Note di Matematica Vol. 17, 121-142 (1997)

PAIRS OF FINITE-TYPE POWER SERIES SPACES

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Abstract. *Let $a = (a_i)$, $a_i \to \infty$, $\lambda = (\lambda_i)$ be sequences of positive numbers. We study the problem on isomorphic classification of pairs*

$$F = (K(\exp(-\frac{1}{p}a_i)),\ K(\exp(-\frac{1}{p}a_i + \lambda_i))).$$

For this purpose we introduce the sequence of so-called m-rectangle characteristics μ_m^F. It is shown that the system of all these characteristics is a complete quasidiagonal invariant on the class of pairs of finite-type power series spaces. Using some new linear topological invariants (compound invariants) we prove that m-rectangle characteristics are invariant on the class of such pairs. Some applications to pairs of spaces of analytic functions are considered.

1 Introduction

Pairs of spaces of analytic functions $(A(D_0), A(D_1))$ ($D_0 \subset D_1$ are open sets in $\mathbb{C}^n$), having a common basis, were a subject of studying by many authors (see, e.g., [29, 42, 27, 28, 19, 44, 37, 43, 51, 40, 39, 45, 48, 1, 2, 52, 3]). These investigations are connected in a natural way with the question about isomorphism of such pairs and, more generally, with the isomorphic classification problem for pairs of imbedded Köthe spaces with the common canonical basis. The case of coherently regular Köthe spaces was completely studied by Dragilev ([20, 21, 22]) with the use of so-called *simultaneous diametral dimension* (introduced in [44]), which is a modification of the classical diametral dimension (for more details see below, item 2.4, where these results are discussed in some modified but equivalent form). The isomorphism problem for pairs of Köthe spaces without the assumption about the coherent regularity proved to be quite complicated: pairs $(A(D_0), A(D_1))$, having a common basis and satisfying the condition $\partial D_0 \cap \partial D_1 \neq \emptyset$, represent non-trivial examples of such kind.

We consider here the isomorphic classification on the special class of pairs of Köthe spaces with some new invariants, which allowed to reveal very delicate distinctions in the topological structure of pairs without coherent regularity (including the above-mentioned intricated pairs of spaces of analytic functions).

The main tool in our study of this problem are so-called m-rectangle characteristics of pairs, constructed by analogy with other m-rectangle characteristics, which were considered first for studying of families of Banach spaces ([5, 6, 7, 8, 12, 13, 14, 15]). Characteristics of such type proved also to be a good instrument for isomorphic classification of many non-trivial classes of locally convex spaces ([16, 10, 11, 9, 18, 17, 31, 3]).

The proof, that m-rectangle characteristics are linear topological invariants on the class of considered pairs, is based on so-called compound invariants, which were introduced by Zahariuta [46, 47] and find their development in numerous studies of isomorphic classification of locally convex spaces and families of Banach spaces (see the above-mentioned citations). The information about connections with the preceding investigations of linear topological invariants (Kolmogorov, Pelczynski, Bessaga, Rolevicz, Tikhomirov, Mityagin, Dragilev, Zahariuta, Kondakov, Dubinsky, Vogt, Djakov, Terzioğlu et al) can be found, for example, in [48, 10, 3].

2 Preliminaries

2.1 Spaces

Let X be a locally convex space with an absolute basis $e = \{e_i\}_{i\in I}$, where I is a countable set. Any sequence of positive numbers $a = (a_i)_{i\in I}$ generates the weighted ℓ_1-ball in X, associated with the basis e as follows:

$$B^e(a) = B^e((a_i)) := \{x = \sum_{i=1}^{\infty} \xi_i e_i \in X : \sum_{i\in I} |\xi_i| a_i \leq 1\}. \quad (1)$$

By $K(a_{i,p})$ we denote Köthe space, defined by Köthe matrix $(a_{i,p})_{i,p\in\mathbb{N}}$; $e = \{e_i\}_{i\in\mathbb{N}}$, is its canonical basis. In particular, spaces $E_\alpha(a) := K(\exp(\lambda_p a_i))$, $\lambda_p \nearrow \alpha$, $a = (a_i)$, $a_i > 0$, $\alpha < +\infty$, are called *finite-type power series spaces*.

$A(D)$ is the locally convex space of all analytic functions in the domain $D \subset \mathbb{C}^n$ with the topology of uniform convergence on compact subsets of D.

Denote by Q the set of all non-decreasing natural numbers sequences with infinite sets of values. It is well known that for arbitrary Köthe space the following fact is true.

Proposition 1 *Let M be a set in $X = K(a_{i,p})$. Then M is bounded if and only if there exists $q = (q_i) \in Q$ and a positive constant C such that*

$$M \subset C B^e((a_{i,q_i})).$$

2.2 Pairs

We consider pairs (X,Y) of locally convex spaces X and Y with a linear continuous injection $Y \hookrightarrow X$. Two pairs (X,Y) and $(\tilde{X},\tilde{Y})$ are called *isomorphic* (we write $(X,Y) \simeq (\tilde{X},\tilde{Y})$) if there exists an isomorphism $T : X \to \tilde{X}$ such that its restriction on Y is also an isomorphism from Y onto $\tilde{Y}$.

A system $\{x_i\}_{i\in\mathbb{N}} \subset Y$ is said to be *absolute basis* for a pair (X,Y) if this system constitutes an absolute basis for both of the spaces X and Y.

Suppose (X,Y) and $(\tilde{X},\tilde{Y})$ are two pairs with absolute bases $\{x_i\}$ and $\{\tilde{x}_i\}$, respectively. An isomorphism $T : (X,Y) \to (\tilde{X},\tilde{Y})$ is said to be *quasidiagonal* if there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ and a sequence (t_i) such that $Tx_i = t_i \tilde{x}_{\sigma(i)}$, $i \in \mathbb{N}$. In this case the pairs (X,Y) and $(\tilde{X},\tilde{Y})$ are called *quasidiagonally isomorphic* (with respect to those fixed bases) and we write shortly $(X,Y) \overset{qd}{\simeq} (\tilde{X},\tilde{Y})$; in the particular case $t_i \equiv 1$ for all $i \in \mathbb{N}$, the operator T is said to be *permutational*, the pairs (X,Y) and $(\tilde{X},\tilde{Y})$ are called *permutationally isomorphic* and we write $(X,Y) \overset{p}{\simeq} (\tilde{X},\tilde{Y})$. And when (t_i) is an arbitrary sequence but $\sigma(i) \equiv i$ for all $i \in \mathbb{N}$ the operator T is said to be *diagonal*.

We use the following notation

$$F[\lambda, a] := (K(\exp(-\frac{1}{p}a_i)), K(\exp(-\frac{1}{p}a_i + \lambda_i))), \quad (2)$$

where $a = (a_i)$, $a_i \to \infty$, $\lambda = (\lambda_i)$ are sequences of positive numbers.

2.3 Pairs invariants

Let $\mathcal{E}$ be a class of pairs of locally convex spaces and Γ a set with an equivalence relation $\sim$. We say that $\gamma : \mathcal{E} \to \Gamma$ is a linear topological invariant if $(X,Y) \simeq (\tilde{X},\tilde{Y}) \Longrightarrow \gamma(X,Y) \sim \gamma(\tilde{X},\tilde{Y})$, (X,Y), $(\tilde{X},\tilde{Y}) \in \mathcal{E}$.

The invariants to be studied here are based on the following well-known characteristic of a couple of absolutely convex sets. Let (X,Y) be a pair of linear spaces, U,V absolutely convex subsets in Y. Consider

$$\beta(V,U) := \sup\{\dim L : U \cap L \subset V\}, \quad (3)$$

where L runs through the set of all finite-dimensional subspaces of $Y_V = \overline{\operatorname{span}} V$. This characteristic relates with Bernstein diameters $b_n(V,U)$ [41] in the following way: $\beta(V,U) = |\{n : b_n(V,U) \geq 1\}|$. The following properties follow immediately from the definition (3):

$$\begin{aligned} &V_1 \subset V,\ U \subset U_1 \text{ implies } \beta(V_1,U_1) \leq \beta(V,U); \\ &\beta(\alpha V,U) = \beta(V, \tfrac{1}{\alpha}U),\ \alpha > 0. \end{aligned} \quad (4)$$

For weighted balls (1) the characteristic (3) admits a simple computation.

Proposition 2 *(see, e.g., [34]). For a couple of weights a, b we have*

$$\beta(B^e(b), B^e(a)) = |\{i : b_i \leq a_i\}|.$$

2.4 Classical invariants analogue

Suppose (X,Y) is a pair of F-spaces with $Y \hookrightarrow X$; U_p, $p \in \mathbb{N}$, is a sequence of absolutely convex neighborhoods of 0 in X, which defines the topology in X, i.e. the system $\{t\,U_p : t > 0,\ p \in \mathbb{N}\}$ forms a base of neighborhoods of 0 in X; and V_p, $p \in \mathbb{N}$, is a similar sequence for Y. As a natural analogue of the classical diametral dimension on the class $\mathcal{F}$ of all pairs of F-spaces, we can take the following family of functions ($t > 0$):

$$\beta_{(X,Y)} := (\beta(t\,V_q, U_p))_{p,q \in \mathbb{N}}. \quad (5)$$

Let $(\tilde{X},\tilde{Y})$ be another pair from $\mathcal{F}$ and $\tilde{U}_p$, $\tilde{V}_p$, $p \in \mathbb{N}$, a corresponding systems of neighborhoods, defining the topologies in $\tilde{X}$, $\tilde{Y}$ respectively. Then define the equivalence $\beta_{(X,Y)} \sim \beta_{(\tilde{X},\tilde{Y})}$ by the following relation

$$\forall p \exists p' \forall q'\ \exists q \exists c : \beta(t\,V_q, U_p) \leq \beta(ct\,\tilde{V}_{q'}, \tilde{U}_{p'}),\ \beta(t\,\tilde{V}_q, \tilde{U}_p) \leq \beta(ct\,V_{q'}, U_{p'}). \quad (6)$$

This invariant is introduced by analogy with the corresponding single space invariant β_X, which can be considered as a particular case of (5) $\beta_X := \beta_{(X,X)}$, with $V_p = U_p$, $\tilde{V}_q = \tilde{U}_p$, $p \in \mathbb{N}$ in the above definitions.

Proposition 3 *(cf. [44, 21, 22]) Let* $(X,Y) \simeq (\tilde{X},\tilde{Y})$*. Then* $\beta_{(X,Y)} \sim \beta_{(\tilde{X},\tilde{Y})}$.

We say that a pair (X,Y) of Köthe spaces $X = K(a_{i,p})$, $Y = K(b_{i,p})$ is *coherently regular* (shortly, $(X,Y) \in CR$) if $a_{i,p}/a_{i,q} \downarrow 0$, $b_{i,p}/b_{i,q} \downarrow 0$, $p < q$.

With the use of the invariant $\beta_{(X,Y)}$, (together with the invarint β_X for single spaces), the complete isomorphic classification on the class CR can be established ([23, 24, 26]).

Theorem 4 *Let* $(X,Y), (\tilde{X},\tilde{Y}) \in CR$*. Then the following statements are equivalent:*

(i) $(X,Y) \simeq (\tilde{X},\tilde{Y})$;

(ii) $(X,Y) \overset{qd}{\simeq} (\tilde{X},\tilde{Y})$;

(iii) $\beta_X \sim \beta_{\tilde{X}}, \quad \beta_{(X,Y)} \sim \beta_{(\tilde{X},\tilde{Y})}$.

The following particular case is important because of applications to pairs of spaces of analytic functions (see section 3).

Corollary 5 *Let all the sequences* $a = (a_i)$, $\lambda = (\lambda_i)$, $\tilde{a} = (\tilde{a}_i)$, $\tilde{\lambda} = (\tilde{\lambda}_i)$ *be positive and tend monotonically to* ∞*. Then the following statements are equivalent:*

(i) $F[\lambda,a] \simeq F[\tilde{\lambda},\tilde{a}]$;

(ii) $F[\lambda,a] \overset{qd}{\simeq} F[\tilde{\lambda},\tilde{a}]$;

(iii) $\exists c : \frac{1}{c} a_i \le \tilde{a}_i \le c\, a_i$ and $\frac{\lambda_i - \tilde{\lambda}_i}{a_i} \to 0$.

Corollary 6 *If in the previous corollary* $\lambda_i \sim \sigma a_i$ *then the condition (iii) can be changed by the following*

$$\exists c : \frac{1}{c} a_i \le \tilde{a}_i \le c\, a_i \text{ and } \tilde{\lambda}_i \sim \sigma a_i.$$

2.5 Some geometrical facts

In the construction of compound invariants we shall use the following geometrical facts.

For a couple $A_\nu = B^e(a^{(\nu)})$, $\nu = 0,1$, we consider the following one-parameter family of weighted balls $(A_0)^{1-\alpha}(A_1)^\alpha := B^e(a^{(\alpha)})$, where

$$a^{(\alpha)} := ((a_i^{(0)})^{1-\alpha}(a_i^{(1)})^\alpha)_{i\in\mathbb{N}}, \quad \alpha \in \mathbf{R}.$$

The following elementary fact is well-known (see, for example, [4, 32, 38]).

Proposition 7 *Let* e *and* f *be absolute bases of a locally convex space* X *and* $A_\nu = B^e(a^{(\nu)})$, $\tilde{A}_\nu = B^f(\tilde{a}^{(\nu)})$, $\nu = 0,1$*. Then* $A_\nu \subset \tilde{A}_\nu$, $\nu = 0,1$ *implies*

$$(A_0)^{1-\alpha}(A_1)^\alpha \subset (\tilde{A}_0)^{1-\alpha}(\tilde{A}_1)^\alpha, \quad \alpha \in (0,1).$$

Proposition 8 *Let e be an absolute basis of a locally convex space X, $a^{(j)} = (a_i^{(j)})$, $j = 1,2,\dots,r$, sequences of positive numbers and $c = (c_i)$, $d = (d_i)$ sequences, defined by the following formulae: $c_i = \max\{a_i^{(j)} : j = 1,2,\dots,r\}$, $d_i = \min\{a_i^{(j)} : j = 1,2,\dots,r\}$, $i \in \mathbb{N}$. Then the following relations hold:*

$$B^e(c) \subset \bigcap_{j=1}^{r} B^e(a^{(j)}) \subset rB^e(c), \quad B^e(d) = conv(\bigcup_{j=1}^{r} B^e(a^{(j)})),$$

where $conv(M)$ means the convex hull of a set M.

For $p \in \mathbb{N}$ and $(p_i) \in Q$ we set the notation $[p, p_i] := \max\{p, p_i\}$, $i \in \mathbb{N}$.

Lemma 9 *Let $X = K(a_{i,p})$, $\tilde{X} = K(\tilde{a}_{i,p})$, and e, $\tilde{e}$ be canonical bases in X, $\tilde{X}$, respectively. If $T : \tilde{X} \to X$ is an isomorphism then $\forall (q_i) \in Q\ \exists (r_i) \in Q\ \forall r \in \mathbb{N}\ \exists q \in \mathbb{N},\ C :$*

$$T(B^{\tilde{e}}((\tilde{a}_{i,[q,q_i]}))) \subset C B^e((a_{i,[r,r_i]})). \tag{7}$$

Proof.[Proof] Since $\{\varepsilon T(B^{\tilde{e}}((\tilde{a}_{i,p}))),\ p \in \mathbb{N},\ \varepsilon > 0\}$ is a base of neighborhoods of the origin in X, and the image of any bounded set is bounded, we get, taking into account Proposition 1, that $\forall (q_i) \in Q\ \exists (r_i) \in Q$ and $\forall r \in \mathbb{N}\ \exists q \in \mathbb{N},\ M > 0$ such that

$$T(B^{\tilde{e}}((\tilde{a}_{i,q_i}))) \subset M B^e((a_{i,r_i})), \quad T(B^{\tilde{e}}((\tilde{a}_{i,q}))) \subset M B^e((a_{i,r})).$$

Therefore $T(B^{\tilde{e}}((\tilde{a}_{i,q_i})) \bigcap B^{\tilde{e}}((\tilde{a}_{i,q}))) \subset M B^e((a_{i,r_i})) \bigcap B^e((a_{i,r}))$. Then, using Proposition 8, we get (7) with $C = 2M$.

3 Pairs of spaces of analytic functions

3.1 One variable case

Consider some results about the isomorphism of pairs of spaces of analytic functions of one variable.

First define *Green capacity* $C(D_0, D_1)$ for an arbitrary pair of domains $D_0 \subset D_1 \subset \mathbb{C}$ in the following way. Consider two sequences of bounded domains with smooth boundaries $(D_{\nu,s})_{s=1}^{\infty}$, such that $\overline{D_{\nu,s}} \subset D_{\nu,s+1}$, $D_\nu = \bigcup_{s=1}^{\infty} D_{\nu,s}$, $\nu = 0,1$, and set

$$C(D_0, D_1) := \lim_{s\to\infty} \lim_{r\to\infty} C(\overline{D_{0,s}}, D_{1,r}), \tag{8}$$

where $C(K, D)$ is the Green capacity of a compact set K with respect to a domain D ([33]).

The following statement shows that this characteristic is an invariant on the class of pairs $(A(D_0), A(D_1))$ (the particular case, when the both domains are one-connected and D_0 is relatively compact in D_1, was considered in [44]).

Theorem 10 *Let D_i, $\tilde{D}_i$, $i = 0,1$, be domains in $\mathbb{C}$ such that $D_0 \subset D_1$. Then*

$$(A(D_0), A(D_1)) \simeq (A(\tilde{D}_0), A(\tilde{D}_1)) \Longrightarrow C(D_0, D_1) = C(\tilde{D}_0, \tilde{D}_1). \tag{9}$$

This theorem can be obtained from Proposition 3 by calculating of the invariant $\beta_{(X,Y)}$ through the capacity. In a calculation of this invariant, it is useful the following statement, which is a very particular case of the well-known fact (see, e.g., [44, 37, 51]).

Lemma 11 *Let D_0, D_1 be bounded domains in $\mathbb{C}$ with smooth boundaries, $\overline{D_0} \subset D_1$; H_0, H_1 any Hilbert spaces, complied with the following linear continuous embeddings:*

$$A(\overline{D_1}) \hookrightarrow H_1 \hookrightarrow A(D_1) \hookrightarrow A(\overline{D_0}) \hookrightarrow H_0 \hookrightarrow A(D_0);$$

$e = \{e_k\}$ the common orthogonal basis for the spaces H_0, H_1 such that $\|e_k\|_{H_0} = 1$, $\|e_k\|_{H_1} = \mu_k \uparrow \infty$. Then

$$\lim_{k\to\infty} \frac{\ln \mu_k}{k} = \frac{1}{C(\overline{D_0}, D_1)}.$$

Proof.[Proof of Theorem 10] Let Hilbert spaces $H_\nu^{(s)}$, the basis $e^{(s)} = \{e_k^{(s)}\}$, and the numbers $\mu_k^{(s)}$ be as in Lemma 11 with respect to the pair of domains $D_{0,s}$, $D_{1,s}$, considered in the definition (8), $s \in \mathbb{N}$. Then the unit balls $U_s := B_{H_0^{(s)}}$, $V_s := B_{H_1^{(s)}}$ define the topologies in the spaces $X = A(D_0)$ and $Y = A(D_1)$, respectively. Therefore, by Proposition 2 and Lemma 11,

$$\beta(\exp t V_q, U_p) = |\{k : \ln \mu_k^{(s)} \le t\}| \sim C(\overline{D_{0,p}}, D_{1,q})\, t.$$

Hence, due to Proposition 3 and (8), the conclusion (9) is proved. □

Remark 12 *1) Suppose additionally that D_1 is a regular domain and the boundary ∂D_0 consists of a finite set of closed analytic curves. Then,due Corollary 6, the opposite inclusion $\Longleftarrow$ in Theorem (10) holds, too.*

2) On the subclass of pairs $(A(D_0), A(D_1))$, such that

$$\partial D_0 \cap \partial D_1 \neq \emptyset, \tag{10}$$

the invariant (8) brings no information, since $C(D_0, D_1) = \infty$. The problem on the isomorphic classification within this class remains open.

3.2 Several variables case

Suppose D is a bounded complete n-circular domain (Reinhardt domain) in $\mathbb{C}^n$ and consider its characteristic function

$$h_D(\theta) := \sup\{\theta_1 \ln|z_1| + \theta_2 \ln|z_2| + \ldots \theta_n \ln|z_n| : z = (z_k) \in D\},$$

defined on the simplex $\Sigma = \{\theta = (\theta_k) \in \mathbf{R}_+^n : \theta_1 + \ldots + \theta_n = 1\}$.

Let D_0 and D_1 be a pair of Reinhardt domains such that $D_0 \subset D_1$. The number $M(D_0, D_1)$, defined by the following relation

$$M(D_0, D_1) = \frac{1}{n\sqrt{n}} \int_\Sigma \frac{d\sigma(\theta)}{(h_{D_1}(\theta) - h_{D_0}(\theta))^n},$$

where $d\sigma(\theta)$ is the Lebesgue measure on Σ, may be considered as a natural capacity-like characteristic of the pair (D_0, D_1).

The following theorem shows that this characteristic is invariant under isomorphisms of pairs.

Theorem 13 *Let D_i, $\tilde{D}_i$, $i = 0, 1$, be Reinhardt domains such that D_0 ($\tilde{D}_0$) is relatively compact in D_1 (in $\tilde{D}_1$ respectively). Then*

$$(A(D_0), A(D_1)) \simeq (A(\tilde{D}_0), A(\tilde{D}_1)) \tag{11}$$

if and only if

$$M(D_0, D_1) = M(\tilde{D}_0, \tilde{D}_1). \tag{12}$$

Proof. The system $e = \{e_k\}$ of all monomials $e_k(z) = z^k = z_1^{k_1} \dots z_n^{k_n}, k = (k_1, \dots, k_n) \in \mathbb{N}_0{}^n$ is an absolute basis for the pair $(A(D_0), A(D_1))$. We use the usual notation: $|k| := k_1 + \dots + k_n$, $\theta(k) := \frac{k}{|k|}$. For the pair

$$(X, Y) = (A(D_0), A(D_1))$$

consider the systems of neighborhoods, defining their topologies, like in item 2.5: $U_p = B^e(a_p^{(0)}), V_p = B^e(a_p^{(1)})$, with

$$a_p^{(\nu)} := (a_{k,p}^{(\nu)}), \quad a_{k,p}^{(\nu)} := \exp\left(\left(-\frac{1}{p} + h_{D_\nu}\left(\frac{k}{|k|}\right)\right)|k|\right),\ k \in \mathbb{N}_0{}^n,\ \nu = 0, 1.$$

It is quite easy to check that

$$\beta((\exp t) V_q, U_p) = |\{k \in \mathbb{N}_0^k : |k| \le t\, \varphi_{p,q}(\theta(k))\}| = t^n\, (\mathrm{Vol}(W(p,q) + o(1))),$$

as $t \to \infty$, where

$$\varphi_{p,q}(\theta) = \left(\frac{q-p}{pq} + h_{D_1}(\theta) - h_{D_0}(\theta)\right)^{-1},\ \theta \in \Sigma,$$

and

$$W(p,q) = \{x = (x_i) \in \mathbb{R}^n{}_+ : \rho(x) \le \varphi_{p,q}(\theta(x))\}$$

with $\rho(x) = x_1 + \dots + x_n$, $\theta(x) = \frac{x}{\rho(x)}$.

Therefore after some computations we get

$$\lim_{q\to\infty} \lim_{p\to\infty} \lim_{t\to\infty} t^{-n} \beta(\exp t\, V_q, U_p) = M(D_0, D_1).$$

Since the isomorphism (11) implies the relationship (6), after the analogous computations for the second pair, we get the equality (12). □

Remark 14 *1) Let $\mathcal{E}$ be the class of all pairs $(A(D_0), A(D_1))$ such that $D_0 \subset D_1$ are Reinhardt domains, satisfying the condition (10). Since $M(D_0, D_1) = \infty$ in this case, the invariant (12) gives no answer to the following question: are there non-isomorphic pairs in the class $\mathcal{E}$?*

With the use of invariants, much stronger than that considered above (so-called, compound invariants), it will be shown below (section 7) that, in fact, there exists a continuum non-isomorphic pairs in this class.

4 m-rectangle characteristics

We restrict ourselves by considering the class of pairs (2). Without loss of generality it can be assumed that

$$a_i > 1,\ \lambda_i > 1,\ i \in \mathbb{N}. \tag{13}$$

This class includes the pairs from section 3 (if they have a basis). Obviously, $X = E_0(a)$ and $Y = T(E_0(a))$ is an image of X under the diagonal map $T : Te_i = \exp(\lambda_i)e_i,\ i \in \mathbb{N}$. Here $\{e_i\}$ is the canonical basis for the spaces X and Y, i.e. for the pair $F = F[\lambda, a]$.

Given $a = (a_i)$, $\lambda = (\lambda_i)$, and $m \in \mathbb{N}$, introduce *m-rectangle characteristic of* (λ, a) (or of $F = F[\lambda, a]$), as follows:

$$\mu_m^{(\lambda,a)}(\delta,\varepsilon;\ \tau,t) = \mu_m^F(\delta,\varepsilon;\ \tau,t) := |\bigcup_{k=1}^{m}\{i : \delta_k \le \lambda_i \le \varepsilon_k,\ \tau_k \le a_i \le t_k\}| \tag{14}$$

defined for

$$\delta = (\delta_k),\ \varepsilon = (\varepsilon_k),\ \tau = (\tau_k),\ t = (t_k) \in \mathbf{R}_+^m;\ 1 \le \tau_k \le t_k, k \in \mathbb{N}_m. \tag{15}$$

The name "m-rectangle" is natural because the function (14) calculates how many points (λ_i, a_i) are contained in the union of m rectangles $P_k = (\delta_k, \varepsilon_k] \times (\tau_k, t_k], k \in \mathbb{N}_m$.

Let $\lambda = (\lambda_i)$, $a = (a_i)$, $\tilde{\lambda} = (\tilde{\lambda}_i)$ and $\tilde{a} = (\tilde{a}_i)$, $m \in \mathbb{N}$. We say that *the functions* $\mu_m^{(\lambda,a)}$ *and* $\mu_m^{(\tilde{\lambda},\tilde{a})}$ *are equivalent* and write $\mu_m^{(\lambda,a)} \approx \mu_m^{(\tilde{\lambda},\tilde{a})}$ if there exists a constant $\Delta > 1$, a strictly decreasing function $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$, $\varphi(t) \to 0$ as $t \to \infty$ (in general Δ and φ depend on m) such that the following inequalities

$$\mu_m^{(\lambda,a)}(\delta,\varepsilon;\tau,t) \le \mu_m^{(\tilde{\lambda},\tilde{a})}(\delta - \varphi(\tau)t, \varepsilon + \varphi(\tau)t;\ \frac{\tau}{\Delta}, \Delta t), \tag{16}$$

$$\mu_m^{(\tilde{\lambda},\tilde{a})}(\delta,\varepsilon;\tau,t) \le \mu_m^{(\lambda,a)}(\delta - \varphi(\tau)t, \varepsilon + \varphi(\tau)t;\ \frac{\tau}{\Delta}, \Delta t), \tag{17}$$

with $\varphi(\tau) = (\varphi(\tau_k))$, $\varphi(\tau)t = (\varphi(\tau_k)t_k)$, $\frac{\tau}{\Delta} = (\frac{\tau_k}{\Delta})$, $\Delta t = (\Delta t_k) \in \mathbf{R}_+^m$, hold for all collections of parameters $\delta, \varepsilon, \tau, t$.

If, moreover, the function φ and the constant Δ can be chosen so that the inequalities (16), (17) hold for all $m \in \mathbb{N}$ (i.e. φ and Δ are independent of m), then we say that *the systems of characteristics* $(\mu_m^{(\lambda,a)})_{m\in\mathbb{N}}$ *and* $(\mu_m^{(\tilde{\lambda},\tilde{a})})_{m\in\mathbb{N}}$ *are equivalent* and write $(\mu_m^{(\lambda,a)}) \approx (\mu_m^{(\tilde{\lambda},\tilde{a})})$.

The following statement shows that the system $(\mu_m^{(\lambda,a)})$ is a complete quasidiagonal invariant on the class of pairs $F = F[\lambda, a]$.

Theorem 15 *For pairs F and $\tilde{F}$ the following statements are equivalent:*

(a) $F \overset{p}{\simeq} \tilde{F}$;

(b) $F \overset{qd}{\simeq} \tilde{F}$;

(c) $(\mu_m^F) \approx (\mu_m^{\tilde{F}})$.

Proof. Evidently $(a) \Longrightarrow (b)$. Consider $(b) \Longrightarrow (c)$. Suppose that *(b)* holds, i.e. there exists an isomorphism $T : F \to \tilde{F}$ such that $Te_i = t_i\tilde{e}_{\sigma(i)},\ \ i \in \mathbb{N}$, where $\sigma : \mathbb{N} \to \mathbb{N}$ is a bijection and (t_i) is a scalar sequence. Then, using continuity of operators T and T^{-1}, after some elementary calculations we show that the sequences $a, \tilde{a}, \lambda, \tilde{\lambda}$ satisfy the following relations:

$$\exists \Delta > 0: \ \frac{1}{\Delta} \le \frac{\tilde{a}_{\sigma(i)}}{a_i} \le \Delta, \ \ i \in \mathbb{N} \tag{18}$$

and

$$\lim \frac{\tilde{\lambda}_{\sigma(i)} - \lambda_i}{a_i} = 0.$$

The latter means that there exists a decreasing function $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$, $\varphi(t) \to 0$ as $t \to \infty$ such that

$$\frac{|\tilde{\lambda}_{\sigma(i)} - \lambda_i|}{a_i} \le \varphi(a_i), \ \ i \in \mathbb{N}. \tag{19}$$

Now take any $m \in \mathbb{N}$ and an arbitrary collection of parameters δ, ε, τ, t of the kind (15). From (18) and (19) it follows that

$$\{i : \delta_k \le \lambda_i \le \varepsilon_k, \ \ \tau_k \le a_i \le t_k\} \subset$$
$$\{i : \delta_k - \varphi(\tau_k)t_k \le \tilde{\lambda}_{\sigma(i)} \le \varepsilon_k + \varphi(\tau_k)t_k, \ \ \frac{\tau_k}{\Delta} \le \tilde{a}_{\sigma(i)} \le \Delta t_k\}$$

for each $k \in \mathbb{N}_m$.

This inclusion implies the estimate (16) and, for reasons of symmetry, the relation (17) is also true. Since the function φ and the constant Δ do not depend on m, we get (c). Hence $(b) \Longrightarrow (c)$ is proved.

Now consider $(c) \Longrightarrow (a)$. Suppose the statement (c) holds. This means that there exists a function φ and a constant Δ, both independent of m, such that the estimates (16) and (17) hold for any collection of parameters δ, ε, τ, t of the kind (15).

Define the multiple-valued function $S : \mathbb{N} \to \mathbb{N}$ by the rule

$$S(i) = \{j : \lambda_i - \varphi(a_i)a_i \le \tilde{\lambda}_j \le \lambda_i + \varphi(a_i)a_i, \ \ \frac{a_i}{\Delta} \le \tilde{a}_j \le \Delta a_i\}, \ \ i \in \mathbb{N}.$$

Due to (16) one can check that the map S satisfies the conditions of Hall-König theorem ([30], Ch. 3). By this theorem there exists an injection $s : \mathbb{N} \to \mathbb{N}$ such that $s(i) \in S(i)$, $i \in \mathbb{N}$.

Similarly, from the inequality (17) we deduce that there exists an injection $r : \mathbb{N} \to \mathbb{N}$ such that

$$r(i) \in \{j : \tilde{\lambda}_i - \varphi(\tilde{a}_i)\tilde{a}_i \le \lambda_j \le \tilde{\lambda}_i + \varphi(\tilde{a}_i)\tilde{a}_i, \ \ \frac{\tilde{a}_i}{\Delta} \le a_j \le \Delta\tilde{a}_i\}, \ \ i \in \mathbb{N}.$$

Following Mityagin ([35, 36]), from Cantor-Bernstein set-theory construction we get that there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that

$$\lambda_i - \Delta\varphi\left(\frac{a_i}{\Delta}\right)a_i \le \tilde{\lambda}_{\sigma(i)} \le \lambda_i + \Delta\varphi\left(\frac{a_i}{\Delta}\right)a_i, \ \frac{a_i}{\Delta} \le \tilde{a}_{\sigma(i)} \le \Delta a_i, \ i \in \mathbb{N}.$$

It is easy to show now that the formula $De_i = \tilde{e}_{\sigma(i)}$, $i \in \mathbb{N}$ defines the isomorphism $D : F \to \tilde{F}$. That completes the proof. □

5 Invariance of m-rectangle characteristics and compound invariants

In this section using appropriate compound invariants we prove that each function μ_m^F is a linear topological invariant on the considered class of pairs.

Theorem 16 *If* $F \simeq \tilde{F}$, *then* $\mu_m^F \approx \mu_m^{\tilde{F}}$ *for each* m.

Proof. For simplicity we put $a_{i,p} = \exp(-\frac{1}{p}a_i)$, $b_{i,p} = \exp(-\frac{1}{p}a_i + \lambda_i)$, $\tilde{a}_{i,p} = \exp(-\frac{1}{p}\tilde{a}_i)$, $\tilde{b}_{i,p} = \exp(-\frac{1}{p}\tilde{a}_i + \tilde{\lambda}_i)$, $i, p \in \mathbb{N}$. Take an arbitrary $m \in \mathbb{N}$. Suppose $T : \tilde{F} \to F$ is an isomorphism. Consider the following two absolute bases for F : the canonical basis $e = \{e_i\}_{i\in\mathbb{N}}$ and T-image $\tilde{e} = \{\tilde{e}_i\}_{i\in\mathbb{N}}$ of the canonical basis for $\tilde{F}$. Then each element $x \in F$ has two basis expansions:

$$x = \sum_{i=1}^{\infty} \xi_i e_i = \sum_{i=1}^{\infty} \eta_i \tilde{e}_i,$$

and the system of norms $||x||_p = \sum\limits_{i=1}^{\infty} |\eta_i|\tilde{a}_{i,p}$ is equivalent to the original system of norms in $X : |x|_p = \sum\limits_{i=1}^{\infty} |\xi_i|a_{i,p}$, $x \in X$, $p \in \mathbb{N}$, and the system of norms $||x||_p = \sum\limits_{i=1}^{\infty} |\eta_i|\tilde{b}_{i,p}$ is equivalent to the original system of norms in $Y : |x|_p = \sum\limits_{i=1}^{\infty} |\xi_i|b_{i,p}$, $x \in Y$, $p \in \mathbb{N}$.

Therefore, using also Proposition 1, we can choose numbers $r, p, s \in \mathbb{N}$ and increasing sequences of natural numbers $(s_i^{(k)})$, $(q_i^{(k)})$, $(r_i^{(k)})$, $k \in \mathbb{N}_m$, such that the following inclusions

$$\begin{array}{ccccc} C^{-1}B^{\tilde{e}}((\tilde{a}_{i,s})) & \subset & B^{e}((a_{i,q})) & \subset & CB^{\tilde{e}}((\tilde{a}_{i,r})), \\ C^{-1}B^{\tilde{e}}((\tilde{a}_{i,s_i^{(k)}})) & \subset & B^{e}((a_{i,q_i^{(k)}})) & \subset & CB^{\tilde{e}}((\tilde{a}_{i,r_i^{(k)}})), \\ C^{-1}B^{\tilde{e}}((\tilde{b}_{i,s})) & \subset & B^{e}((b_{i,q})) & \subset & CB^{\tilde{e}}((\tilde{b}_{i,r})), \\ C^{-1}B^{\tilde{e}}((\tilde{b}_{i,s_i^{(k)}})) & \subset & B^{e}((b_{i,q_i^{(k)}})) & \subset & CB^{\tilde{e}}((\tilde{b}_{i,r_i^{(k)}})), \end{array} \tag{20}$$

are valid with some constant C for $k \in \mathbb{N}_m$. Therewith, taking into account Lemma 9 and $a_i \to \infty$, we may assume that ($i \in \mathbb{N}, k \in \mathbb{N}_m$, $l \in \mathbb{N}_{m-1}$)

$$\begin{array}{l} 2r < q,\ 2q < s,\ 2s < r_i^{(m)},\ 2r_i^{(k)} < q_i^{(k)} \\ 2q_i^{(k)} < s_i^{(k)},\ 2s_i^{(l+1)} < r_i^{(l)},\ s_i^{(1)}/\tilde{a}_i \to 0. \end{array} \tag{21}$$

To prove $\mu_m^F \approx \mu_m^{\tilde{F}}$ we compare the value of the function β for two pairs compound absolutely convex sets U, V and $\tilde{U}$, $\tilde{V}$, with the balls from (20) as a row material for their construction.

Take an arbitrary collection of parameters δ, ε, τ, t of the kind (15). Without loss of generality we assume that the sequence $\{\tau_k, k \in \mathbb{N}_m\}$ is *non-decreasing*.

Now introduce some new weight-sequences by combining (including an interpolation) of the weights from (20), which are immediately connected with the original topologies in F and $\tilde{F}$. The first set of such weights, related with the basis e,

$$v_j^{(k)} = (v_{i,j}^{(k)}), u_j^{(k)} = (u_{i,j}^{(k)}),\ j \in \mathbf{N_4};\ k \in \mathbb{N}_m, \tag{22}$$

we define as follows :

$$\begin{aligned}
v_{i,1}^{(k)} &= a^{\frac{1}{2}}_{i,q_i^{(k)}} b^{\frac{1}{2}}_{i,q_i^{(k)}}, & u_{i,1}^{(k)} &= a^{\frac{1}{2}}_{i,q_i^{(k)}} b^{\frac{1}{2}}_{i,q_i^{(k)}},\\
v_{i,2}^{(k)} &= \exp(\frac{\tau_k}{2q})\ a^{\frac{1}{2}}_{i,q} b^{\frac{1}{2}}_{i,q}, & u_{i,2}^{(k)} &= \exp(\frac{t_k}{q}) a^{\frac{1}{2}}_{i,q} b^{\frac{1}{2}}_{i,q},\\
v_{i,3}^{(k)} &= \exp(\frac{\delta_k}{2}) a_{i,q_i^{(k)}}, & u_{i,3}^{(k)} &= \exp(-\frac{\delta_k}{2}) b_{i,q_i^{(k)}},\\
v_{i,4}^{(k)} &= \exp(-\frac{\varepsilon_k}{2}) b_{i,q_i^{(k)}}, & u_{i,4}^{(k)} &= \exp(\frac{\varepsilon_k}{2}) a_{i,q_i^{(k)}}.
\end{aligned} \tag{23}$$

The second set of weights

$$\tilde{v}_j^{(k)} = (\tilde{v}_{i,j}^{(k)}), \tilde{u}_j^{(k)} = (\tilde{u}_{i,j}^{(k)}), j \in \mathbf{N_4};\ k \in \mathbb{N}_m, \tag{24}$$

is produced in a very similar way from the weights, related with the second basis $\tilde{e}$,

$$\begin{aligned}
\tilde{v}_{i,1}^{(k)} &= \tilde{a}^{\frac{1}{2}}_{i,r_i^{(k)}} \tilde{b}^{\frac{1}{2}}_{i,r_i^{(k)}}, & \tilde{u}_{i,1}^{(k)} &= \tilde{a}^{\frac{1}{2}}_{i,s_i^{(k)}} \tilde{b}^{\frac{1}{2}}_{i,s_i^{(k)}},\\
\tilde{v}_{i,2}^{(k)} &= \exp(\frac{\tau_k}{2q}) \tilde{a}^{\frac{1}{2}}_{i,r} \tilde{b}^{\frac{1}{2}}_{i,r}, & \tilde{u}_{i,2}^{(k)} &= \exp(\frac{t_k}{q}) \tilde{a}^{\frac{1}{2}}_{i,s} \tilde{b}^{\frac{1}{2}}_{i,s},\\
\tilde{v}_{i,3}^{(k)} &= \exp(\frac{\delta_k}{2}) \tilde{a}_{i,r_i^{(k)}}, & \tilde{u}_{i,3}^{(k)} &= \exp(-\frac{\delta_k}{2}) \tilde{b}_{i,s_i^{(k)}},\\
\tilde{v}_{i,4}^{(k)} &= \exp(-\frac{\varepsilon_k}{2}) \tilde{b}_{i,r_i^{(k)}}, & \tilde{u}_{i,4}^{(k)} &= \exp(\frac{\varepsilon_k}{2}) \tilde{a}_{i,s_i^{(k)}}.
\end{aligned} \tag{25}$$

Using these weights we construct now two pairs of compound absolutely convex sets in Y:

$$V = \operatorname{conv}\left(\bigcup_{k=1}^{m} \bigcap_{j=1}^{4} B^e(v_i^{(k)})\right),\ U = \bigcap_{k=1}^{m} \operatorname{conv}\left(\bigcup_{j=1}^{4} B^e(u_i^{(k)})\right); \tag{26}$$

$$\tilde{V} = \operatorname{conv}\left(\bigcup_{k=1}^{m} \bigcap_{j=1}^{4} B^{\tilde{e}}(\tilde{v}_i^{(k)})\right),\ \tilde{U} = \bigcap_{k=1}^{m} \operatorname{conv}\left(\bigcup_{j=1}^{4} B^{\tilde{e}}(\tilde{u}_i^{(k)})\right). \tag{27}$$

From (20) it follows that $V \subset C\tilde{V},\ \tilde{U} \subset CU$, therefore, by (4), we get the estimate:

$$\beta(V,U) \leq \beta(C^2\tilde{V},\tilde{U}). \tag{28}$$

Since the sets (26), (27) are not ℓ_1-balls, we cannot immediately apply Proposition 2 for calculation of the function β in (28). Therefore we consider first some auxiliary ℓ_1-balls, inscribed into and circumscribed about these sets. Namely, by Proposition 8, we get the following inclusions

$$B^e(c) \subset V,\ \ U \subset mB^e(d),\ \ \tilde{V} \subset 4B^{\tilde{e}}(\tilde{c}),\ \ B^{\tilde{e}}(\tilde{d}) \subset \tilde{U}, \tag{29}$$

where

$$c = (c_i), \quad c_i = \min_{i\in\mathbb{N}_m} \{c_i^{(k)}\}, \quad c_i^{(k)} = \max_{j\in\mathbf{N_4}} \{v_{i,j}^{(k)}\}; \tag{30}$$

$$\tilde{c} = (\tilde{c}_i), \quad \tilde{c}_i = \min_{i\in\mathbb{N}_m} \{\tilde{c}_i^{(k)}\}, \quad \tilde{c}_i^{(k)} = \max_{j\in\mathbf{N_4}} \{\tilde{v}_{i,j}^{(k)}\}; \tag{31}$$

$$d = (d_i), \quad d_i = \max_{i\in\mathbb{N}_m} \{d^{(k)}{}_i\}, \quad d^{(k)}{}_i = \min_{j\in\mathbf{N_4}} \{u_{i,j}^{(k)}\}; \tag{32}$$

$$\tilde{d} = (\tilde{d}_i), \quad \tilde{d}_i = \max_{i\in\mathbb{N}_m} \{\tilde{d}_i^{(k)}\}, \quad \tilde{d}_i^{(k)} = \min_{j\in\mathbf{N_4}} \{\tilde{u}_{i,j}^{(k)}\}. \tag{33}$$

Using (28), (29), and the properties (4) of the characteristic $\beta(V,U)$, we get the inequality

$$\beta(B^e(c), B^e(d)) \leq \beta(M B^{\tilde{e}}(\tilde{c}), B^{\tilde{e}}(\tilde{d})), \tag{34}$$

where $M := 4mC^2$. First we estimate the left side of this inequality from below. From Proposition 2 and definitions of the weights (30), (32) we obtain

$$\beta(B^e(c), B^e(d)) = |\bigcup_{k=1}^{m}\bigcup_{l=1}^{m}\{i : c_i^{(k)} \leq d_i^{(l)}\}| \geq |\bigcup_{k=1}^{m}\{i : c_i^{(k)} \leq d_i^{(k)}\}|. \tag{35}$$

Fix any $k \in \mathbb{N}_m$. Taking into account the definitions of the sequences $c^{(k)}$, $d^{(k)}$, and the equality $v_1^{(k)} = u_1^{(k)}$, we get

$$\{i : c_i^{(k)} \leq d_i^{(k)}\} = \bigcap_{j=2}^{4}\{i : v_{i,j}^{(k)} \leq u_{i,1}^{(k)}, \ v_{i,1}^{(k)} \leq u_{i,j}^{(k)}\}. \tag{36}$$

One can check that the construction of the weights (22), (23), (30), (32) have been chosen just for the following relations to be true:

$$\begin{array}{ll} \{i : v_{i,2}^{(k)} \leq u_{i,1}^{(k)}\} \supset \{i : a_i \leq \tau_k\}, & \{i : v_{i,1}^{(k)} \leq u_{i,2}^{(k)}\} \supset \{i : a_i \leq t_k\}, \\ \{i : v_{i,3}^{(k)} \leq u_{i,1}^{(k)}\} \supset \{i : \lambda_i \leq \delta_k\}, & \{i : v_{i,1}^{(k)} \leq u_{i,3}^{(k)}\} \supset \{i : \lambda_i \leq \delta_k\}, \\ \{i : v_{i,4}^{(k)} \leq u_{i,1}^{(k)}\} \supset \{i : \lambda_i \leq \varepsilon_k\}, & \{i : v_{i,1}^{(k)} \leq u_{i,4}^{(k)}\} \supset \{i : \lambda_i \leq \varepsilon_k\}. \end{array} \tag{37}$$

We display only the proof of the first conclusion in (37), since the rest of them can be proved quite similarly. It is easy to see that the inequality $v_{i,2}^{(k)} \leq u_{i,1}^{(k)}$ is equivalent to the following inequality

$$(\frac{1}{q} - \frac{1}{q_i^{(k)}})a_i \geq \frac{\tau_k}{2q}. \tag{38}$$

Due to the assumptions (21), $\dfrac{1}{q} - \dfrac{1}{q_i^{(k)}} \geq \dfrac{1}{2q}$, therefore the inequality (38) is weaker than the inequality $a_i \geq \tau_k$. This implies the desired inclusion.

From (37) it follows that

$$\{i : c_i^{(k)} \leq d_i^{(k)}\} \supset \{i : \delta_k \leq \lambda_i \leq \varepsilon_k;\ \tau_k \leq a_i \leq t_k\}. \tag{39}$$

Combining (35) and (39) we obtain finally

$$\mu_m^F(\delta,\varepsilon;\tau,t) \le \beta(B^e(c),B^e(d)). \tag{40}$$

Now we proceed with the estimation of the right-hand side of the inequality (34) from above. Using Proposition 2 and (31), (33), we obtain

$$\beta(MB^{\tilde{e}}(\tilde{c}),B^{\tilde{e}}(\tilde{d})) = |\bigcup_{k,l=1}^{m} S_{k,l}|, \tag{41}$$

where $S_{k,l} := \{i : \tilde{c}_i^{(k)} \le M\tilde{d}_i^{(l)}\}$.

By the definition of the sequences $\tilde{c}^{(k)}$ and $\tilde{d}^{(l)}$ we conclude that

$$S_{k,l} \subset \bigcap_{j=2}^{4}\{i : \tilde{v}_{i,j}^{(k)} \le M\tilde{u}_{i,1}^{(l)},\ \tilde{v}_{i,1}^{(k)} \le M\tilde{u}_{i,j}^{(l)}\}\bigcap\{i : \tilde{v}_{i,1}^{(k)} \le M\tilde{u}_{i,1}^{(l)}\}. \tag{42}$$

First we consider the inequality $\tilde{v}_{i,1}^{(k)} \le M\tilde{u}_{i,1}^{(l)}$. Taking into account the definitions of the sequences $\tilde{v}_1^{(k)}$ and $\tilde{d}_1^{(l)}$ we see that this inequality is equivalent to the inequality

$$(\frac{1}{s_i^{(l)}} - \frac{1}{r_i^{(k)}})\tilde{a}_i \le L, \tag{43}$$

where $L := \ln M$.

Begin with the case $l > k$. Then, recalling the suppositions (21), we conclude that the inequality (43) is not weaker than the inequality

$$\tilde{a}_i \le 2Ls_i^{(1)}. \tag{44}$$

Due to the assumptions (13), (15), (21), the relation (44) implies

$$\{i : \tilde{v}_{i,1}^{(k)} \le M\tilde{u}_{i,1}^{(l)}\} \subset \{i : \tilde{a}_i \le A\}, \tag{45}$$

where $A := \max\{\tilde{a}_i : \tilde{a}_i \le 2Ls_i^{(1)}\}$.

On the other hand, it is obvious that in the case $l \le k$ the following relation holds:

$$\{i : \tilde{v}_{i,1}^{(k)} \le M\tilde{u}_{i,1}^{(l)}\} = \mathbb{N}. \tag{46}$$

Further, it is easy to check that

$$\{i :\ \tilde{v}_{i,2}^{(k)} \le M\tilde{u}_{i,1}^{(l)}\} \subset \{i :\ \frac{\tau_k}{2q} - L \le \frac{1}{r}\tilde{a}_i\}.$$

Therefore, recalling (13), (15), we get

$$\{i : \tilde{v}_{i,2}^{(k)} \le M\tilde{u}_{i,1}^{(l)}\} \subset \{i : \frac{\tau_k}{4qL} \le \tilde{a}_i\} \tag{47}$$

Arguing in a similar way, we obtain the relation

$$\{i : \tilde{v}^{(k)}_{i,1} \le M\tilde{u}^{(l)}_{i,2}\} \subset \{i : \tilde{a}_i \le 4rLt_l\}. \tag{48}$$

Consider the set

$$S'_{k,l} := \{i : \tilde{v}^{(k)}_{i,1} \le M\tilde{u}^{(l)}_{i,1},\ \tilde{v}^{(k)}_{i,2} \le M\tilde{u}^{(l)}_{i,1},\ \tilde{v}^{(k)}_{i,1} \le M\tilde{u}^{(l)}_{i,2},\}$$

and define a constant $\Delta := \max\{A, 4qL\}$.

Then, combining (45), (46), (47), (48), and taking into account (21) and non-decreasing of τ_k, we obtain

$$S'_{k,l} \subset \begin{cases} \{i : \frac{\tau_k}{\Delta} \le \tilde{a}_i \le \Delta t_k\} & \text{if } l > k \\ \{i : \frac{\tau_l}{\Delta} \le \tilde{a}_i \le \Delta t_l\} & \text{if } l \le k. \end{cases} \tag{49}$$

Taking into account the definitions of the sequences $\tilde{v}^{(k)}_j$ and $\tilde{u}^{(l)}_j$, $j \in \mathbf{N_4}$, and the matrices $(a_{i,p})$ and $(b_{i,p})$, we see that

$$\begin{array}{l} \{\tilde{v}^{(k)}_{i,3} \le M\tilde{u}^{(l)}_{i,1}\} \bigcap \{\tilde{v}^{(k)}_{i,4} \le M\tilde{u}^{(l)}_{i,1}\} \subset \{i : \delta_k - T\tilde{a}_i \le \tilde{\lambda}_i \le \varepsilon_k + T\tilde{a}_i\}, \\ \{\tilde{v}^{(k)}_{i,1} \le M\tilde{u}^{(l)}_{i,4}\} \bigcap \{\tilde{v}^{(k)}_{i,1} \le M\tilde{u}^{(l)}_{i,3}\} \subset \{i : \delta_l - T\tilde{a}_i \le \tilde{\lambda}_i \le \varepsilon_l + T\tilde{a}_i\}, \end{array} \tag{50}$$

where $T = T(i) := 2\left(\frac{L}{\tilde{a}_i} + \frac{1}{r_i^{(m)}}\right)$.

Now it is easy to see how to choose an appropriate function φ. Taking any non-decreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $r_i^{(m)} \ge \psi(a_i)$, we can use any decreasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(t) \to 0$ as $t \to \infty$, satisfying the estimate:

$$\varphi(\tau) \ge 2\Delta(\frac{\Delta L}{\tau} + \frac{1}{\psi(\tau)}).$$

Since (42) and (50), the inclusion holds:

$$S_{k,l} \subset S'_{k,l} \bigcap \{i : \max\{\delta_k,\ \delta_l\} - T\tilde{a}_i < \tilde{\lambda}_i \le \min\{\varepsilon_k,\ \varepsilon_l\} + T\tilde{a}_i\}. \tag{51}$$

Taking into account (49), (51) and the definition of the function φ, we get

$$\begin{array}{rl} S_{k,l} \subset & \{i : \delta_k - \varphi(\tau_k)t_k \le \tilde{\lambda}_i \le \varepsilon_k + \varphi(\tau_k)t_k,\ \frac{\tau_k}{\Delta} \le \tilde{a}_i \le \Delta t_k\} \bigcup \\ & \{i : \delta_l - \varphi(\tau_l)t_l \le \tilde{\lambda}_i \le \varepsilon_l + \varphi(\tau_l)t_l\ \ \frac{\tau_l}{\Delta} \le \tilde{a}_i \le \Delta t_l\} \end{array} \tag{52}$$

for any $k, l \in \mathbb{N}_m$.

Finally, from (41),(52),(42), we get

$$\beta(MB^{\tilde{e}}(\tilde{c}), B^{\tilde{e}}(\tilde{d})) \le \mu^{\tilde{F}}_m(\delta - \varphi(\tau)t,\ \varepsilon + \varphi(\tau)t;\ \frac{\tau}{\Delta},\ \Delta t). \tag{53}$$

Combining (40) and (53), we get the inequality (16).

Similarly, we obtain the inequality (17). Thus, $\mu^F_m \approx \mu^{\tilde{F}}_m$. This concludes the proof of Theorem 16. □

6 Comparison of m-rectangular invariants

Here we compare the strength of the invariants μ_m^F for different $m \in \mathbb{N}$, and show also that the equivalence of all individual invariants μ_m^F does not imply the equivalence of the systems of invariants (μ_m^F). Thus, there is a gap between the results about isomorphic and qusidiagonally isomorphic classification of pairs.

Theorem 17 *For each m there exist pairs F and $\tilde{F}$ satisfying the following conditions:*
(a) $\mu_l^F \approx \mu_l^{\tilde{F}}$, $l = 1,2,\dots,m$;
(b) $\mu_{m+1}^F \not\approx \mu_{m+1}^{\tilde{F}}$.

Theorem 18 *There exist pairs F and $\tilde{F}$ such that*
(a) $\mu_m^F \approx \mu_m^{\tilde{F}}$, *for each* $m \in \mathbb{N}$;
(b) $(\mu_m^F) \not\approx (\mu_m^{\tilde{F}})$.

Since the analogy with the case of so-called first type power spaces, considered thoroughly in [16], we describe only how to construct the corresponding pairs F, $\tilde{F}$, avoiding the proofs.

In each example the required pairs $F[\lambda,a]$ and $F[\tilde{\lambda},\tilde{a}]$ will be collected from special finite-dimensional blocks, as follows.

We begin with the notion of α-dense set. Suppose $\alpha > 1$; we say that the set $A \subset \mathbb{R}$ is α-dense in $B \subset \mathbb{R}$ if for each point $x \in B$ there exists a point $\tilde{x} \in A$ such that $\frac{\tilde{x}}{\sqrt{\alpha}} \leq x \leq \sqrt{\alpha}\tilde{x}$.

Further, we consider $m \in \mathbb{N}$, independent of j, for the first theorem and $m = j$, $j \in \mathbb{N}$ for the second one.

Let us take an arbitrary number $\alpha > 1$, a sequence $(\beta_j) \uparrow \infty$, $\beta_1 > \alpha$, and a sequence (η_j) defined as follows $\eta_1 \geq 1$, $\eta_{j+1} \geq \beta_{j+1}^3 \eta_j$, $j \in \mathbb{N}$.

For each $j \in \mathbb{N}$ we choose $2m$ natural numbers $k_{j,1}, k_{j,2}, \dots, k_{j,2m}$ so that

$$k_{j,1} = 1,\ k_{j,l} > k_{j,l-1} + \beta_j^2 \eta_j,\ l = 2,3,\dots,2m.$$

On the Figure 1 it is drawn the set S_j, consisting of horizontal and vertical segments (two "combs" with m "cogs" on each of them), in which all the points (λ_i, a_i), $(\tilde{\lambda}_i, \tilde{a}_i)$, corresponding to the j-th block will be located.

On the segment $[\eta_j, \beta_j^2 \eta_j]$ we select a finite α-dense point set M_j, including the points η_j, $\beta_j \eta_j$, $\beta_j^2 \eta_j$.

Consider the set L_j of all the points $(l, y_{j,s}) \in S_j$ such that $1 \leq l \leq k_{j,2m}$, $l \in \mathbb{N}$, $y_{j,s} \in M_j$. After enumeration we get

$$L_j = \{(x_{j,i}, y_{j,i}),\ i = 1,2,\dots,n_j - 1\}$$

with some number n_j.

Define the vectors $\lambda^{(j)} = (\lambda_i^{(j)})_{i=1}^{n_j}$, $a^{(j)} = (a_i^{(j)})_{i=1}^{n_j}$, $\tilde{\lambda}^{(j)} = (\tilde{\lambda}_i^{(j)})_{i=1}^{n_j}$, $\tilde{a}^{(j)} = (\tilde{a}_i^{(j)})_{i=1}^{n_j}$ by the following formulae

$$\lambda_i^{(j)} = \begin{cases} x_{j,i} & \text{if } i = 1,2,\dots,n_j - 1, \\ k_{j,1} & \text{if } i = n_j; \end{cases}$$

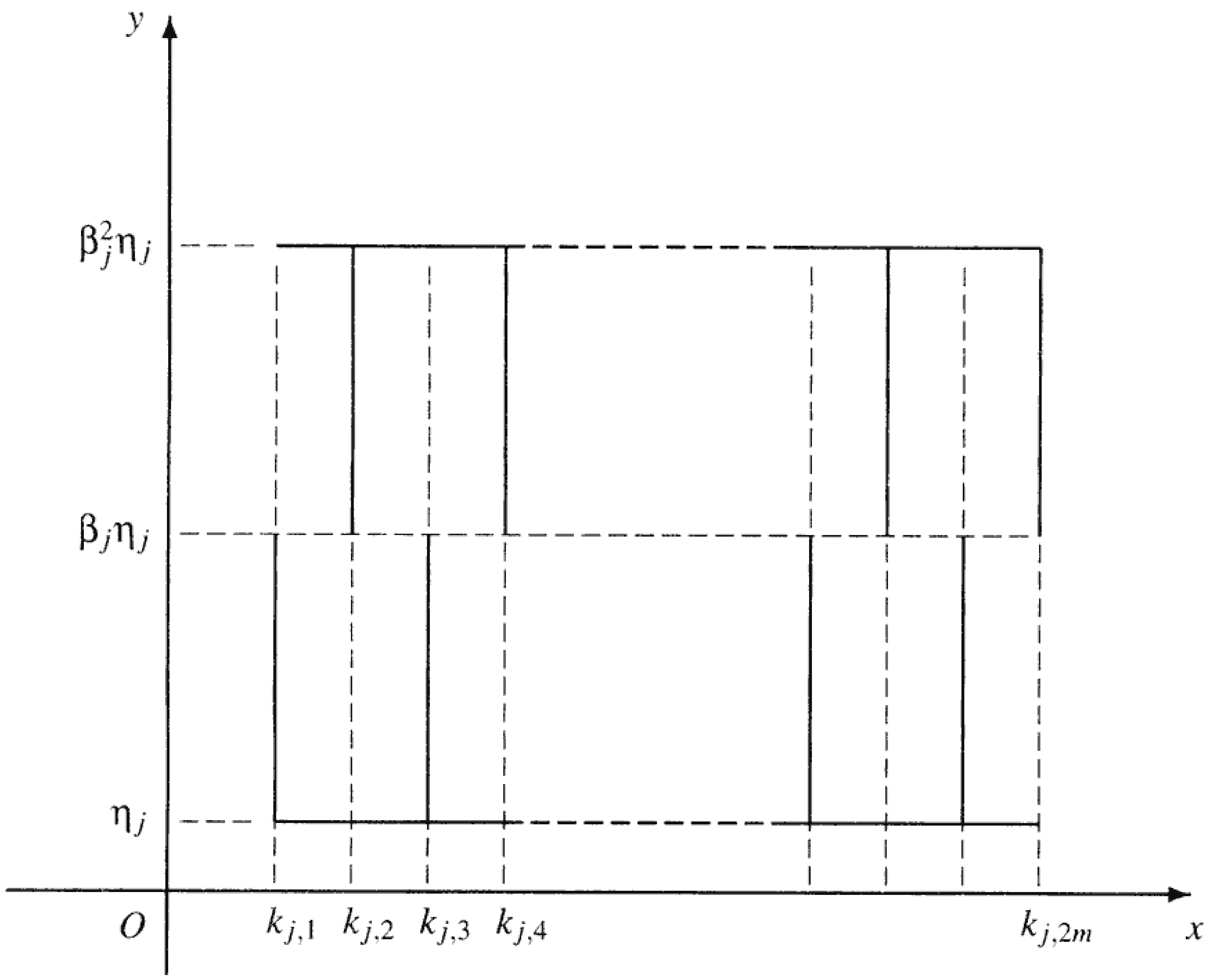

Figure 1: The set S_j.

$$\tilde{\lambda}_i^{(j)} = \begin{cases} x_{j,i} & \text{if } i = 1,2,\ldots,n_j-1, \\ k_{j,2} & \text{if } i = n_j; \end{cases}$$

$$a_i^{(j)} = \tilde{a}_i^{(j)} = \begin{cases} y_{j,i} & \text{if } i = 1,2,\ldots,n_j-1, \\ \beta_j\eta_j & \text{if } i = n_j. \end{cases}$$

Now we construct the sequences $\lambda = (\lambda_i)$, $a = (a_i)$, $\tilde{\lambda} = (\tilde{\lambda}_i)$, $\tilde{a} = (\tilde{a}_i)$. The sequence $\lambda = (\lambda_i)$ we define by the rule $\lambda_i = \lambda^{(j)}_{i-(n_0+n_1+\ldots+n_{j-1})}$ for $n_{j-1} < i \le n_j$ $(n_0 := 0)$. The sequences a, $\tilde{\lambda}$, $\tilde{a}$ are defined in the same manner.

One can show, using the considerations of [16] as a hint, that for the constructed pairs the statements (a) and (b) in the theorems are valid.

7 Applications

We apply here the simplest of the compound invariants, studied in sections 4,5, namely the one-rectangle invariant, to the class of pairs of spaces of analytic functions in Reinhardt domains (item 3.2); this gives some isomorphic classification on this class, essentially stronger than that considered in 3.2.

Define a characteristic of a pair of Reinhardt domains $D_0 \subset D_1$, as follows:

$$\psi(\alpha) = \text{ mes } \{\theta \in \Sigma : \ h_{D_1}(\theta) - h_{D_0}(\theta) \le \alpha\}, \tag{54}$$

where mes means the Lebesgue measure on Σ.

Theorem 19 *Let D_i, $\tilde{D}_i$, $i = 0,1$, be Reinhardt domains such that $D_0 \subset D_1$, $\tilde{D}_0 \subset \tilde{D}_1$, and $\psi(\alpha)$, $\tilde{\psi}(\alpha)$, be the characteristics (54) for the pairs (D_0, D_1) and $(\tilde{D}_0, \tilde{D}_1)$, respectively. If (11) is true then there exists a constant c such that*

$$1/c\ \psi(\varepsilon/c) \le \tilde{\psi}(\varepsilon) \le \psi(c\varepsilon),\ \varepsilon > 0. \tag{55}$$

Proof. Let $h(\theta) = h_{D_1}(\theta) - h_{D_0}(\theta)$, $\tilde{h}(\theta) = h_{\tilde{D}_1}(\theta) - h_{\tilde{D}_0}(\theta)$ and

$$F := (A(D_0), A(D_1)), \quad \tilde{F} := (A(\tilde{D}_0), A(\tilde{D}_1)).$$

Then, putting $m = 1$, $\delta_1 = 0$, $\varepsilon_1 = \varepsilon t$, $t_1 = t, \tau_1 = t/2$, we get

$$\begin{aligned} \tilde{N} := \mu_m^{\tilde{F}}(\delta_1, \varepsilon_1;\ \tau_1, t_1) &= |\{k \in \mathbb{N}_0^n : |k|\tilde{h}(\theta(k)) \le \varepsilon t, \quad t/2 \le |k| \le t\}| \\ &\ge |\{k : \tilde{h}(\theta(k)) \le \varepsilon, \quad t/2 \le |k| \le t\}|. \end{aligned}$$

Using some elementary estimates for the number of integer points in a given domain of $\mathbb{R}^n$, we get

$$\tilde{N} \ge \frac{3t^n}{4n\sqrt{n}}(1 + o(1)), \quad t \to \infty. \tag{56}$$

By Theorem 16, there exists a constant $\Delta > 1$, and a sequence $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, tending to 0 as $t \to 0$, such that

$$\begin{aligned} \tilde{N} &\le \mu_m^F(\delta_1 - \varphi(\tau_1)t_1, \varepsilon_1 + \varphi(\tau_1)t_1; \quad \tau_1/\Delta, \Delta t_1) \\ &\le |\{k : h(\theta(k)) \le \Delta(\varepsilon + \varphi(t/2))t/2\Delta \le |k| \le \Delta t\}|. \end{aligned}$$

After some elementary estimations we get from here the following asymptotical inequality

$$\tilde{N} \leq \frac{1}{n\sqrt{n}} \Delta^n \psi(2\Delta(\varepsilon + \varphi(t/2))) t^n (1 + o(1)), \quad t \to \infty. \tag{57}$$

Combining (56), (57), we get

$$\tilde{\psi}(\varepsilon) \leq 2\Delta^n \psi(2\Delta(\varepsilon + \varphi(t/2)))(1 + o(1)), \quad t \to \infty.$$

Thus, after turning $t \to \infty$, we obtain (55) with $c = 2\Delta^n$. Because of the symmetry with respect to F and $\tilde{F}$, the proof is completed. □

Example Consider the one-parameter family of pairs of Reinhardt domains in $\mathbb{C}^2$, defined as follows:

$$\begin{aligned} D_0^{(u)} &:= \{(z_1, z_2) : |z_1| < 1, |z_2| < 1\}, \\ D_1^{(u)} &:= \{(z_1, z_2) : |z_1|^{\lambda} |z_2|^{1-\lambda} < \exp \lambda^u, \ \lambda \in [0,1]\}, \ 1 \leq u < \infty. \end{aligned}$$

Then $h_{D_0^{(u)}}(\lambda, 1-\lambda) \equiv 0$, $h_{D_1^{(u)}}(\lambda, 1-\lambda) = \lambda^u, 0 \leq \lambda \leq 1$, and the corresponding characteristic (54) is $\psi^{(u)}(\alpha) = \alpha^{1/u}$. Therefore, by Theorem 19, the pairs $F^{(u)} := (A(D_0^{(u)}), A(D_1^{(u)}))$, $1 \leq u < \infty$, are mutually non-isomorphic.

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