varphi}$ is symmetric if $\varphi \in \mathcal{L}(^nE)$ is. This follows easily from the following

Observation. *Let E be norrned and $\varphi \in \mathcal{L}_s(^nE)$. Then $\overline{\varphi}$ is symmetric if and only if $\overline{\varphi}$ is separately weak-*-continuous.*

(This is an immediate consequence of $(*)$ in 6.1. and proposition 6.2..) Therefore it is enough to find a symmetric $\varphi \in \mathcal{L}(^2E)$ such that $L_\varphi \in \mathcal{L}(E; E')$ is *not* weakly compact. The typical non-weakly compact operator is the summing operator $\ell_1 \longrightarrow \ell_1' = \ell_\infty$ (see [LP, 8.1.]).

Arens [Ar] considered $\varphi \in \mathcal{L}_s({}^2\ell_1)$ having the representing matrix

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & \cdots \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \ddots \end{pmatrix}$$

Its $2m$-th row $\varphi(e_{2m},\cdot) = L_\varphi(e_{2m}) =: x_m$ $\sigma(\ell_\infty,\ell_1)$-converges for $m \longrightarrow \infty$ to $(1,0,1,0,1,0,\ldots) =: x''$; if $b \in \ell'_\infty$ is a Banach-limit on the odd components, then $\langle b, x_m\rangle = 0$, but $\langle b, x''\rangle = 1$. It follows that $L_\varphi(B_{\ell_1})$ is not $\sigma(\ell_\infty,\ell'_\infty)$-compact and the Arens-extension $\overline{\varphi}$ is not symmetric. Another but related example was given in [ACG].

6.4. Using the same ideas as in 6.2. it is straightforward to verify that the following holds true:

Proposition. [[ACG]]. *For every normed space E the following statements are equivalent:*

(a) *For every $n \geq 2$ and every $\varphi \in \mathcal{L}_s({}^nE)$ the Arens-extension $\overline{\varphi}$ is symmetric (equivalently: separately weak-*-continuous).*

(b) *The same as* (a) *for $n = 2$ only.*

(c) Every symmetric $T \in \mathcal{L}(E;E')$ (i.e. $\langle Tx,y\rangle = \langle Ty,x\rangle$) is weakly compact.

Normed spaces E satisfying one of these equivalent conditions are called *Arens-regular* or symmetrically regular; E is called *regular* if all $T \in \mathcal{L}(E;E')$ are weakly compact. Pisier's factorization theorem [Pi, 4.1.] implies that E is regular if E' has cotype 2 and E has the approximation property, since in this case all operators $E \longrightarrow E'$ even factor through a Hilbert space. The Haagerup-Pisier-Grothendieck inequality (see [H]) implies in the same way that every C*-algebra is regular.

If $E \cong E^2$, then regular = Arens-regular, but Leung [L] showed that the dual of the James space is Arens-regular but not regular; Harmand gave an example of an Arens-regular space, the bidual of which is not (see [AGGM] for these and other results on Arens-regularity).

6.5. If $q \in \mathcal{P}^n(E)$, then $\overline{q}$ defined by $\overline{q}(x'') := \check{\overline{q}}(x'',\ldots,x'')$ extends q to a continuous n-homogeneous polynomial on E''. Since Aron and Berner [AB] used this extension for extending holomorphic functions $E \longrightarrow \mathbb{C}$ to E'' (via their Taylor-expansion) $\overline{q}$ is called nowadays the *Aron-Berner extension* of q. Though $\check{\overline{q}}$ is not symmetric, it is immediate from 6.1.(*) that $\overline{q}(x'') = (\check{q})^{\wedge(\lambda)}(x'',\ldots,x'')$ for all $\lambda \in S_n$ in other words: the Aron-Bemer extension is independent from the order of extending $\check{q} \in \mathcal{L}(E,\ldots,E)$ to the bidual.

Example. *For $q \in \mathcal{P}^n(E)$ and $T \in \mathcal{L}(E;F)$ one has $(q \circ T)^- = \overline{q} \circ T''$. This follows from the example at the end of 6.1..*

Example. *Let μ be a finite signed measure and $q_n(\widetilde{f}) := \int f^n d\mu$ the n-th integrating polynomial on L_∞ (see 3.5.). Then $\overline{q}_n = q_n \circ \kappa'_{L_1}$ where $\kappa_{L_1}: L_1(|\mu|) \hookrightarrow L_1(|\mu|)'' = L_\infty(|\mu|)'$ is the canonica1 embedding.*

Proof. It is enough to show that the extension ψ of $\check{q}_n$ defined by

$$\psi(x''_1,\dots,x''_n) := \int \prod_{j=1}^{n} \kappa'_{L_1}(x''_j)\,d\mu$$

satisfies the continuity-condition $(**)$ at the end of 6.1.. For this take $\widetilde{f}_1,\dots,\widetilde{f}_{j-1} \in L_\infty$ and $x''_{j+1},\dots,x''_n \in L''_\infty$. For $g := \widetilde{f}_1 \cdots \widetilde{f}_{j-1} \cdot \kappa'_{L_1}(x''_{j+1}) \cdots \kappa'_{L_1}(x''_n) \in L_\infty \subset L_1$ and $x'' \in L''_\infty$ one has

$$\psi(\widetilde{f}_1,\dots \widetilde{f}_{j-1}, x'', x''_{j+1},\dots,x''_n) = \langle g; \kappa'_{L_1}(x'')\rangle_{L_1,L_\infty} = \langle \kappa_{L_1}(g), x''\rangle_{L'_\infty,L''_\infty}$$

which proves the desired continuity. □

An obvious modification of this proof shows that $\check{\overline{q}}$ is even separately weak-$*$-continuous.

6.6. If $P(E) := \bigoplus_{n=0}^{\infty} \mathcal{P}^n(E)$ (with $\mathcal{P}^0(E) := \mathbb{K}$) is the space of all polynomials, then for $q = c + x' + \sum_{n=2}^{m} q_n \in \mathcal{P}(E)$

$$\overline{q} := c + x' + \sum_{n=2}^{m} \overline{q}_n$$

defines a linear extension map $\mathcal{P}(E) \longrightarrow \mathcal{P}(E'')$ which, by 6.1.$(*)$, is multiplicative.

6.7. While it is obvious from the definition that $||\varphi|| = ||\overline{\varphi}||$ it is not at all trivial that $||q||_{\mathcal{P}^n(E)} = ||\overline{q}||_{\mathcal{P}^n(E'')}$. This was proved by Davie and Gamelin; the key for the proof is the following approximation result (see [DG] for a proof):

Theorem. *Let E be normed, $S \subset E$ bounded and $x''_0 \in \overline{S}^{\sigma(E'',E')}$. Then there is a net (x_α) in conv (S) such that*

$$q(x_\alpha) \longrightarrow \overline{q}(x''_0)$$

for all polynomials $q \in \mathcal{P}(E)$.

Applying this to $S = B_E$ gives the

Corollary. *For every $q \in \mathcal{P}^n(E)$ one has $||q||_{\mathcal{P}^n(E)} = ||\overline{q}||_{\mathcal{P}^n(E'')}$.*

Corollary. *The natura1 embedding $\otimes^{n,s}\kappa_E : \otimes^{n,s}_{\pi_s} E \longrightarrow \otimes^{n,s}_{\pi_s} E''$ is an isometty*

Proof. Cleary $\|\otimes^{n,s} \kappa_E\| \le 1$ and $\langle q, z\rangle = \langle \overline{q}, \otimes^{n,s}\kappa_E(z)\rangle$ gives the remaining inequality. □

There is a natura1 duality bracket between $\mathcal{P}^n(E)$ and $\otimes^{n,s}E''$:

$$\langle q, z''\rangle := \langle \overline{q}, z''\rangle_{\mathcal{P}^n(E''),\otimes^{n,s}E''}$$

the restriction of which to $\otimes^{n,s}E$ gives the duality $\mathcal{P}^n(E) = \left(\otimes^{n,s}_{\pi_s} E\right)'$; hence the bipolar theorem implies the

Corollary. *The unit ball $B_{\otimes^{n,s}_{\pi_s}E}$ is $\sigma(\otimes^{n,s}E'', P(E))$-dense in $B_{\otimes^{n,s}_{\pi_s}E''}$.*

6.8. It is well-known (see e.g. [DF, 6.7.] for $n = 2$, the extension to $n > 2$ is easy) that $\overline{\varphi} \in [\otimes_\varepsilon(E_1'', .\,, E_n'')]$ if $\varphi \in [\otimes_\varepsilon(E_1, \ldots, E_n)]'$ and the "integral" norm is the same. This and the fact that ε_s respect subspaces (3.2.(4)) implies that $q \in \mathcal{P}^n(E)$ is integra1 if and only if $\overline{q}$ is.

Actually also the norm remains unchanged:

Proposition. [Carando-Zalduendo [CZ]]. *Let $q \in P(E)$. Then q is integral if and only if $\overline{q}$ is. Moreover, $\|q\|_{\text{int}} = \|\overline{q}\|_{\text{int}}$ holds in this case.*

Proof. To see the norm equality, factor $q = q_n \circ T$ according to Corollary 3.5. with $\|q\|_{\text{int}} = \|q_n\|\,\|T\|^n$. The Examples 1 and 2 in 6.5. give $\overline{q} = q_n \circ \kappa'_{L_1} \circ T''$ hence, again by 3.5.

$$\|\overline{q}\|_{\text{int}} \leq \|q_n\|\,\|\kappa'_{L_1}\|^n\,\|T''\|^n = \|q\|_{\text{int}}.$$

The other inequality is obvious from 3.2.(4). ✍

6.9. The Arens- and Aron-Bemer extensions can also be obtained using ultrapowers. For this, take for a normed space E the index set $I := \text{FIN}(E'') \times \text{FIN}(E') \times]0,1]$ and choose (with the strong principle of loca1 reflexivity) for every $\iota = (M, N, \varepsilon)$ an operator $T_\iota \in \mathcal{L}(M; E)$ with $T_\iota x = x$ for all $x \in M \cap E$, having $\|T_\iota\| \leq 1 + \varepsilon$ and satisfying $\langle T_\iota x'', x' \rangle = \langle x'', x' \rangle$ for all $(x'', x') \in M \times N$; for $x'' \in E''$ define $f_\iota(x'') := T_\iota x''$ if $x'' \in M$ and $:= 0$ otherwise. Take an ultrafilter $\mathfrak{U}$ on I which is finer than the order filter; $\mathfrak{U}$ is usually called a *local ultrafilter* of E.

For the ultrapower $(E)_\mathfrak{U}$ define the following two natura1 mappings:

$$\begin{array}{ccccccc} J : E'' & \longrightarrow & (E)_\mathfrak{U} & \quad\text{and}\quad & Q : (E)_\mathfrak{U} & \longrightarrow & E \\ x'' & \rightsquigarrow & (f_\iota(x))_\mathfrak{U} & & (x_\iota)_\mathfrak{U} & \rightsquigarrow & \lim_{\iota,\mathfrak{U}} x_\iota \end{array}$$

($\sigma(E'', E')$-limit). It is easy to see that the (linear) isometry J extends the embedding $E \ni x \rightsquigarrow (x)_\mathfrak{U} \in (E)_\mathfrak{U}$ and $QJ = \text{id}_{E''}$; since $\|Q\| \leq 1$ it follows that JQ is a norm-1-projection of $(E)_\mathfrak{U}$ onto $\text{im}J$.

If $\mathfrak{U}_j$ is a loca1 ultrafilter of E_j and $\varphi \in \mathcal{L}(E_1, \ldots, E_n)$, then

$$\varphi_{(\mathfrak{U}_j)}\left((x^1_{\iota_1})_{\mathfrak{U}_1}, \ldots, (x^n_{\iota_n})_{\mathfrak{U}_n}\right) := \lim_{\iota_1,\mathfrak{U}_1} \ldots \lim_{\iota_n,\mathfrak{U}_n} \varphi(x^1_{\iota_1}, \ldots, x^n_{\iota_n})$$

is in $\mathcal{L}\left((E_1)_{\mathfrak{U}_1}, \ldots, (E_n)_{\mathfrak{U}_n}\right)$ with norm $\|\varphi\|$. The special continuity of the Arens-extension (6.1.(∗)) gives that

$$\varphi_{(\mathfrak{U}_j)} = \overline{\varphi} \circ (Q_1, \ldots, Q_n) \quad \textit{and} \quad \overline{\varphi} = \varphi_{(\mathfrak{U}_j)} \circ (J_1, \ldots, J_n) \qquad (*)$$

in particular

Proposition. [[LR]]. *If $q \in \mathcal{P}^n(E)$ and $\mathfrak{U}$ a local ultrafilter of E, then*

$$\overline{q}(x'') = \lim_{\iota_1,\mathfrak{U}} \dots \lim_{\iota_n,\mathfrak{U}} \check{q}\big(f_{\iota_1}(x''),\dots,f_{\iota_n}(x'')\big).$$

If $q \in \mathcal{P}^n(E)$ and if $q_{\mathfrak{U}} := (\check{q})^{\triangle}_{\mathfrak{U}} \in \mathcal{P}^n((E)_{\mathfrak{U}})$ is the polynomial associated to $(\check{q})_{\mathfrak{U}}$, then the proposition, $(*)$ and Corollary 1 in 6.7. imply that $\|q_{\mathfrak{U}}\| = \|q\|$.

6.10. For ultrapowers, however, it seems more natura1 not to use an iterated limit (see [DT], [LR]): for a local ultrafilter $\mathfrak{U}$ of E and $\varphi \in \mathcal{L}({}^nE)$ define

$$\widetilde{\varphi}_{\mathfrak{U}}\big((x_\iota^1)_{\mathfrak{U}},\dots,(x_\iota^n)_{\mathfrak{U}}\big) := \lim_{\iota,\mathfrak{U}} \varphi(x_\iota^1,\dots,x_\iota^n)$$
$$\widetilde{q}_{\mathfrak{U}}\big((x_\iota)_{\mathfrak{U}}\big) := \lim_{\iota,\mathfrak{U}} q(x_\iota)$$

Obviously, $\|\widetilde{\varphi}_{\mathfrak{U}}\| = \|\varphi\|$ and $\|\widetilde{q}_{\mathfrak{U}}\| = \|q\|$.

Observation. [[LR]]. *$q_{\mathfrak{U}} \neq \widetilde{q}_{\mathfrak{U}}$ on $(E)_{\mathfrak{U}}$ in general*

Proof. Take Arens' example from 6.3. and $U_n \in \mathfrak{U}$ with $U_n \supset U_{n+1}$ and $\cap U_n = \emptyset$. Fory, := 0 if $\iota \notin U$, and := $e_{2n} - e_{2n+1}$ if $\iota \in U_n \backslash U_{n+1}$ one gets

$$\widetilde{q}_{\mathfrak{U}}((y_\iota))_{\mathfrak{U}} = \lim_{\iota_1,\mathfrak{U}} \lim_{\iota_2,\mathfrak{U}} \varphi(y_{\iota_1},y_{\iota_2}) = 0 \neq -1 = \lim_{\iota,\mathfrak{U}} \varphi(y_\iota,y_\iota).$$

□

It is likely that also $\overline{q}^{\mathfrak{U}} := \widetilde{q}_{\mathfrak{U}} \circ J \neq \overline{q} \in \mathcal{P}^n(E'')$ in general. In any case it is clear that this "uniterated Aron-Berner" extension $\overline{q}^{\mathfrak{U}}(x'') = \lim_{\iota,\mathfrak{U}} q(f_\iota(x''))$ is also a natura1 and useful extension of $q \in \mathcal{P}^n(E)$. Note that $\|\overline{q}^{\mathfrak{U}}\| = \|q\|$ is obvious (but $\|\overline{q}\| = \|q\|$ was rather involved).

Using the ultrastability of maximal operator ideals (due to Kürsten, see [Ku] and [He]) one can show that φ is integra1 if and only if $\widetilde{\varphi}_{\mathfrak{U}}$ is – with the same norm; it follows that q is integra1 if and only if $\widetilde{q}_{\mathfrak{U}}$ (and hence also $\overline{q}^{\mathfrak{U}}$) is integral. In [FH] it will be shown in the more general context of s-tensor norms that even $\|\overline{q}^{\mathfrak{U}}\|_{\mathrm{int}} = \|q\|_{\mathrm{int}}$ holds.

6.11. If one has fixed an extension procedure $\mathcal{L}^n(E) \ni \varphi \rightsquigarrow \overset{\bullet}{\varphi} \in \mathcal{L}({}^nE'')$ (either the Arens-extension or the "uniterated" ultrapower extension from 6.10.), then every $\varphi \in \mathcal{L}({}^nE;G)$ (where G is normed as well) has an extension $\overset{\bullet}{\varphi} \in \mathcal{L}({}^nE'';\mathrm{G}")$ defined as follows

$$\langle \overset{\bullet}{\varphi}(x_1'',\dots,x_n''),y'\rangle_{G'',G'} := [y' \circ \varphi]^{\bullet}(x_1'',\dots,x_n'')$$

which, clearly, has a characterization as in $6.1.(*)$ but with the $\sigma(G'',\mathrm{G}')$-topology on G". It follows that there is also an extension $\mathcal{P}^n(E;G) \ni q \rightsquigarrow \overset{\bullet}{q} \in \mathcal{P}^n(E'';G'')$. Reca11 that Arens used his extension to extend the multiplication on a Banach algebra to the bidual.

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