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HANKEL OPERATORS ON GENERALIZED BERGMAN-HARDY SPACES[1]

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Abstract. *We study Hankel operators* $H_f : H_2 \to H_2$ *on a class of spaces* H_2 *of analytic functions which includes, among many other examples, the Hardy space and the Bergman spaces on the unit disk as well as the Fock space on* $\mathbb{C}$. *We derive compactness conditions for* H_f *and describe the essential spectrum of* $H_f^* H_f$. *Moreover we investigate Schatten class Hankel operators. The main objects of study are those Hankel operators* H_f *which admit a sequence of vector-valued trigonometric polynomials* f_j *with* $\lim_j ||H_f - H_{f_j}|| = 0$.

1 Introduction

We study Hankel operators on various spaces of holomorphic functions defined as follows:

Let $\mathbb{T} = \{ z \in \mathbb{C} \ : \ |z| = 1 \}$ and consider the normalized Haar measure $d\varphi$ on $\mathbb{T}$ (i.e. $\int_{\mathbb{T}} g d\varphi = (2\pi)^{-1} \int_{-\pi}^{\pi} g(e^{i\theta}) d\theta$). Moreover, let μ be a bounded positive Borel measure on $\mathbb{R}_+$ with supp $\mu \neq \{0\}$ and consider the scalar product

$$\langle f, g \rangle = \int_{\mathbb{R}_+} \int_{\mathbb{T}} f(\, re^{i\varphi}\,) \overline{g(\, re^{i\varphi}\,)} d\varphi d\mu(r).$$

Denote the space of all (classes) of measurable functions $f : \mathbb{C} \to \mathbb{C}$ with $||f||_2 = \sqrt{\langle f, f \rangle} < \infty$ by $L_2 = L_2(d\varphi \otimes d\mu)$. We only study measures μ which are such that all polynomials on $\mathbb{C}$ are elements of $L_2(d\varphi \otimes d\mu)$. (This is always satisfied, for example, if μ has compact support.) Then put

$$H_2(\mu) = \text{ closure of } \{ p : \mathbb{C} \to \mathbb{C} \ : \ p \text{ a polynomial } \} \subset L_2.$$

These notions include the following classical examples.

Examples 1.1.: Put $D = \{ z \in \mathbb{C} \ : \ |z| < 1 \}$.
(1) If $\mu = \delta_1$ is the Dirac measure at 1 then

$$H_2(\mu) \simeq \{ \, h : D \to \mathbb{C} \ : \ h \text{ holomorphic}, \ \sup_{0<r<1} \int_{\mathbb{T}} |h(re^{i\varphi})|^2 d\varphi < \infty \, \}$$

is the classical Hardy space.
(2) If $d\mu(r) = r 1_{[0,1]}(r) dr$ and λ is the two-dimensional Lebesgue measure then $H_2(\mu) \simeq \{ \, h : D \to \mathbb{C} \ : \ h$ holomorph, $\int_D |h|^2 d\lambda < \infty \}$ is the classical Bergman space on D.
(3) If $d\mu(r) = r \exp(-r^2/2) dr$ then $H_2(\mu)$ is a space of entire functions usually called Fock

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space ([3, 11, 14]).

Of course, there are plenty of possibilities to construct further examples, e.g. the preceding measures with special densities ("weighted Bergman spaces") or purely atomic measures. We feel that these various spaces of holomorphic functions and their operators should be given a common unifying approach. The goal of this paper is to contribute to such a program by laying the emphasis on compactness conditions for Hankel operators.

Let $L_\infty = L_\infty(d\varphi \otimes d\mu)$ be the space of all (classes of) measurable functions $f(re^{i\varphi})$ on $\mathbb{C}$ which are essentially bounded with respect to $d\varphi \otimes d\mu$. Finally, consider the orthogonal projection $P : L_2(d\varphi \otimes d\mu) \to H_2(\mu)$.

Definition 1.2. For $f \in L_\infty$, $g \in L_2$ and $h \in H_2(\mu)$ put

$$\begin{aligned} M_f g &= f \cdot g \quad \text{(multiplication operator)} \\ T_f h &= P(f \cdot h) \quad \text{(Toeplitz operator)} \\ H_f h &= f \cdot h - P(f \cdot h) \quad \text{(Hankel operator)} \end{aligned}$$

Clearly, $M_f : L_2 \to L_2$, $T_f : H_2(\mu) \to H_2(\mu)$ and $H_f : H_2(\mu) \to L_2$ are bounded operators.

2 Matrix representations of H_f

For $k \in \mathbb{Z}$ let $\xi_k : \mathbb{C} \to \mathbb{C}$ be the function with

$$\xi_k(z) = \left(\frac{z}{|z|}\right)^k, \text{ if } z \neq 0, \text{ and } \xi_k(0) = 0.$$

If $m \in \mathbb{Z}_+$ put

$$e_m(re^{i\varphi}) = \frac{r^m e^{im\varphi}}{\sqrt{\int_{\mathbb{R}_+} s^{2m} d\mu}}.$$

Then, clearly, $\{ e_m \ : \ m \in \mathbb{Z}_+ \}$ is an orthonormal basis of $H_2(\mu)$.

The main technique in the following proofs is based on a formal Fourier series expansion of a given $f \in L_2$. Here put

$$F_k(r) = \int_{\mathbb{T}} f(re^{i\varphi}) e^{-ik\varphi} d\varphi \ \text{ if } \ k \in \mathbb{Z}.$$

Then we have $f \overset{(L_2)}{=} \sum_k F_k \xi_k$ where $\overset{(L_2)}{=}$ indicates convergence with respect to $||\cdot||_2$. Note, F_k only depends on r. Such functions will be called radial functions.

Proposition 2.1. *Let $f \in L_\infty$ be such that $f \overset{(L_2)}{=} \sum_k F_k \xi_k$. Then, for any $h = \sum_{l \in \mathbb{Z}_+} \beta_l e_l \in H_2(\mu)$ (finite sum), we have*

$(H_f h)(re^{i\varphi}) =$

$$\sum_{m\in\mathbb{Z}_+}\sum_{l\in\mathbb{Z}_+}\left(F_{m-l}(r)r^l - \frac{\int_{\mathbb{R}_+} F_{m-l}(s)s^{m+l}d\mu(s)}{\int_{\mathbb{R}_+} s^{2m}d\mu(s)} r^m\right)\frac{\beta_l}{\sqrt{\int_{\mathbb{R}_+} s^{2l}d\mu(s)}}\xi_m(re^{i\varphi})$$
$$+\sum_{m\in\mathbb{Z}\setminus\mathbb{Z}_+}\sum_{l\in\mathbb{Z}_+} F_{m-l}(r)r^l\frac{\beta_l}{\sqrt{\int_{\mathbb{R}_+} s^{2l}d\mu}}\xi_m(re^{i\varphi}).$$

Proof. We have

$$(fh)(re^{i\varphi}) = \sum_{m\in\mathbb{Z}}\sum_{l\in\mathbb{Z}_+} F_{m-l}(r)r^l\frac{\beta_l}{\sqrt{\int_{\mathbb{R}_+} s^{2l}d\mu(s)}}\xi_m(re^{i\varphi})$$

and, using the orthogonality of the ξ_k,

$$P(fh)(re^{i\varphi}) = \sum_{m\in\mathbb{Z}_+}\langle fh, e_m\rangle e_m(re^{i\varphi}) =$$

$$\sum_{m\in\mathbb{Z}_+}\sum_{l\in\mathbb{Z}_+}\frac{\int_{\mathbb{R}_+} F_{m-l}(s)s^{m+l}d\mu(s)}{\int_{\mathbb{R}_+} s^{2m}d\mu(s)}\frac{\beta_l}{\sqrt{\int_{\mathbb{R}_+} s^{2l}d\mu(s)}}r^m\xi_m(re^{i\varphi})$$

which proves the claim. □

Definition 2.2. Let $f\in L_2$ and $k\in\mathbb{Z}$.

For $m\in\mathbb{Z}_+$ with $m\geq k$ put

$$\begin{aligned} p_{m,k}[f](e^{i\varphi}) &= \frac{\int_{\mathbb{R}_+}\left|f(re^{i\varphi})r^{m-k} - \frac{\int_{\mathbb{R}_+} f(se^{i\varphi})s^{2m-k}d\mu(s)}{\int_{\mathbb{R}_+} s^{2m}d\mu(s)}r^m\right|^2 d\mu(r)}{\int_{\mathbb{R}_+} r^{2m-2k}d\mu(r)} \\ &= \frac{\int_{\mathbb{R}_+}|f(re^{i\varphi})|^2 r^{2m-2k}d\mu(r)}{\int_{\mathbb{R}_+} r^{2m-2k}d\mu(r)} - \frac{\left|\int_{\mathbb{R}_+} f(re^{i\varphi})r^{2m-k}d\mu(r)\right|^2}{\int_{\mathbb{R}_+} r^{2m}d\mu(r)\int_{\mathbb{R}_+} r^{2m-2k}d\mu(r)}\end{aligned}$$

and

$$p_m[f] = p_{m,0}[f].$$

If $m\in\mathbb{Z}\setminus\mathbb{Z}_+$ and $m\geq k$ then put

$$p_{m,k}[f](e^{i\varphi}) = \frac{\int_{\mathbb{R}_+}|f(re^{i\varphi})|^2 r^{2m-2k}d\mu(r)}{\int_{\mathbb{R}_+} r^{2m-2k}d\mu(r)}.$$

We obtain that, for all cases, $p_{m,k}[f]\geq 0$. Moreover,

$$p_m[f](e^{i\varphi}) = \frac{\int_{\mathbb{R}_+}|f(re^{i\varphi})|^2 r^{2m}d\mu(r)}{\int_{\mathbb{R}_+} r^{2m}d\mu(r)} - \left|\frac{\int_{\mathbb{R}_+} f(re^{i\varphi})r^{2m}d\mu(r)}{\int_{\mathbb{R}_+} r^{2m}d\mu(r)}\right|^2.$$

Corollary 2.3. *Let $h = \sum_{l\in\mathbb{Z}_+} \beta_l e_l \in H_2(\mu)$ and $F \in L_\infty$ be radial. Then*

$$(H_F h)(re^{i\varphi}) = \sum_{m\in\mathbb{Z}_+} \Big(F(r) - \frac{\int_{\mathbb{R}_+} F(s)s^{2m}d\mu(s)}{\int_{\mathbb{R}_+} s^{2m}d\mu(s)}\Big) r^m \frac{\beta_m}{\sqrt{\int_{\mathbb{R}_+} s^{2m}d\mu(s)}}\xi_m(re^{i\varphi})$$

and, for any $k \in \mathbb{Z}$,

$$(H_{F\xi_k}h)(re^{i\varphi}) =$$

$$\sum_{\substack{m\in\mathbb{Z}_+\\ m\geq k}} \Big(F(r)r^{m-k} - \frac{\int_{\mathbb{R}_+} F(s)s^{2m-k}d\mu(s)}{\int_{\mathbb{R}_+} s^{2m}d\mu(s)} r^m\Big) \frac{\beta_{m-k}}{\sqrt{\int_{\mathbb{R}_+} s^{2m-2k}d\mu(s)}}\xi_m(re^{i\varphi})$$

$$+ \sum_{\substack{m\in\mathbb{Z}\setminus\mathbb{Z}_+\\ m\geq k}} F(r)r^{m-k} \frac{\beta_{m-k}}{\sqrt{\int_{\mathbb{R}_+} s^{2m-2k}d\mu}}\xi_m(re^{i\varphi}).$$

In particular,

$$||H_F|| = \sup_{m\in\mathbb{Z}_+} \sqrt{p_m[F]} \text{ and } ||H_{F\xi_k}|| = \sup_{m\geq k} \sqrt{p_{m,k}[F]}.$$

3 The space Y

Here we study Hankel operators with symbols taken from a special subspace Y of L_∞.

For any $f \in L_2$ and $\lambda \in \mathbb{T}$ put $f_\lambda(z) = f(\lambda z)$, $z \in \mathbb{C}$. Let $\gamma_j(\psi) = \sum_{|k|<j} \frac{j-|k|}{j} e^{ik\psi}$. If $f \overset{(L_2)}{=} \sum_k F_k \xi_k$, put

$$\sigma_j f = \sum_{|k|<j} \frac{j-|k|}{j} F_k \xi_k = \int_{\mathbb{T}} f_{e^{i\psi}} \gamma_j(-\psi) d\psi.$$

Finally, let

$$\mathcal{L} = \{ \ T \ : H_2(\mu) \to L_2 \ : \ T \text{ linear and bounded } \},$$

$$\mathcal{K} = \{ \ T \in \mathcal{L} \ : \ T \text{ compact } \}$$

and let $q : \mathcal{L} \to \mathcal{L}/\mathcal{K}$ be the quotient map.

Lemma 3.1. *For any $f \in L_\infty$ and $j \in \mathbb{Z}_+$ we have*

$$||H_{\sigma_j f}|| \leq ||H_f|| \text{ and } ||qH_{\sigma_j f}|| \leq ||qH_f||.$$

Proof. For any $T \in \mathcal{L}$ and $\lambda \in \mathbb{T}$ put $T_\lambda h = (Th_{\bar{\lambda}})_\lambda$ if $h \in H_2(\mu)$. Using 2.1. and $T = H_f$ we obtain $H_{(f_\lambda)} = (H_f)_\lambda$. Put

$$(\sigma_j T)h = \int_{\mathbb{T}} (Th_{e^{-i\psi}})_{e^{i\psi}} \gamma_j(-\psi) d\psi.$$

Then we have $\sigma_j H_f = H_{\sigma_j f}$. Moreover we easily derive

$$||(\sigma_j T)h||_2 \leq \sup_{\lambda \in \mathbb{T}} ||Th_\lambda||_2$$

and hence $||\sigma_j T|| \leq ||T||$. In particular, $||H_{\sigma_j f}|| \leq ||H_f||$. If $K \in \mathcal{K}$ then also $\sigma_j K \in \mathcal{K}$ and

$$||H_f + K|| \geq ||H_{\sigma_j f} + \sigma_j K|| \geq ||qH_{\sigma_j f}||.$$

This implies $||qH_f|| \geq ||qH_{\sigma_j f}||$. □

A function of the form $\sum_{|k|<j} F_k \xi_k$ for some radial $F_k \in L_\infty$ and $j \in \mathbb{Z}_+$ will be called $L_\infty(d\mu)-$valued trigonometric polynomial.

Definition 3.2. We put

$$Y = \{ f \in L_\infty \ : \ \lim_{j \to \infty} ||H_f - H_{\sigma_j f}|| = 0 \}.$$

Proposition 3.3. *Let*

$$\begin{aligned} Y_1 &= \{ f \in L_\infty \ : \ \textit{there is a sequence } (f_j) \textit{ of } L_\infty - \textit{valued} \\ &\qquad \textit{trigonometric polynomials with } \lim_{j\to\infty} ||qH_f - qH_{f_j}|| = 0 \} \\ Y_2 &= \{ f \in L_\infty \ : \lim_{j\to\infty} ||qH_f - qH_{\sigma_j f}|| = 0 \} \\ Y_3 &= \{ f \in L_\infty \ : \ \textit{there is a sequence } (f_j) \textit{ of } L_\infty - \textit{valued} \\ &\qquad \textit{trigonometric polynomials with } \lim_{j\to\infty} ||H_f - H_{f_j}|| = 0 \} \end{aligned}$$

Then $Y = Y_1 = Y_2 = Y_3$.

Proof. We have $Y \subset Y_1$. It remains to show $Y_1 \subset Y_2 \subset Y_3 \subset Y$.

a) Let $f \in Y_1$. Take $\varepsilon > 0$. Find an $L_\infty-$valued trigonometric polynomial $\tilde{f}$ with $||qH_f - qH_{\tilde{f}}|| \leq \varepsilon/3$. Moreover, find $j_0 \in \mathbb{Z}_+$ with $||\tilde{f} - \sigma_j \tilde{f}||_\infty \leq \varepsilon/3$ if $j \geq j_0$. The we obtain, with Lemma 3.1.,

$$\begin{aligned} ||qH_f - qH_{\sigma_j f}|| &\leq ||qH_f - qH_{\tilde{f}}|| + ||qH_{\tilde{f}} - qH_{\sigma_j \tilde{f}}|| + ||qH_{\sigma_j \tilde{f}} - qH_{\sigma_j f}|| \\ &\leq \frac{\varepsilon}{3} + ||\tilde{f} - \sigma_j \tilde{f}||_\infty + ||qH_{\tilde{f}} - qH_f|| \\ &\leq \varepsilon. \end{aligned}$$

Thus $f \in Y_2$.

b) Let $f \in Y_2$. Find $K_j \in \mathcal{K}$ such that $\lim_{j\to\infty} ||H_f - H_{\sigma_j f} + K_j|| = 0$. Since $\lim_{j\to\infty} ||f \cdot h - (\sigma_j f) \cdot h||_2 = 0$ we conclude $\lim_{j\to\infty} ||K_j h||_2 = 0$ for every $h \in H_2(\mu)$. For any $K \in \mathcal{K}$ the operator $KP : L_2 \to L_2$ is compact. Hence $\mathcal{K}$ can be isometrically embedded into the space $\mathcal{K}(L_2)$ of all compact linear operators on L_2. It is well-known that the dual space of $\mathcal{K}(L_2)$ as a Banach space can be identified with the trace class operators on L_2 ([10]). This implies that $K_jP \to 0$ with respect to the weak topology $\sigma(\mathcal{K}(L_2), \mathcal{K}(L_2)^*)$. So, by Mazur's theorem, the weak and the norm closure of the convex hull of $\{ K_j \ : \ j = n, n+1, \dots \}$ coincide and these sets contain 0. We obtain convex combinations $\tilde{K}_k = \sum_{j=m_k}^{n_k} \lambda_{k,j} K_j$ with $\lim_{k\to\infty} ||\tilde{K}_k|| = 0$ and $\lim_{k\to\infty} m_k = \infty$. Put $f_k = \sum_{j=m_k}^{n_k} \lambda_{k,j}\sigma_j(f)$. Then f_k is an L_∞−valued trigonometric polynomial and we obtain

$$\begin{aligned} ||H_f - H_{f_k}|| &\leq ||H_f - H_{f_k} + \tilde{K}_k|| + ||\tilde{K}_k|| \\ &\leq \max_{m_k \leq j \leq n_k} ||H_f - H_{\sigma_j f} + K_j|| + ||\tilde{K}_k|| \end{aligned}$$

which implies $\lim_{k\to\infty} ||H_f - H_{f_k}|| = 0$. Hence $f \in Y_3$.

c) If $f \in Y_3$ then using an argument analoguous to a) we see that $f \in Y$. □

Corollary 3.4. *Let $f \in L_\infty$.*
(i) If $\lim_{j\to\infty} ||f - \sigma_j f||_\infty = 0$ then $f \in Y$.
(ii) If H_f is compact then $f \in Y$.

Proof. (i) follows from $||H_f - H_{\sigma_j f}|| \leq ||f - \sigma_j f||_\infty$ while, under the assumption of (ii), we have $qH_f = 0$. In the latter case $\lim_{j\to\infty} ||qH_f - qH_{f_j}|| = 0$ if $f_j = 0$.

Let $\mathcal{L}(H_2(\mu))$ and $\mathcal{K}(H_2(\mu))$ be the spaces of all linear, bounded and linear, compact operators $H_2(\mu) \to H_2(\mu)$, resp. For $f \in L_\infty$ we obtain that $H_f^* H_f \in \mathcal{L}(H_2(\mu))$. Let $Q : \mathcal{L}(H_2(\mu)) \to \mathcal{L}(H_2(\mu))/\mathcal{K}(H_2(\mu))$ be the quotient map and $\sigma_{ess}(H_f^* H_f)$ the essential spectrum of $H_f^* H_f$, i.e. the spectrum of $QH_f^* H_f$ in $\mathcal{L}(H_2(\mu))/\mathcal{K}(H_2(\mu))$. □

Definition 3.5. We say that μ satisfies condition (I) if

$$\lim_{m\to\infty} \int_{\mathbb{R}_+} \left| \frac{s^{m-k}}{\int_{\mathbb{R}_+} r^{m-k} d\mu} - \frac{s^m}{\int_{\mathbb{R}_+} r^m d\mu} \right| d\mu(s) = 0 \text{ for all } k \in \mathbb{Z}.$$

μ satisfies condition (II) if

$$\lim_{m\to\infty} \frac{\int_{\mathbb{R}_+} r^{m-k} d\mu \int_{\mathbb{R}_+} r^{m-l} d\mu}{\int_{\mathbb{R}_+} r^m d\mu \int_{\mathbb{R}_+} r^{m-k-l} d\mu} = 1 \text{ for all } k, l \in \mathbb{Z}.$$

It is easily seen that all examples in section 1 satisfy (I) and (II).

Recall that, for $f \in L_\infty$, the functions $p_m[f]$ are uniformly bounded in $L_\infty(d\varphi)$ (see Definition 2.2.) We regard $L_\infty(d\varphi)$ as dual space of $L_1(d\varphi)$ in the usual way and consider the w^*-topology $\sigma(L_\infty(d\varphi), L_1(d\varphi))$. For any free ultrafilter $\mathcal{U}$ on $\mathbb{Z}_+$ the w^*−limit along $\mathcal{U}$ will be denoted by $\lim_{m,\mathcal{U}}$. We put $p_\mathcal{U}[f] = \lim_{m,\mathcal{U}} p_m[f]$, i.e.

$$\int_{\mathbb{T}} p_\mathcal{U}[f](e^{i\varphi}) g(e^{i\varphi}) d\varphi = \lim_{m,\mathcal{U}} \int_{\mathbb{T}} p_m[f](e^{i\varphi}) g(e^{i\varphi}) d\varphi$$

for any $g \in L_1(d\varphi)$. Let

$$\Phi_m[f](e^{i\varphi}) = \frac{\int_{\mathbb{R}_+} f(re^{i\varphi}) r^{2m} d\mu}{\int_{\mathbb{R}_+} r^{2m} d\mu}.$$

Then $\Phi_m[f] \in L_\infty(d\varphi)$ if $f \in L_\infty$. Furthermore let $\Phi_{\mathcal{U}}[f]$ be the w^*-limit of the $\Phi_m[f]$ along $\mathcal{U}$. Then

$$p_m[f] = \Phi_m[|f|^2] - |\Phi_m[f]|^2 \text{ and } p_{\mathcal{U}}[f] = \Phi_{\mathcal{U}}[|f|^2] - |\Phi_{\mathcal{U}}[f]|^2$$

Theorem 3.6. *Let μ satisfy (I) and (II). If $f \in Y$ then the uniform limit $\lim_{j\to\infty} p_{\mathcal{U}}[\sigma_j f]$ exists on $\mathbb{T}$. We have*

$$\sigma_{ess}(H_f^*H_f) = \{ \lim_{j\to\infty} p_{\mathcal{U}}[\sigma_j f](\lambda) \ : \lambda \in \mathbb{T}, \mathcal{U} \textit{ a free ultrafilter on } \mathbb{Z}_+ \}$$

Moreover,

$$||qH_f|| = \lim_{j\to\infty} \sup\{ \sqrt{||p_{\mathcal{U}}[\sigma_j f]||_\infty} \ : \ \mathcal{U} \textit{ a free ultrafilter on } \mathbb{Z}_+ \}$$

Proof. At first assume that f is an $L_\infty(d\mu)-$valued trigonometric polynomial. Then $|f|^2$ is an $L_\infty(d\mu)-$valued trigonometric polynomial, too. We obtain

$$f \in X := \{ g \in L_\infty \ : \ \lim_{j\to\infty} ||T_g - T_{\sigma_j g}|| = 0 \}$$

and also $|f|^2 \in X$. (Recall, T_g is the Toeplitz operator with symbol g.) It was shown in [9] that, under conditions (I) and (II), the C^*-subalgebra $\mathcal{A}$ of $\mathcal{L}(H_2(\mu))/\mathcal{K}(H_2(\mu))$ generated by $\{QT_g : g \in X\}$ is commutative. If Spec $\mathcal{A}$ is the set of all multiplicative linear non-zero functionals on $\mathcal{A}$ then, for any $g \in X$,

$$\{ \Phi(QT_g) \ : \ \Phi \in \text{ Spec } \mathcal{A} \} = \{ \Phi_{\mathcal{U}}[g](\lambda) \ : \mathcal{U} \text{ a free ultrafilter on } \mathbb{Z}_+, \lambda \in \mathbb{T} \}.$$

Finally, for any $g \in X$, $\Phi_{\mathcal{U}}[g]$ is continuous on $\mathbb{T}$.

A straightforward computation yields, for arbitrary $f \in L_\infty$,

$$H_f^*H_f = T_{|f|^2} - T_{\bar{f}}T_f.$$

So, if f is an $L_\infty(d\mu)-$valued trigonometric polynomial then $Q(H_f^*H_f) \in \mathcal{A}$. If $f \in Y$ is arbitrary then $\lim_{j\to\infty} ||H_f^*H_f - H_{\sigma_j f}^*H_{\sigma_j f}|| = 0$. Hence again $Q(H_f^*H_f) \in \mathcal{A}$. Furthermore, if $\Phi \in$ Spec $\mathcal{A}$ such that $\Phi(QT_g) = \Phi_{\mathcal{U}}[g](\lambda)$ for all $g \in X$, then

$$\Phi(QH_f^*H_f) = \lim_{j\to\infty} (\Phi_{\mathcal{U}}[|\sigma_j f|^2](\lambda) - |\Phi_{\mathcal{U}}[\sigma_j f]|^2(\lambda)) =: \gamma(\lambda).$$

γ is the uniform limit of the trigonometric polynomials

$$\Phi_{\mathcal{U}}[|\sigma_j f|^2] - |\Phi_{\mathcal{U}}[\sigma_j f]|^2 = p_{\mathcal{U}}[\sigma_j f]$$

on $\mathbb{T}$, hence γ is continuous. We conclude

$$\sigma_{ess}(H_f^*H_f) = \{ \lim_{j\to\infty} p_{\mathcal{U}}[\sigma_j f](\lambda) \ : \ \lambda \in \mathbb{T}, \mathcal{U} \text{ a free ultrafilter on } \mathbb{Z}_+ \}.$$

Finally, since $\sqrt{||QH_f^*H_f)||} = ||qH_f||$, we obtain the last part of the theorem. □

Corollary 3.7. *Let $f \in L_\infty$ be such that $\lim_{j\to\infty} ||f - \sigma_j f||_\infty = 0$. If μ satisfies (I) and (II) then*

$$\sigma_{ess}(H_f^* H_f) = \{\ p_{\mathcal{U}}[f](\lambda)\ :\ \mathcal{U} \text{ a free ultrafilter on } \mathbb{Z}_+,\ \lambda \in \mathbb{T}\ \}$$

and

$$||qH_f|| = \limsup_{m\to\infty} \sqrt{||p_m[f]||_\infty}.$$

Proof. We have $||qH_f - qH_{\sigma_j f}|| \le ||f - \sigma_j f||_\infty$, and hence, by assumption, $f \in Y$. Moreover $|||\sigma_j f|^2 - |f|^2||_\infty \le 2||f||_\infty ||\sigma_j f - f||_\infty$ which implies

$$||p_m[f] - p_m[\sigma_j f]||_\infty \ \le 4||f||_\infty ||f - \sigma_j f||_\infty$$

and

$$||p_{\mathcal{U}}[f] - p_{\mathcal{U}}[\sigma_j f]||_\infty \ \le 4||f||_\infty ||f - \sigma_j f||_\infty.$$

We obtain $p_{\mathcal{U}}[f] = \lim_{j\to\infty} p_{\mathcal{U}}[\sigma_j f]$ and, in view of the theorem,

$$\sigma_{ess}(H_f^* H_f) = \{\ p_{\mathcal{U}}[f](\lambda)\ :\ \mathcal{U} \text{ a free ultrafilter on } \mathbb{Z}_+,\ \lambda \in \mathbb{T}\ \}.$$

$\sigma_j f$ is an $L_\infty(d\mu)-$valued trigonometric polynomial and $\{p_m[\sigma_j f] : m \in \mathbb{Z}_+\}$ is equicontinuous on $\mathbb{T}$ since it consists of trigonometric polynomials with a uniformly bounded number of summands. Theorem 3.6. yields

$$\begin{aligned} ||qH_f|| &= \lim_{j\to\infty} ||qH_{\sigma_j f}|| \\ &= \lim_{j\to\infty} \sup\{\ \sqrt{||p_{\mathcal{U}}[\sigma_j f]||_\infty}\ :\ \mathcal{U} \text{ a free ultrafilter on } \mathbb{Z}_+\ \} \\ &= \lim_{j\to\infty} \limsup_{m\to\infty} \sqrt{||p_m[\sigma_j f]||_\infty} \\ &= \limsup_{m\to\infty} \sqrt{||p_m[f]||_\infty}. \end{aligned}$$

It seems likely that Corollary 3.7. remains true for arbitrary $f \in Y$. □

4 Compactness conditions

We study compactness conditions for H_f which are ultimately consequences of Theorem 3.6. However, here we want to derive our results under assumptions which are weaker than conditions (I) and (II).

Definition 4.1. We say that μ satisfies condition (III) if

$$\lim_{m\to\infty} \frac{(\int_{\mathbb{R}_+} r^{2m-k} d\mu)^2}{\int_{\mathbb{R}_+} r^{2m} d\mu \int_{\mathbb{R}_+} r^{2m-2k} d\mu} = 1$$

for all $k \in \mathbb{Z}$.

Condition (III) is a special case of condition (II) (see Definition 3.5).

Lemma 4.2. *Let $k \in \mathbb{Z}$. Then H_{ξ_k} is compact if and only if*

$$\lim_{m\to\infty} \frac{(\int_{\mathbb{R}_+} r^{2m-k} d\mu)^2}{\int_{\mathbb{R}_+} r^{2m} d\mu \int_{\mathbb{R}_+} r^{2m-2k} d\mu} = 1.$$

Proof. Corollary 2.3. yields, for $h = \sum_{l\in\mathbb{Z}_+} \beta_l e_l \in H_2(\mu)$,

$$||H_{\xi_k} h||_2^2 = \sum_{m\in\mathbb{Z}_+, m\geq k} p_{m,k}[1]|\beta_{m-k}|^2 + \sum_{m\in\mathbb{Z}\setminus\mathbb{Z}_+, m\geq k} p_{m,k}[1]|\beta_{m-k}|^2.$$

There are at most finitely many $m \in \mathbb{Z}\setminus\mathbb{Z}_+$ with $m \geq k$. Hence H_{ξ_k} is compact if and only if

$$\lim_{m\to\infty} p_{m,k}[1] = \lim_{m\to\infty}(1 - \frac{(\int_{\mathbb{R}_+} r^{2m-k} d\mu)^2}{\int_{\mathbb{R}_+} r^{2m} d\mu \int_{\mathbb{R}_+} r^{2m-2k} d\mu}) = 0.$$

□

Theorem 4.3. *Let μ satisfy (III). Then, for any $f \in L_\infty$ with $f \overset{(L_2)}{=} \sum_k F_k \xi_k$, the following are equivalent:*
a. H_f is compact
b. $f \in Y$ and one of the following (equivalent) conditions is satisfied:
(i) $H_{F_k \xi_k}$ is compact for all $k \in \mathbb{Z}$
(ii) H_{F_k} is compact for all $k \in \mathbb{Z}$
(iii) $\lim_{m\to\infty} p_m[F_k] = 0$ for all $k \in \mathbb{Z}$
(iv) $\lim_{m\to\infty} \int_{\mathbb{T}} p_m[\sigma_j f](e^{i\varphi}) g(e^{i\varphi}) d\varphi = 0$ for all $g \in L_1(d\varphi)$ and all $j \in \mathbb{Z}_+$
(v) $\lim_{m\to\infty} \int_{\mathbb{T}} p_m[\sigma_j f]^p d\varphi = 0$ for all $p \in]0,\infty[$ and all $j \in \mathbb{Z}_+$
(vi) $\lim_{m\to\infty} \int_{\mathbb{T}} p_m[\sigma_j f]^p d\varphi = 0$ for all $j \in \mathbb{Z}_+$ and some $p \in]0,\infty[$

Proof. $a. \Rightarrow b.(i)$: Assume that H_f is compact. Then Proposition 2.1. yields

$$H_{F_k\xi_k} h = \int_{\mathbb{T}} (H_f h_{e^{-i\varphi}})_{e^{i\varphi}} e^{-ik\varphi} d\varphi, \quad h \in H_2(\mu).$$

So, if $h_\alpha \to 0$ weakly then $\lim_\alpha ||H_{F_k\xi_k} h_\alpha||_2 = 0$ which implies that $H_{F_k\xi_k}$ is compact. Moreover, $f \in Y_2 = Y$ by Proposition 3.3.

From now on assume that $f \in Y$. Then we show
$(i) \Leftrightarrow (ii)$: By Definition 1.2., for $h \in H_2(\mu)$,

$$\begin{aligned}(H_{F_k\xi_k} - H_{F_k} T_{\xi_k})h &= (M_{F_k\xi_k} - PM_{F_k\xi_k} - (M_{F_k} - PM_{F_k})PM_{\xi_k})h \\ &= (M_{F_k} - PM_{F_k})H_{\xi_k} h.\end{aligned}$$

(We used $M_{F_k\xi_k} = M_{f_k} M_{\xi_k}$). By Lemma 4.2., in view of (III), H_{ξ_k} is compact. Hence $H_{F_k\xi_k} - H_{F_k} T_{\xi_k}$ is compact. This implies $q(H_{F_k\xi_k}) = q(H_{F_k}) q(T_{\xi_k})$. By (III) and Lemma 4.2., $0 = qH_{\xi_k} = qM_{\xi_k} - qT_{\xi_k}$. Hence $qT_{\xi_k} = qM_{\xi_k}$ is invertible and $(qT_{\xi_k})^{-1} = qT_{\xi_{-k}}$. We obtain $qH_{F_k} = 0$ if and only if $qH_{F_k\xi_k} = 0$.
$(ii) \Leftrightarrow (iii)$: By Corollary 2.3., for $h = \sum_{l\in\mathbb{Z}_+} \beta_l e_l \in H_2(\mu)$,

$$\langle H_{F_k} h, H_{F_k} h\rangle = \sum_{m\in\mathbb{Z}_+} p_m[F_k]|\beta_m|^2.$$

This implies that H_{F_k} is compact if and only if $\lim_{m\to\infty} p_m[F_k] = 0$.
$(iii) \Rightarrow (iv)$: By definition, $\sqrt{p_m[\cdot](e^{i\varphi})}$ is subadditive, i.e.

$$\sqrt{p_m[\sigma_j f](e^{i\varphi})} \le \sum_{|k|<j} \frac{j-|k|}{j} \sqrt{p_m[F_k]} \text{ for any } j.$$

(We use $p_{\mathcal{U}}[F_k \xi_k] = p_{\mathcal{U}}[F_k]$.) This implies

$$p_m[\sigma_j f](e^{i\varphi}) \le \sum_{|k|<j} c p_m[F_k]$$

where $c > 0$ is a suitable constant depending on j but not on m. Using (iii) and the fact that $p_m[\sigma_j f] \ge 0$ we obtain (iv).
$(iv) \Rightarrow (v)$: We have, using Hölder inequality, since $p_m[f] \ge 0$,

$$0 \le \int_{\mathbb{T}} p_m[f]^p d\varphi \le \begin{cases} (\int_{\mathbb{T}} p_m[f] d\varphi)^p & \text{if} \quad 0 < p \le 1 \\ (2||f||_\infty^2)^{p-1} \int_{\mathbb{T}} p_m[f] d\varphi & \text{if} \quad 1 < p < \infty \end{cases}$$

Hence (iv) with $g = 1$ implies (v).
$(v) \Rightarrow (vi)$ is clear.
$(vi) \Rightarrow a.$: We claim $\lim_{m\to\infty} p_m[F_k] = 0$ for all k. Indeed, fix $k \in \mathbb{Z}$ and let

$$G_j(r) = \sum_{l=1}^{j} \frac{1}{j} f(re^{i2\pi l/j}) e^{-i2\pi lk/j}.$$

Then

$$\lim_{j\to\infty} G_j(r) = \int_{\mathbb{T}} f(re^{i\varphi}) e^{-ik\varphi} d\varphi = F_k(r).$$

The dominated convergence theorem implies

$$\lim_{j\to\infty} ||F_k - G_j||_4 = \lim_{j\to\infty} (\int_{\mathbb{R}_+} |G_j(r) - F_k(r)|^4 d\mu)^{1/4} = 0.$$

Moreover

$$|\sqrt{p_m[F_k]} - \sqrt{p_m[G_j]}| \le \sqrt{p_m[F_k - G_j]} \le \sqrt{2} ||F_k - G_j||_4 \frac{(\int_{\mathbb{R}_+} r^{4m} d\mu)^{1/4}}{(\int_{\mathbb{R}_+} r^{2m} d\mu)^{1/2}}.$$

Hence

$$\begin{aligned} \sqrt{p_m[F_k]} &= \lim_{j\to\infty} \sqrt{p_m[G_j]} \\ &\le \lim_{j\to\infty} \sum_{l=1}^{j} \frac{1}{j} \sqrt{p_m[f](e^{i2\pi l/j})} \\ &= \int_{\mathbb{T}} \sqrt{p_m[f](e^{i\varphi})} d\varphi \\ &\le \begin{cases} (\int_{\mathbb{T}} p_m[f]^p d\varphi)^{1/(2p)} & \text{if} \quad \frac{1}{2} \le p \\ (2||f||_\infty^2)^{1/2-p} \int_{\mathbb{T}} p_m[f]^p d\varphi & \text{if} \quad 0 < p < \frac{1}{2} \end{cases} \end{aligned}$$

We conclude $\lim_{m\to\infty} p_m[F_k] = 0$. By what we have proved already we obtain that $H_{F_k\xi_k}$ is compact for all k. Therefore $H_{\sigma_j f}$ is compact for all j. Since $f \in Y$, H_f is compact. □

Corollary 4.4. *Let $f \in L_\infty$ such that $\lim_{j\to\infty} ||f - \sigma_j f||_\infty = 0$. Then the following are equivalent*
(a) H_f is compact
(b) $\lim_{m\to\infty} p_m[f] = 0$ uniformly on $\mathbb{T}$
(c) $\lim_{m\to\infty} \int_{\mathbb{T}} p_m[f](e^{i\varphi})g(e^{i\varphi})d\varphi = 0$ for all $g \in L_1(d\varphi)$
(d) $\lim_{m\to\infty} \int_{\mathbb{T}} p_m[f]^p d\varphi = 0$ for all $p \in]0,\infty[$
(e) $\lim_{m\to\infty} \int_{\mathbb{T}} p_m[f]^p d\varphi = 0$ for some $p \in]0,\infty[$.

Proof. Everything follows from Theorem 4.3. and the fact that here

$$||p_m[f] - p_m[\sigma_j f]||_\infty \leq 4||f||_\infty ||f - \sigma_j f||_\infty \quad \text{i.e.}$$

$$\lim_{j\to\infty} ||p_m[f] - p_m[\sigma_j f]||_\infty = 0 \quad \text{uniformly in } m.$$

□

Examples 4.5. (1) We start with the Hardy space $H_2 = H_2(\mu)$ where $\mu = \delta_1$. Here, for any $f \in L_\infty$ and all m, we have $p_m[f] = 0$. Hence $Y = \{ f \in L_\infty : H_f \text{ is compact} \}$.

Let $L_1 = L_1(d\varphi \otimes d\mu)$ and put

$$\begin{aligned} H_\infty &= w^* - \text{closure of } \{ p : \mathbb{C} \to \mathbb{C} : p \text{ a polynomial} \} \text{ in } L_\infty = L_1^*, \\ C &= \text{closure of } \{ p : \mathbb{C} \to \mathbb{C} : p \text{ a polynomial} \} \text{ in } L_\infty. \end{aligned}$$

Hence

$$\begin{aligned} C &= \{ f \in L_\infty : \lim_{j\to\infty} ||f - \sigma_j f||_\infty = 0 \} \\ &\cong \{ f : \mathbb{T} \to \mathbb{C} : f \text{ continuous} \}. \end{aligned}$$

Clearly, $C \subset Y$ and $H_\infty \subset Y$ since $H_h = 0$ if $h \in H_\infty$.

Moreover, put

$$\begin{aligned} H_1 &= \text{closure of } \{ p : \mathbb{C} \to \mathbb{C} : p \text{ a polynomial} \} \text{ in } L_1 \text{ and} \\ H_1^0 &= \{ h \in H_1 : \langle h, e_0 \rangle = 0 \}. \end{aligned}$$

It is easy to see that $(H_1^0)^* \cong L_\infty / H_\infty$. It is well-known that, for each $h \in H_1$, there are $h_1, h_2 \in H_2$ with $h = h_1 \cdot h_2$ and $||h_1||_2 = ||h_2||_2 = \sqrt{||h||_1}$. If in addition $\langle h, e_0 \rangle = 0$ then we can assume $\langle h_2, e_0 \rangle = 0$ (see [7]). Finally, here we have

$$(id - P)L_2 = \{ \bar{h} : h \in H_2, \langle h, e_0 \rangle = 0 \}.$$

Hence, for $f \in L_\infty$, we derive the Nehari theorem,

$$\begin{aligned} ||H_f|| &= \sup\{ |\int_{\mathbb{R}_+} \int_{\mathbb{T}} f h_1 h_2 d\varphi d\mu| : h_1, h_2 \in H_2, \\ &\qquad ||h_1||_2, ||h_2||_2 \leq 1, \langle h_2, e_0 \rangle = 0 \} \\ &= ||f + H_\infty||_\infty \text{ (norm in } L_\infty / H_\infty). \end{aligned}$$

Thus, $\{H_f : f \in Y\}$ is isometric to a subspace of the closure of $\{g + H_\infty : g \in C\} \subset (H_\infty + C)/H_\infty$. We conclude

$$Y = H_\infty + C = \{ f \in L_\infty \ : \ H_f \text{ compact } \}.$$

The second equality constitutes the well-known compactness characterization of Hankel operators on the Hardy space (see [6]).

(2) Consider the Bergman space where $d\mu = 1_{[0,1]} r dr$. There is a known characterization of compact Hankel operators, due to Stroethoff and Zheng (see [12]):

Let, for some $\lambda \in D$, $\varphi_\lambda(z) = (\lambda - z)/(1 - \bar{\lambda} z)$, $z \in D$. Then, for $f \in L_\infty$, H_f is compact if and only if

$$\lim_{|\lambda| \to 1} ||(id - P)(f \circ \varphi_\lambda)||_2 = 0.$$

(Recall, $f \in L_2$, hence $(id - P)f$ makes sense.) There are other, more complicated compactness characterizations involving the bounded mean oscillation with respect to the Bergman metric (see [4], [2] and [14], Chap. 7).

Our condition here reads, for $f \in L_\infty$, H_f is compact if and only if $f \in Y$ and

$$\lim_{m \to \infty} ((2m+2) \int_0^1 |F_k(r)|^2 r^{2m+1} dr - (2m+2)^2 | \int_0^1 F_k(r) r^{2m+1} dr|^2) = 0$$

for all Fourier coefficients F_k of f. This condition is quite convenient if one wants to check explicitly the compactness of H_f for those f which are known to be the elements of Y, for example where $\lim_{j \to \infty} ||f - \sigma_j f||_\infty = 0$. This includes many discontinuous f.

We demonstrate this at the following example of a radial function F (which is automatically in Y) with $F(r) \in \{1, -1\}$ for all r. Use induction to find integers $0 < m_1 < m_2 \dots$ such that, with $a_n = 1 - 1/n$,

$$|\Phi_{m_n}[F \cdot 1_{[0,a_n]}]| = (2m_n + 2) | \int_0^{a_n} F(r) r^{2m_n+1} dr| \leq \frac{1}{n}.$$

Indeed, if we have $F|_{[0,a_n]}$ already, we use $|\Phi_m[F 1_{[0,a_n]}]| \leq \Phi_m[1_{[0,a_n]}]$ and $\lim_{m \to \infty} \Phi_m[1_{[0,a_n]}] = 0$ to find $m_{n+1} > m_n$ with $|\Phi_{m_{n+1}}[F 1_{[0,a_n]}]| \leq (n+1)^{-1}$. Then we define $F(r) \in \{1, -1\}$ for $r \in]a_n, a_{n+1}]$ such that $|\Phi_{m_{n+1}}[F 1_{]a_n, a_{n+1}]}]| = 0$. This implies $|F| = 1$ and $\lim_{n \to \infty} \Phi_{m_n}[F] = 0$. Hence $\lim_{n \to \infty} p_{m_n}[F] = 1$. Thus $F \in Y$ but H_F is not compact. On the other hand, $H_{|F|} = H_1 = 0$ is compact. (Such an example was mentioned without proof in [1] where it was attributed to Sarason.)

(3) Let μ be such that, for some $a > 0$, $a \in$ supp $\mu \ \subset [0, a]$. It was shown in [9] that μ satisfies conditions (I) and (II). Let $F \in L_\infty$ be radial such that $\lim_{r \to a} F(r) = F(a)$. Then an elementary computation shows that $\lim_{m \to \infty} \Phi_m[F] = F(a)$. This implies $\lim_{m \to \infty} p_m[F] = 0$. Hence H_F is compact. Let $f \in Y$ be such that $\lim_{r \to a} f(re^{i\varphi}) = f(ae^{i\varphi})$ for all φ. If $f \overset{(L_2)}{=} \sum_k F_k \xi_k$ then $\lim_{r \to a} F_k(r) = F_k(a)$ for all k. Hence H_f is compact (cf. [13]).

(4) Now we turn to the Fock space where $d\mu = re^{-r^2/2} dr$. In [11] it was shown that, for

$f \in L_\infty$, H_f is compact if and only if

$$\lim_{|\lambda|\to\infty} ||(id - P)(f \circ \tau_\lambda)||_2 = 0$$

where $\tau_\lambda(z) = z + \lambda$.

The following compactness characterization is due to Berger and Coburn ([3]). If $f \in L_\infty$ then H_f is compact if and only if $f = f_1 + f_2$ where $f_1, f_2 \in L_\infty$,

$$\lim_{|w|\to\infty} \int_0^\infty \int_0^{2\pi} |f_1(re^{i\varphi})|^2 \exp(-|re^{i\varphi} - w|^2/2) r d\varphi dr = 0$$

and

$$\lim_{R\to\infty} \sup_{\substack{|z-w|\le 1 \\ |z|\ge R}} |f_2(z) - f_2(w)| = 0.$$

Our condition reads, H_f is compact if and only if $f \in Y$ and

$$\lim_{m\to\infty} \Big(\frac{\int_0^\infty |F_k(r)|^2 r^{2m+1} e^{-r^2/2} dr}{m!2^m} - \frac{|\int_0^\infty F_k(r) r^{2m+1} e^{-r^2/2} dr|^2}{(m!)^2 2^{2m}} \Big) = 0$$

for all Fourier coefficients F_k of f which is particularly handy in special situations. (We used $\int_0^\infty r^{2m+1} e^{-r^2/2} dr = m!2^m$.) For example, similarly as in (1), we find a radial $F \in Y$ where H_F is non-compact. Moreover, if $f(z) = e^{z/|z|}$ for $z \neq 0$, then we check easily that H_f is compact since here $\lim_{j\to\infty} ||f - \sigma_j f||_\infty = 0$ and the Fourier coefficients of f are $F_k(r) = 1_{]0,1]}/(k!)$, $k = 0, 1, 2, \ldots$.

5 Schatten class Hankel operators

Let $f \in L_\infty$. If $H_f^* H_f$ is compact then there are numbers $\lambda_k > 0$ and an orthonormal system (h_k) in $H_2(\mu)$ such that

$$H_f^* H_f h = \sum_{k\in\mathbb{Z}_+} \lambda_k \langle h, h_k \rangle h_k \quad \text{for all } h \in H_2(\mu).$$

Fix $p \in]0, \infty[$. Following [14] we say that $H_f \in \mathcal{S}_p$ if $\sum_{k\in\mathbb{Z}_+} |\lambda_k|^{p/2} < \infty$.

Theorem 5.1. *Let $f \in L_\infty$, $f \overset{(L_2)}{=} \sum_k F_k \xi_k$. Then $H_f \in \mathcal{S}_2$ (i.e. H_f is a Hilbert-Schmidt operator) if and only if*

$$\sum_{k\in\mathbb{Z}} \sum_{\substack{m\in\mathbb{Z} \\ m\ge k}} p_{m,k}[F_k] < \infty.$$

Proof. We have $H_f \in \mathcal{S}_2$ if and only if

$$\sum_{l\in\mathbb{Z}_+} \langle H_f^* H_f e_l, e_l \rangle = \sum_{l\in\mathbb{Z}_+} \langle H_f e_l, H_f e_l \rangle < \infty$$

(see e.g. [14], Theorems 1.4.3, 1.4.7.). Using Proposition 2.1. we see that the latter condition is equivalent to

$$\sum_{l\in\mathbb{Z}_+}\sum_{m\in\mathbb{Z}} p_{m,m-l}[F_{m-l}] < \infty.$$

Applying Proposition 2.1. once more we also obtain □

Corollary 5.2. *Let $f \in L_\infty$, $f \overset{(L_2)}{=} \sum_k F_k\xi_k$. If $H_f \in \mathcal{S}_2$ then $H_f = \sum_{k\in\mathbb{Z}} H_{F_k\xi_k}$ (convergence with respect to the operator norm).*

Proposition 5.3. *(a) Let $F \in L_\infty$ be radial and let $k \in \mathbb{Z}$. Then, for any $p \in]0,\infty[$,*

$$H_{F\xi_k} \in \mathcal{S}_p \quad \text{if and only if} \quad \sum_{\substack{m\in\mathbb{Z}\\ m\geq k}} p_{m,k}[F]^{p/2} < \infty.$$

(b) Let $f \overset{(L_2)}{=} \sum_k F_k\xi_k \in L_\infty$. If $p \geq 2$ and $H_f \in \mathcal{S}_p$ then $H_{F_k\xi_k} \in \mathcal{S}_p$ for all k.

Proof. (a): Using Proposition 2.1. we obtain, for any $h \in H_2(\mu)$,

$$\begin{aligned} H^*_{F\xi_k}H_{F\xi_k}h &= \sum_{l\in\mathbb{Z}_+}\langle H_{F\xi_k}h, H_{F\xi_k}e_l\rangle e_l \\ &= \sum_{\substack{m\in\mathbb{Z}\\ m\geq k}} p_{m,k}[F]\langle h, e_{m-k}\rangle e_{m-k} \end{aligned}$$

Hence $H_{F\xi_k} \in \mathcal{S}_p$ if and only if

$$\sum_{\substack{m\in\mathbb{Z}\\ m\geq k}} p_{m,k}[F]^{p/2} < \infty.$$

(b): Here, in view of Proposition 2.1., we have

$$\sum_{l\in\mathbb{Z}_+}\langle H_f^*H_fe_l, e_l\rangle^{p/2} = \sum_{l\in\mathbb{Z}_+}\Big(\sum_{m\in\mathbb{Z}} p_{m,m-l}[F_{m-l}]\Big)^{p/2} < \infty$$

by [14], Theorem 1.4.7. With $k = m-l$, (b) follows from (a).

Since $\mathcal{S}_p$ is a vector space we obtain □

Corollary 5.4. *Let $f = \sum_{|k|<j} F_k\xi_k$ be an $L_\infty(d\mu)$−valued trigonometric polynomial and let $p \geq 2$. Then*

$$H_f \in \mathcal{S}_p \quad \text{if and only if} \quad \sum_{|k|<j}\sum_{m\geq k} p_{m,k}[F_k]^{p/2} < \infty.$$

6 Hankel operators with unbounded symbols

Let $\operatorname{supp}\mu \subset \mathbb{R}_+$ be bounded. Then any polynomial on $\mathbb{C}$ is essentially bounded with respect to $d\varphi \otimes d\mu$. In this case we can extend Definition 1.2. to the case of $f \in L_2$. Then M_f, H_f and T_f are densely defined. Proposition 2.1. remains true for $f \in L_2$ (where $h \in H_2(\mu)$

is a polynomial). Definition 2.2. can be extended literally to $f \in L_2$. As a consequence of Proposition 2.1. we obtain

Proposition 6.1. *Let supp μ be bounded. Moreover let $F \in L_2$ be radial and $k \in \mathbb{Z}$. Then*
(a) $H_{F\xi_k}$ is bounded if and only if $\sup_{m \geq k} p_{m,k}[F] < \infty$.
(b) $H_{F\xi_k}$ is compact if and only if $\lim_{m \to \infty} p_{m,k}[F] = 0$.

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