Note di Matematica Vol. 18 - n. 1, 131-141 (1998)

NEW STRUCTURES ON TANGENT BUNDLES

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Abstract. *We construct three kinds of almost complex structures compatible with some naturally defined Riemannian metrics on the tangent bundle of an almost Hermitian manifold and in particular, of a complex space form. We use these structures to give a class of hyperhermitian structures and a one-parameter family of hyperkähler structures on the tangent bundle of a complex space form of positive holomorphic sectional curvature.*

0. INTRODUCTION

Let TM be the tangent bundle of a connected Riemannian manifold (M,g). Recently, starting from the construction of Riemannian metrics on TM, Oproiu [9] has given almost Hermitian metrics on TM and moreover, by computing the Kähler condition and the Ricci tensors, he has constructed a Kähler-Einstein metric on TM where M is a space form of negative curvature. On the other hand, the first and third author [13] have independently constructed a larger class of almost Hermitian structures and of Hermitian structures on TM by a different approach, starting from the construction of almost complex structures.

The purpose of the present paper is to determine new geometric structures defined on the tangent bundles of general almost Hermitian manifolds and of complex space forms.

In Section 1, we recall some preliminaries about tangent bundles. Then, in Sections 2, 3 and 4, we construct three kinds of almost Hermitian structures on the tangent bundle TM of an almost Hermitian manifold (M,J,g). The first class of almost Hermitian structures on TM is an almost Hermitian version of the Riemannian case treated in [13]. The second class of almost Hermitian structures on TM is constructed by using the almost Hermitian structure of the base manifold. The last class is given by a generalization of the horizontal lift of J. In Section 5, we use these structures to determine almost hypercomplex structures and also hyperhermitian structures when the base manifold is a complex space form. This yields a one-parameter family of hyperkähler structures when the constant holomorphic sectional curvature is positive.

Our main purpose is to present these examples in a concise way. For that reason we omit most of the tedious but rather straightforward computations.

1. PRELIMINARIES

First, we recall some basic facts concerning the Riemannian geometry of the tangent bundle of a Riemannian manifold. Let (M,g) be an n-dimensional Riemannian manifold with Levi

[1] 1991 *Mathematics Subject Classification.* 53C15. *Keywords and phrases.* Tangent bundles; (almost) complex and Hermitian structures; Kähler structures; (almost) hypercomplex and hyperhermitian structures; hyperkähler structures.

Civita connection ∇. The Riemannian curvature tensor R is given by

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

for all vector fields $X, Y \in \mathcal{X}(M)$, the Lie algebra of smooth vector fields on M. We denote by T_pM the tangent space of M at $p \in M$, by TM the tangent bundle of M and by π the natural projection of TM onto M.

An almost Hermitian structure on a manifold M is, by definition, a pair (J, g) formed by a tensor field J of type (1,1) and a Riemannian metric g, satisfying

$$J^2(X) = -X, \; g(JX, JY) = g(X, Y)$$

for all vector fields $X, Y \in \mathcal{X}(M)$. J (resp. g) is called an almost complex structure (resp. an almost Hermitian metric). We denote by Ω the Kähler form, given by $\Omega(X, Y) = g(JX, Y)$ for all vector fields $X, Y \in \mathcal{X}(M)$. The almost Hermitian structure (J, g) is said to be almost Kählerian if Ω is closed, that is, $d\Omega$=0.

Furthermore, if J is an almost complex structure on M and $p \in M$, let $T_p^+(M, J)$ denote the eigenspace of J_p corresponding to the eigenvalue $\sqrt{-1}$ and $\chi^+(M, J)$ the set of complex tangent vector fields of type (1,0) on M. We note that J is a complex structure if and only if $[\chi^+(M, J), \chi^+(M, J)] \subset \chi^+(M, J)$ (cf. [8, Chap.IX, Theorems 2.5 and 2.8]). Such an almost complex structure is called integrable. If an almost complex structure J is integrable, then the almost Hermitian structure (J, g) is said to be Hermitian. Moreover, if the Kähler form of such a structure is closed, it is called a Kähler structure. It is well-known that an almost Hermitian structure (J, g) is a Kähler structure if and only if J is parallel with respect to ∇, that is, $\nabla J = 0$. Furthermore, a Kähler manifold (M, J, g) is called a complex space form of constant holomorphic sectional curvature $4c$ if the curvature tensor R satisfies

$$\begin{aligned} R(X,Y)Z = c\{&g(Y,Z)X - g(X,Z)Y \\ &-g(Y,JZ)JX + g(X,JZ)JY + 2g(X,JY)JZ\} \end{aligned} \tag{1.1}$$

for all vector fields $X, Y, Z \in \mathcal{X}(M)$.

Now, let $K : TTM \longrightarrow TM$ be the connection map corresponding to ∇ [5]. For each $u \in T_pM$, $p \in M$, we denote by $T_u^H TM$ (resp. $T_u^V TM$) the kernel of $K|_{T_uTM}$ (resp. $d\pi|_{T_uTM}$). This is an n-dimensional subspace of T_uTM, called the horizontal subspace (resp. the vertical subspace) of T_uTM. We then have a direct sum decomposition

$$T_uTM = T_u^H TM \oplus T_u^V TM.$$

The elements of $T_u^H TM$ (resp. $T_u^V TM$) are said to be horizontal vectors (resp. vertical vectors) at u. For each $u, X \in T_pM$, X_u^H (resp. X_u^V) denotes the horizontal lift (resp. the vertical lift) of X to T_uTM. These lifts are determined by

$$K(X_u^H) = 0, \; d\pi(X_u^H) = X, \; K(X_u^V) = X, \; d\pi(X_u^V) = 0. \tag{1.2}$$

The canonical almost complex structure J_0 on the tangent bundle TM of a Riemannian manifold (M, g) is defined by

$$J_0(X_u^H) = X_u^V, \; J_0(X_u^V) = -X_u^H. \tag{1.3}$$

J_0 is integrable if and only if M is locally flat [5].

Recently, by a modification of J_0, Kähler structures have been constructed on the tangent bundle of a real space form, a complex projective n-space $\mathbf{C}P^n$ and a quaternionic projective n-space $\mathbf{H}P^n$ (see [6],[7],[9],[11] and references therein, also for other constructions).

Furthermore, Sasaki introduced in [10] the well-known canonical metric G_0 of the tangent bundle of a Riemannian manifold (M,g). This Sasaki metric is given by

$$G_0(X^H,Y^H) = G_0(X^V,Y^V) = g(X,Y)\circ\pi,\ G_0(X^H,Y^V) = 0 \tag{1.4}$$

for all vector fields $X, Y \in \mathcal{X}(M)$ [5]. Tachibana and Okumura [5] remarked that the almost Hermitian structure (J_0, G_0) is an almost Kähler structure which is not Kählerian unless the base manifold M is locally flat.

A vector field along π is, by definition, a map $A : TM \longrightarrow TM$ satisfying $\pi\circ A = \pi$. If A is a vector field along π, A^H (resp. A^V) denotes the vector field on TM obtained by the horizontal lift (resp. the vertical lift) of A, that is, $A^H : u \longmapsto A^H_u := A(u)^H_u$ (resp. $A^V : u \longmapsto A^V_u := A(u)^V_u$). Any horizontal vector field (resp. vertical vector field) X on TM can be written as $X = A^H$ (resp. $X = A^V$) for a unique vector field A along π. Such an A is given by $A(u) = d\pi(X_u)$ (resp. $A(u) = K(X_u)$). The composition of two vector fields along π is also a vector field along π. If A is a vector field along π, then the covariant derivative of A in the direction of $\xi \in T_uTM$ is, by definition, the tangent vector to M at $p = \pi(u)$ given by

$$\nabla_\xi A = (K\circ dA)(\xi) \tag{1.5}$$

where dA is the differential of A. If A is a vector field along π and if X is a vector field on TM, the map $\nabla_X A : u \longmapsto \nabla_{X_u}A$ is a vector field along π. By generalizing Dombrowski's lemma [5], Ii [6] has shown the following.

Lemma 1.1. *If A and B are vector fields along π, then the brackets of the horizontal and vertical lifts are given by*

(1) $[A^H,B^H]_u = (\nabla_{A^H}B)^H_u - (\nabla_{B^H}A)^H_u - (R(A(u),B(u))u)^V_u$ *for* $u \in TM$,

(2) $[A^H,B^V] = (\nabla_{A^H}B)^V - (\nabla_{B^V}A)^H$,

(3) $[A^V,B^V] = (\nabla_{A^V}B)^V - (\nabla_{B^V}A)^V$.

For vector fields A and B along π, let $g(A,B)$ denote the function on TM given by $u \longmapsto g(A(u),B(u))$. Then we have

$$\xi g(A,B) = g(\nabla_\xi A, B) + g(A, \nabla_\xi B) \tag{1.6}$$

for any tangent vector ξ to TM.

Every tensor field T of type (1,1) on M is a vector field along π. Then we have

Lemma 1.2. *Let T be a tensor field of type* (1,1) *on M and $\xi \in TM$, $u \in TM$. Then the covariant derivative $\nabla_\xi T$ of T (as vector field along π) and the covariant derivative $\nabla_{d\pi(\xi)}T$ of T (as a tensor field of type* (1,1) *on M) are related by*

$$\nabla_\xi T = (\nabla_{d\pi(\xi)}T)(u) + T(K(\xi)). \tag{1.7}$$

Corollary 1.3. *Let T be a tensor field of type $(1,1)$ on M and A a vector field along π. Then we have*
(1) $\nabla_{A^V} T = T \circ A$,
(2) $\nabla_{A^H} T = 0$ *if T is parallel.*

The identity map $\mathrm{id} : u \longmapsto u$ on TM is a parallel tensor field of type $(1,1)$ on M. Let $\|\cdot\|$ be the function $u \longmapsto \|u\| = \sqrt{g(u,u)}$ on TM. If A is a vector field along π, then we have by (1.6) and Corollary 1.3,

$$A^H \|\cdot\|^2 = 0, \ A^V \|\cdot\|^2 = 2g(A, \mathrm{id}) \tag{1.8}$$

where $g(A, \mathrm{id})$ denotes the function $u \longmapsto g(A(u), u)$ on TM.

The canonical symplectic form on TM is the closed 2-form Ω_0 given by

$$\Omega_0(X, Y) = g(d\pi(X), K(Y)) - g(d\pi(Y), K(X)) \tag{1.9}$$

where X, Y are any vector fields on TM. Note that

$$\Omega_0(X_u^H, Y_u^V) = g(X, Y), \ \Omega_0(X_u^H, Y_u^H) = \Omega_0(X_u^V, Y_u^V) = 0 \tag{1.10}$$

where $X, Y \in T_{\pi(u)}M$.

2. THE FIRST CLASS OF ALMOST COMPLEX STRUCTURES

We begin our considerations about almost complex structures with the construction of a general class of examples on the tangent bundle TM of an almost Hermitian manifold (M, J, g). Putting $t = \|u\|^2$, we define an almost complex structure J_1 on TM by

$$\begin{cases} J_1(X_u^H) = fX_u^V + \dfrac{h-f}{t} g(X, u)u^V + \dfrac{k-f}{t} g(X, Ju)(Ju)^V, \\ J_1(X_u^V) = -\dfrac{1}{f} X_u^H + \dfrac{h-f}{tfh} g(X, u)u^H + \dfrac{k-f}{tfk} g(X, Ju)(Ju)^H \end{cases} \tag{2.1}$$

for all vectors $X \in T_{\pi(u)}M$ and where f, h and k are positive, differentiable real functions on $[0,\infty)$, satisfying the initial conditions

$$\begin{cases} \text{the function } \dfrac{h-f}{t} \text{ is differentiable at } 0, \\ \text{the function } \dfrac{k-f}{t} \text{ is differentiable at } 0. \end{cases} \tag{2.2}$$

Now, let (M, J, g) be a complex space form. We compute the integrability condition for the almost complex structure J_1 on TM. Using Lemmas 1.1, 1.2, (1.1) and (1.8), a long but rather straightforward computation, using the criterion given in Section 1, gives the following:

Proposition 2.1. *Let (M, J, g) be a complex space form of holomorphic sectional curvature $4c$. Then the almost complex structure J_1 given by (2.1) is integrable if and only if the functions f, h and k satisfy the conditions*
(1) $f^2 = ct + fh - 2thf'$,

(2) $f^2 = -ct + fk$,
(3) $k^2 = 4ct + hk - 2thk'$
where f' (resp. k') is the derivative of f (resp. k) with respect to t.

Remark. It is easily seen that (3) holds when (1) and (2) are satisfied.

In what follows, we mention some special cases.

Example 2.1 Let (M, J, g) be a complex space form of holomorphic sectional curvature $4c$ $(c > 0)$. For $f = h = \sqrt{ct}$ and $k = 2\sqrt{ct}$, the complex structure J_1 on the punctured tangent bundle $T^0M = TM \setminus M$ coincides with the one given in Theorem 3.3 of [6].

Example 2.2 Let (M, J, g) be a complex space form of holomorphic sectional curvature $4c$ $(c > 0)$. Put

$$\rho_c = \begin{cases} \dfrac{\sqrt{ct}}{\tanh(\sqrt{ct})} & \text{if } t > 0, \\ 1 & \text{if } t = 0. \end{cases}$$

For $f = \rho_c$, $h = 1$ and $k = \rho_{4c}$, the complex structure J_1 on the tangent bundle TM coincides with that of Theorem 3.5 in [6].

Example 2.3 Let (M, J, g) be a complex space form of holomorphic sectional curvature $4c$ $(c > 0)$. For $f = \sqrt{e^{-t} + ct}$, $h = \frac{\sqrt{e^{-t}+ct}}{1+t}$ and $k = \frac{e^{-t}+2ct}{\sqrt{e^{-t}+ct}}$, we obtain a new complex structure J_1 on TM.

Next, let (M, J, g) be again an almost Hermitian manifold. We define a Riemannian metric G_1 on TM which is compatible with J_1 by

$$\begin{cases} G_1(X^H, Y^H) = \alpha g(X, Y) + \beta g(X, u)g(Y, u) + \gamma g(X, Ju)g(Y, Ju), \\ G_1(X^V, Y^V) = \dfrac{\alpha}{f^2} g(X, Y) + \dfrac{(f^2 - h^2)\alpha + tf^2\beta}{tf^2h^2} g(X, u)g(Y, u) \\ \qquad\qquad\qquad + \dfrac{(f^2 - k^2)\alpha + tf^2\gamma}{tf^2k^2} g(X, Ju)g(Y, Ju), \\ G_1(X^H, Y^V) = 0 \end{cases} \tag{2.3}$$

for all vectors $X, Y \in T_{\pi(u)}M$ and where the differentiable functions α, β and γ satisfy the conditions

$$\alpha = \alpha(t) > 0, \ \alpha(t) + t\beta(t) > 0, \ \alpha(t) + t\gamma(t) > 0. \tag{2.4}$$

To express that the almost Hermitian structure (J_1, G_1) must be almost Kählerian, we use the exterior derivative $d\Omega_1$ of the Kähler form Ω_1 of (J_1, G_1) on TM, given by the formula

$$(d\Omega_1)(X, Y, Z) = XG_1(J_1Y, Z) - YG_1(J_1X, Z) + ZG_1(J_1X, Y)$$

$$+ G_1(J_1Z, [X, Y]) - G_1(J_1Y, [X, Z]) + G_1(J_1X, [Y, Z])$$

for all vector fields $X,Y,Z \in \mathcal{X}(TM)$. A detailed computation as in [13], which we omit here, shows that $d\Omega_1 = 0$ if the functions $\alpha, \beta, \gamma, f, h$ and k satisfy

$$\begin{cases} (1) \quad \alpha = af \quad (a = \text{constant}), \\ (2) \quad \dfrac{\alpha}{f} = \dfrac{\alpha + t\beta}{h}, \ \dfrac{\alpha}{f} = \dfrac{\alpha + t\gamma}{k}, \\ (3) \quad 2\dfrac{d}{dt}\left(\dfrac{(\alpha + t\gamma)t}{k}\right) - \dfrac{1}{k}(\alpha + t\gamma) - \dfrac{1}{h}(\alpha + t\beta) = 0. \end{cases} \tag{2.5}$$

Since (3) holds when (1) and (2) are satisfied, we obtain

Theorem 2.2. *Let TM be the tangent bundle of an almost Hermitian manifold (M,J,g). Let f, h and k be positive differentiable functions on $[0,\infty)$ which satisfy (2.2). Then the almost Hermitian structure given by $\alpha = f$, $\beta = \dfrac{h-f}{t}$ and $\gamma = \dfrac{k-f}{t}$ in (2.1) and (2.3) is also almost Kählerian on TM.*

Remark. This theorem is an almost Hermitian version of Theorem 2.2 in [13]. It is easily seen from (1.9) and (1.10) that the Kähler form of (TM, J_1, G_1), mentioned above, coincides with the canonical symplectic form.

Combining Theorem 2.2 and Proposition 2.1, we obtain

Theorem 2.3. *Let TM be the tangent bundle of a complex space form of holomorphic sectional curvature $4c$. If the almost Kähler structure on TM in Theorem 2.2 satisfies $h = \dfrac{f^2 - ct}{f - 2tf'}$ and $k = \dfrac{f^2 + ct}{f}$, then it is Kählerian.*

3. THE SECOND CLASS OF ALMOST COMPLEX STRUCTURES

The second class of almost complex structures J_2 on the tangent bundle TM of an almost Hermitian manifold (M,J,g) is defined, in terms of the structure J of the base manifold, by

$$\begin{cases} J_2(X_u^H) = r(JX)_u^V + \dfrac{q-r}{t} g(X,u)(Ju)^V - \dfrac{p-r}{t} g(X,Ju)u^V, \\ J_2(X_u^V) = \dfrac{1}{r}(JX)_u^H + \dfrac{r-p}{trp} g(X,u)(Ju)^H - \dfrac{r-q}{trq} g(X,Ju)u^H \end{cases} \tag{3.1}$$

for all vectors $X \in T_{\pi(u)}M$ where r, p and q are differentiable real functions on $[0,\infty)$, satisfying the initial conditions

$$\begin{cases} \text{the function } \dfrac{p-r}{t} \text{ is differentiable at } 0, \\ \text{the function } \dfrac{q-r}{t} \text{ is differentiable at } 0. \end{cases} \tag{3.2}$$

Suppose now that (M,J,g) is a complex space form. In a similar way as for the structure J_1, we may compute the integrability condition of the almost complex structure J_2 on TM. The result of the detailed computation gives

Proposition 3.1. *Let (M, J, g) be a complex space form of holomorphic sectional curvature $4c$. Then the almost complex structure J_2 given by* (3.1) *is integrable if and only if the functions r, p and q must satisfy the conditions*

(1) $r^2 = ct + rp - 2tpr'$,
(2) $r^2 = -ct + rq$,
(3) $q^2 = 4ct + pq - 2tpq'$.

Remark. Here again, (3) holds when (1) and (2) are satisfied.

4. THE THIRD CLASS OF ALMOST COMPLEX STRUCTURES

Finally, we define the third class of almost complex structures on the tangent bundle TM of an almost Hermitian manifold (M, J, g) by

$$\begin{cases} J_3(X_u^H) = \lambda(JX)_u^H + \dfrac{a-\lambda}{t} g(X,u)(Ju)^H + \dfrac{a-\lambda}{t\lambda a} g(X,Ju)u^H, \\ J_3(X_u^V) = \mu(JX)_u^V + \dfrac{b-\mu}{t} g(X,u)(Ju)^V + \dfrac{b-\mu}{t\mu b} g(X,Ju)u^V \end{cases} \tag{4.1}$$

for all vectors $X \in T_{\pi(u)}M$ and where $\lambda = \pm 1$, $\mu = \pm 1$, and a, b are positive, differentiable real functions on $[0,\infty)$, satisfying the initial conditions

$$\begin{cases} \text{the function } \dfrac{a-\lambda}{t} \text{ is differentiable at } 0, \\ \text{the function } \dfrac{b-\mu}{t} \text{ is differentiable at } 0. \end{cases} \tag{4.2}$$

Remark. If $\lambda = \mu = 1$, $a = \lambda$, $b = \mu$ in (4.1), the structure J_3 is the horizontal lift J^H of J (see [14]).

Now, suppose that (M, J, g) is a Kähler manifold and compute the integrability condition of the almost complex structure J_3 on TM. Using the vanishing of the well-known Nijenhuis tensor [8], we obtain

Proposition 4.1. *Let (M, J, g) be a Kähler manifold. Then the almost complex structure J_3 given by* (4.1) *is integrable if and only if $a = \lambda$.*

5. HYPERHERMITIAN AND HYPERKÄHLER STRUCTURES

Using the three kinds of structures J_α, $\alpha = 1, 2, 3$, defined above on the tangent bundle TM of an almost Hermitian manifold (M, J, g) and putting

$$r = f,\ q = \frac{hk}{p},\ \lambda = -1,\ \mu = 1,\ a = -\frac{h}{p},\ b = \frac{k}{p},$$

a direct computation shows that

$$J_\alpha J_\beta = J_\gamma$$

where (α, β, γ) is a cyclic permutation of (1,2,3). Therefore, $(J_\alpha)_{\alpha=1,2,3}$ is an almost hypercomplex structure on the tangent bundle TM (see, for example, [1], [2]). Hence, we have

Proposition 5.1. *Let (M, J, g) be an almost Hermitian manifold. Then there exists a class of almost hypercomplex structures $(J_\alpha)_{\alpha=1,2,3}$ on the tangent bundle TM.*

Next, we consider the case of a complex space form of holomorphic sectional curvature $4c$. By Propositions 2.1, 3.1 and 4.1, we obtain the following hypercomplex structures $(J_\alpha)_{\alpha=1,2,3}$ on TM:

$$\begin{cases} J_1(X_u^H) = fX^V + \dfrac{h-f}{t} g(X,u)u^V + \dfrac{k-f}{t} g(X,Ju)(Ju)^V, \\ J_1(X_u^V) = -\dfrac{1}{f}X^H + \dfrac{h-f}{tfh} g(X,u)u^H + \dfrac{k-f}{tfk} g(X,Ju)(Ju)^H, \end{cases}$$

$$\begin{cases} J_2(X_u^H) = f(JX)^V + \dfrac{k-f}{t} g(X,u)(Ju)^V - \dfrac{h-f}{t} g(X,Ju)u^V, \\ J_2(X_u^V) = \dfrac{1}{f}(JX)^H + \dfrac{f-h}{tfh} g(X,u)(Ju)^H - \dfrac{f-k}{tfk} g(X,Ju)u^H, \end{cases} \tag{5.2}$$

$$\begin{cases} J_3(X_u^H) = -(JX)^H, \\ J_3(X_u^V) = (JX)^V + \dfrac{k-h}{th} g(X,u)(Ju)^V + \dfrac{k-h}{tk} g(X,Ju)u^V, \end{cases}$$

for all vectors $X \in T_{\pi(u)}M$ and where $h = \dfrac{f^2 - ct}{f - 2tf'}$ *and* $k = \dfrac{ct + f^2}{f}$. So, we have

Theorem 5.2. *Let TM be the tangent bundle of a complex space form of holomorphic sectional curvature $4c$. The structures given by (5.2) determine a hypercomplex structure on TM.*

Now, we define a Riemannian metric $\hat{G}$ on TM by

$$\begin{cases} \hat{G}(X^H, Y^H) = \alpha g(X,Y) + \beta\{g(X,u)g(Y,u) + g(X,Ju)g(Y,Ju)\}, \\ \hat{G}(X^V, Y^V) = \dfrac{\alpha}{f^2} g(X,Y) + \dfrac{(f^2 - h^2)\alpha + tf^2\beta}{tf^2h^2} g(X,u)g(Y,u) \\ \qquad\qquad + \dfrac{(f^2 - k^2)\alpha + tf^2\beta}{tf^2k^2} g(X,Ju)g(Y,Ju), \\ \hat{G}(X^H, Y^V) = 0 \end{cases} \tag{5.3}$$

for all vectors $X, Y \in T_{\pi(u)}M$ and where the functions α, β, f, h and k satisfy the conditions

$$\alpha(t) > 0, \quad \alpha(t) + t\beta(t) > 0, \tag{5.4}$$

$$h = \frac{f^2 - ct}{f - 2tf'}, \quad k = \frac{ct + f^2}{f}. \tag{5.5}$$

Then it is easily seen that $(J_\gamma, \hat{G})_{\gamma=1,2,3}$ is a hyperhermitian structure.

Furthermore, for the Kähler forms $\hat{\Omega}_\gamma$, defined by

$$\hat{\Omega}_\gamma(X,Y) = \hat{G}(J_\gamma X, Y), \quad \gamma = 1,2,3$$

for all vector fields $X, Y \in \mathcal{X}(TM)$, we compute $d\hat{\Omega}_\gamma$. These tedious computations, which we omit here, show that $(J_\gamma, \hat{G})_{\gamma=1,2,3}$ is a hyperkähler structure if and only if the functions α, β, f, h and k satisfy

$$\alpha = bf \;\; b = \text{constant}), \quad \frac{\alpha}{f} = \frac{1}{h}(\alpha + \beta t), \quad \frac{\alpha}{f} = \frac{1}{k}(\alpha + \beta t). \tag{5.6}$$

Put $b = 1$ for the homothetic factor in (5.3). Then the equations (5.5) and (5.6) imply the equation

$$(f^2 + ct)f' = cf. \tag{5.7}$$

The general solution of the ordinary differential equation (5.7) is given by

$$f = \frac{1}{2}\left(-A \pm \sqrt{4ct + A^2}\right), \tag{5.8}$$

where A is an integration constant. We now assume that $c > 0$ and $A < 0$. Therefore, putting $a = -A > 0$ and $f = \frac{1}{2}\left(a + \sqrt{4ct + a^2}\right)$, we have

$$\begin{aligned} h &= k = \sqrt{4ct + a^2}, \\ \alpha &= \frac{1}{2}\left(a + \sqrt{4ct + a^2}\right), \quad \beta = \frac{1}{2t}\left(-a + \sqrt{4ct + a^2}\right). \end{aligned}$$

Moreover, $\dfrac{h-f}{t}$ is differentiable at 0. So, from (5.2) and (5.3) we have

Theorem 5.3. *Let TM be the tangent bundle of a complex space form (M, J, g) of holomorphic sectional curvature $4c$ $(c > 0)$. Then there exists a one-parameter family of hyperkähler structures $(J_\gamma(a), \hat{G}(a))_{\gamma=1,2,3}$ on TM.*

Remarks.

(1) Calabi [3] constructed already a hyperkähler structure on the cotangent bundle of a complex projective n-space $\mathbf{C}P^n$.

(2) We also refer to [4] for results about the existence of hypercomplex structures on TM and for further references.

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