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ON THE COMMUTANT OF THE IDEAL CENTRE

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Abstract. *The centre $Z(E)$ of a Riesz space E is the collection of all operators T on E that satisfy $-\lambda I \leq T \leq \lambda I$ for some $\lambda \in \mathbb{R}$, where I is the identity operator. We study the commutant $Z(E)_C$ of $Z(E)$ in the order bounded operators on E. If E is a Riesz space with topologically full centre, we identify $Z(E)_C$ with the orthomorphisms of E.*

PRELIMINARIES

If E is a Riesz space $E^{\sim}$, the order dual of E will be the Riesz space of all order bounded linear functionals on E. Riesz spaces considered in this note are assumed to have separating order duals. $Z(E)$ will denote the ideal centre, Orth (E), will denote the orthomorphisms of E. If E is a topological Riesz space E' will denote continuous dual of E. When $T : E \to F$ is an order bounded operator between two Riesz spaces, the adjoint of T carries $F^{\sim}$ into $E^{\sim}$ and it will be denoted by $T^{\sim}$. In all undefined terminology concerning Riesz spaces we will adhere to the definitions in [1],[6] and [11].

When the order dual $E^{\sim}$ separates the points of the Riesz space E, an order bounded operator $T : E \to E$ is an orthomorphism if and only if its adjoint $T^{\sim} : E^{\sim} \to E^{\sim}$ is an orthomorphism. Moreover, the operator $\psi : T \to T^{\sim}$ from Orth (E) into Orth $(E^{\sim})$ is a Riesz homomorphism [1]. The image under ψ of the centre $Z(E)$ will be denoted by $Z^{\sim}(E)$. $Z^{\sim}(E)$ is a Riesz subspace of $Z(E^{\sim})$.

Definition: A Riesz space E, with separating order dual $E^{\sim}$, is said to have topologically full centre if, for each pair x, y in E with $0 \leq y \leq x$, there exists a net (π_α) in $Z(E)$ with $0 \leq \pi_\alpha \leq I$ for each α, such that $\pi_\alpha x \to y$ in $\sigma(E, E^{\sim})$.

Banach lattices with topologically full centre were initiated in [10]. These spaces were also studied in [2], [3] and [8]. The class of Riesz spaces and the class of Banach lattices that have topologically full centre are quite large. σ-Dedekind complete Riesz spaces have topologically full centres. However, not all Riesz spaces have topologically full centres.

Example: [12] Let E be the Riesz space of piecewise affine, continuous functions on $[0, 1]$. Considered as a sublattice of $C[0, 1]$, E is cofinal in $C[0, 1]$. That is to say for each $0 \leq f$ in $C[0, 1]$ there exists $e \in E$ with $f \leq e$. Hence each positive functional on E can be extended to a positive functional on $C[0, 1]$ by Theorem 83.15 in [11]. Therefore, we have $E^{\sim} = C[0, 1]^{\sim}$. But $Z(E)$ consists only of multiples of identity and $Z(E)$ is not topologically full. To see this let $T \in Z(E)$ and $f \in E$. Let $x \in [0, 1]$ be arbitrary and define $f_x = f - f(x) \cdot 1$. $f_x \in E$ and $|Tf_x| \leq \lambda |f_x|$ for some $\lambda \in \mathbb{R}_+$. $|Tf_x|(x) = 0$ as $f_x(x) = 0$. Hence $(Tf)(x) = (T1)(x)f(x)$ for all $x \in [0, 1]$. $T(1)$ is a constant function on $[0, 1]$. If $T(1)$ were not a constant function then $(T1)(x) = ax + b$ on some of open interval. Then we would have $(T(T1))(x) = a^2x^2 + 2abx + b^2$ on some interval. This would violate the linearity of $T(T1)$

on that interval. Hence $(Tf)(x) = (T1)(x)f(x)$ so that $Tf = \alpha_T f$ with $\alpha_T = T(1)$.

There exists an AM-space E whose centre consists of multiples of the identity therefore $Z(E)$ is not topologically full [9]. As Banach lattices are relatively uniformly complete and have separating order duals, we see that relative uniform completess does not, in general, imply topological fullness of the centre. We now give an example of a Riesz space which has topologically full centre but is not Dedekind or σ-Dedekind complete.

Example: Let X be a topological space. We denote by $C_b(X)$ the AM-space of all bounded, continuous real valued functions on X. It is well-known that if X is also normal $C_b(X)$ is σ-Dedekind (Dedekind) complete if and only if each open F_σ-subset of X has open closure (each subset of X has open closure). However, unlike these completeness properties, $C_b(X)$ always have topologically full centre for a normal topological space X.

Let us note that $C_b(X)$ can be identified with its centre. Clearly, each $f \in C_b(X)$ defines an operator $T(g) = f \cdot g$, $g \in C_b(X)$, in the ideal centre. For a proof that this correspondence is an onto lattice isomorphism we refer the reader to Theorem 8.27 in [1]. To see that $C_b(X)$ has topologically full centre, we let $0 \leq y \leq x$ in $C_b(X)$ and $0 < \epsilon$ be arbitrary. Let $D = \{t \in X : \epsilon \leq x(t)\}$. The function $z(t) = y(t) / x(t)$ is continuous and bounded on the closed set D. Tietze extension theorem allows us to extend $z = z(t)$ to an element in $C_b(X)$. We denote the extension by $z = z(t)$ again and consider the operator T defined by z. Clearly, $\| y - T(x) \| < \epsilon$ and $C_b(X)$ has topologically full centre.

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MAIN RESULTS

Order bounded maps on the Riesz space E will be denoted by $L_b(E)$. $Z(E)_C$ will denote the commutant of $Z(E)$ in $L_b(E)$. That is, $Z(E)_C = \{T \in L_b(E) : T\pi = \pi T$ for each $\pi \in Z(E)\}$. Since $E^\sim$ is assumed to separate the points of E, E is Archimedean. It follows that the Riesz space Orth (E) under composition is an Archimedean f-algebra and therefore it is commutative. Hence $\mathrm{Orth}(E) \subset Z(E)_C$.

In this note we study the relationship between $\mathrm{Orth}(E)$ and $Z(E)_C$. The notion of relative uniform convergence for lattices is due to Kantorovitch [5]. As is done by Gordon in [4], the notion of relative uniform convergence can be generalized so as to apply to spaces that are not vector lattices.

Definition: A sequence (x_n) in an ordered vector space E is said to converge relative uniform to $x \in E$ if there exists $e \in E_+$ and $(\lambda_n) \subset \mathbb{R}_+$ with $\lambda_n \to 0$ such that, for all $n \in \mathbb{N}$,

$$-\lambda_n e \leq x - x_n \leq \lambda_n e.$$

A subset Y of E is called relatively uniformly closed if whenever a sequence (x_n) in Y converges relative uniform to x, x must also be in Y.

$Z(E)_C$ is a relatively uniformly closed subset of $L_b(E)$. Let (T_n) be a sequence in $Z(E)_C$ which converges relative uniform to T. To show $T \in Z(E)_C$, it is enough to show $T\pi = \pi T$ whenever $0 \leq \pi \leq I$. Let us choose $(\lambda_n) \subset \mathbb{R}_+$ with $\lambda_n \to 0$ and $U \in L(E)_+$ with $-\lambda_n U \leq T - T_n \leq \lambda_n U$ for each n. Then for each $\pi \in Z(E)_+$ with $0 \leq \pi \leq I$, we have

$-2\lambda_n U \leq T\pi - \pi T = T\pi - \pi T + \pi T_n - T_n\pi \leq 2\lambda_n U$ for each n. Since $L_b(E)$ is Archimedean, we have $T\pi = \pi T$ and $Z(E)_C$ is relatively uniformly closed.

Let T_α and T be order bounded operators on the Riesz space E. If $\sup_\alpha\{T_\alpha x\} = Tx$ for each $x \in E_+$, we write $T = \sup_\alpha T_\alpha$. Since $Z(E)$ is positively generated and consists of order continuous operators on E, it is easy to see that if $T = \sup_\alpha T_\alpha$ with $T_\alpha \in Z(E)_C$ for each α then $T \in Z(E)_C$.

Let E, F and G be Riesz spaces. A bilinear map $\phi : E \times F \to G$ is called a bi-lattice homomorphism if for each $f \in F_+$, the maps $y \to \phi(y,f)$ of E into G and the maps $F \to G$ defined by $g \to \phi(x,g)$ for each $x \in E_+$ are lattice homomorphisms. In particular, we shall consider the bilinear map $\phi : E \times E^\sim \to Z(E)'$ defined by $\phi(x,f) = \mu_{x,f}$, where $\mu_{x,f}(\pi) = f(\pi(x))$ for $\pi \in Z(E)$. For each $x \in E_+$, the maps $f \to \phi(x,f)$ and for each $f \in E^\sim_+$ the maps $x \to \phi(x,f)$ are positive and we have $|\phi(x,f)| \leq \phi(|x|,|f|)$ for each (x,f) in $E \times E^\sim$.

If E has topologically full centre, we can say more about the positivity of the map ϕ.

Lemma: Let E be a Riesz space with separating order dual $E^\sim$. If E has topologically full centre, then $\phi : E \times E^\sim \to Z(E)'$ is a bi-lattice homomorphism.

Proof: First, we show $\phi(x,\cdot) : E^\sim \to Z(E)'$ is a lattice homomorphism for each $x \in E_+$. Positivity of the map $\phi(x,\cdot) : E^\sim \to Z(E)'$ implies that $\phi(x,f)^+ \leq \phi(x,f^+)$ for each $x \in E_+$ and $f \in E^\sim$. Let $\pi \in Z(E)_+$, and $x \in E_+$ be arbitrary, then

$$\phi(x,f^+)(\pi) = \mu_{x,f^+}(\pi) = f^+(\pi x) = \sup\{f(y) : 0 \leq y \leq \pi(x)\}.$$

For each y with $0 \leq y \leq \pi(x)$, we can find (π_α) in $Z(E)$ such that $0 \leq \pi_\alpha \leq I$ for each α and $\pi_\alpha(\pi x) \to y$ in $\sigma(E, E^\sim)$ as E has topologically full centre. As $0 \leq \pi_\alpha(\pi x) \leq \pi(x)$ for each α, we have

$$f(\pi_\alpha(\pi x)) \leq \phi(x,f)(\pi_\alpha \pi) \leq \phi(x,f)^+(\pi) \quad \text{for each} \quad \alpha.$$

Hence $f(y) \leq \phi(x,f)^+(\pi)$ for each y with $0 \leq y \leq \pi(x)$. Taking supremum over all such y, we obtain $\phi(x,f^+) \leq \phi(x,f)^+$ for an arbitrary $f \in E^\sim$.

Now we show that $\phi(\cdot,f) : E \to Z(E)'$ is a lattice homomorphism for each $f \in E^\sim_+$. It is enough to show that $\phi(x,f) \wedge \phi(y,f) = 0$ for each x, y in E satisfying $x \wedge y = 0$. As I, the identity operator on E, is a strong order unit in $Z(E)$ it suffices to show

$$\phi(x,f) \wedge \phi(y,f)(I) = (\mu_{x,f} \wedge \mu_{y,f})(I) = 0.$$

It is well-known that

$$\begin{aligned} (\mu_{x,f} \wedge \mu_{y,f})(I) &= \inf\{\mu_{x,f}(\pi_1) + \mu_{y,f}(\pi_2) : 0 \leq \pi_1, \pi_2 \in Z(E), \quad \pi_1 + \pi_2 = I\} \\ &= \inf\{f(\pi_1 x) + f(\pi_2 x) : 0 \leq \pi_1, \pi_2 \in Z(E), \quad \pi_1 + \pi_2 = I\}. \end{aligned}$$

Let $z = x + y$. Let I_x, I_y and I_z be, respectively, the order ideals generated by x, y and z. Then I_z is actually the order direct sum of I_x and I_y by Theorem 17.6 in [10] . We denote by P the order projection of I_z onto I_x. Let J be the restriction to I_z of order bounded functionals on E. Then J is an order ideal in $I_z^\sim$ because if $f \in I_z^\sim$ satisfies $0 \leq f \leq g_{|_{I_z}}$ for some $g \in E^\sim$, then f has an extention to a positive functional on E by Theorem 2.3 in [1]. The adjoint $P^\sim : I_z^\sim \to I_z^\sim$ of P satisfies $0 \leq P^\sim \leq I$ and as a consequence we obtain $P^{\sim}(J) \subset J$. As a result of these simple observations we obtain that the pair $< I_z, J >$ constitutes a Riesz pair and $P : I_z(\sigma(I_z, J)) \to I_z(\sigma(I_z, J))$ continuous. Since $0 \le P(z) \le z$, there exists (π_α) in $Z(E)$, $0 \le \pi_\alpha \le I$ such that $\pi_\alpha(z) \to P(z)$ in $\sigma(E, E^{\sim})$ and therefore in $\sigma(I_z, J)$. We have $P(\pi_\alpha z) = \pi_\alpha(Pz) = \pi_\alpha(x)$, and the continuity of P now yields $\pi_\alpha x \to x$ in $\sigma(I_z, J)$. Since we have $\pi_\alpha(z) = \pi_\alpha(x) + \pi_\alpha(y)$ for each α, we have $\pi_\alpha(y) \to 0$ in $\sigma(I_z, J)$. As $\mu_{x,f} \wedge \mu_{y,f}(I) \le f((I - \pi_\alpha)(x) + \pi_\alpha(y))$ for each α, we obtain $(\mu_{x,f} \wedge \mu_{y,f})(I) \le \lim_\alpha f((I - \pi_\alpha)(x) + \pi_\alpha(y)) = 0$ which completes the proof. □

A well-known Theorem of H.Nakano (cf. [1], Theorem 8.3) states that if E has the principal projection property then the band preserving operators are precisely the ones that commute with order projections. Thus, for Riesz spaces having the principal projection property, $Z(E)_C$ is precisely Orth E.

Since $Z(E)_C =$ Orth E for a Dedekind complete Riesz space E, we have $Z(E)_C$ is relatively uniformly complete. Thus it is natural to ask whether $Z(E)_C$ is relatively uniformly complete whenever E is. Let $\{\pi_n\}$ be a relatively uniformly Cauchy sequence of positive operators in $Z(E)_C$. That is to say, there exists $T \in L_b(E)_+$ such that for every $\epsilon > 0$, there exists an integer $N = N(\epsilon)$ with

$$-\epsilon T \le \pi_n - \pi_m \le \epsilon T \text{ for all } n, m \ge N.$$

Hence, we have $-\epsilon T(x) \le \pi_n(x) - \pi_m(x) \le \epsilon T(x)$ for each $x \in E_+$ and all $n, m \ge N$. Since E is relatively uniformly complete and E_+ is relatively uniformly closed, there exists a unique $\pi_0(x) \epsilon E_+$ such that $\pi_n(x) \to \pi_0(x)$ relatively uniformly. π_0 extends uniquely to a positive operator on E. Using that π_n is order bounded for each n, it is straightforward to prove that π_0 is order bounded. To show $\pi_0 \in Z(E)_C$ it is enough to show π_0 commutes with each $0 \le T \epsilon Z(E)$, which follows easily from the fact that a positive operator is sequentially relatively uniformly continuous.

When E is Dedekind complete, commutativity of a positive operator with the operators in the ideal centre implies that it is an interval preserving operator. Let $0 \le S \epsilon Z(E)_C$ be arbitrary, then for each $z \in E^+$, $S[0, z] \subseteq [0, Sz]$. Thus, it is enough to verify $[0, Sz] \subseteq S[0, z]$. Let $0 \le y \le Sz$ be arbitrary. Then there exists some $\pi \in Z(E)$, $0 \le \pi \le I$ satisfying $\pi(Sz) = S(\pi z) = y$. Thus $y \in S[0, z]$.

Proposition: Let E be a Riesz space with separating order dual $E^{\sim}$ and topologically full centre. Then $Z(E)_C =$ Orth E.

Proof: Let $T \in Z(E)_C$ be arbitrary. It is easy to see that $\mu_{Tx,f} = \mu_{x,\tilde{T}f}$ for each $x \in E$ and $f \in E^{\sim}$. On the other hand if $x \perp y$ in E, then positivity of the map $\phi : E \times E^{\sim} \to Z(E)'$ implies that $|\mu_{x,f}| \le \mu_{|x|,|f|} \le \mu_{|x|,|f| \vee |g|}$ and $|\mu_{y,g}| \le \mu_{|y|,|g|} \le \mu_{|y|,|f| \vee |g|}$ for each f, g in $E^{\sim}$. Hence $0 \le |\mu_{x,f}| \wedge |\mu_{y,g}| \le \mu_{|x|,|f| \vee |g|} \wedge \mu_{|y|,|f| \vee |g|} = \mu_{|x| \wedge |y|,|f| \vee |g|} = 0$ by the lemma. Thus we obtain that, if $x \perp y$ in E, then $\mu_{x,f} \perp \mu_{y,g}$ in $Z(E)'$ for each f, g in $E^{\sim}$. In particular, we have $\mu_{x,\tilde{T}f} \perp \mu_{y,f}$ for each $f \in E^{\sim}$. Hence $\mu_{Tx,f} \perp \mu_{y,f}$ for each x, y in E with $x \perp y$ and $f \in E^{\sim}$. Since $E^{\sim}$ separates the points of E, the result $Tx \perp y$ follows from the fact that $x \to \phi(x, f)$ is a lattice homomorphism for each $f \in E^{\sim}_+$. □

Results show that for a Banach lattice with order continuous norm we have $Z(E) = Z(E^{\sim}) = Z(E')$, where E' denotes the topological dual [7]. The next result characterizes Riesz spaces for which $Z(E) = Z(E^{\sim})$.

Proposition: Let E be a Dedekind complete Riesz space. Then $E^{\sim} = E_n^{\sim}$ if and only if the adjoint map $\psi : Z(E) \to Z(E^{\sim})$ is surjective.

Proof: Suppose $E^{\sim} = E_n^{\sim}$. Then the topology $\sigma(E, E^{\sim})$ is order continuous. Therefore E is an order ideal of $(E^{\sim})_n^{\sim}$ by Theorem 11.13 in [1]. Let $T \in Z(E^{\sim})$ be arbitrary. Then $T^{\sim} \in Z((E^{\sim})_n^{\sim})$ and it leaves E invariant. Thus we conclude that $T^{\sim}|_E$ is in $Z(E)$. It is straightforward to see that $(T^{\sim}|_E)^{\sim} = T$. Thus ψ is surjective.

Suppose ψ is surjective. To show that $E^{\sim} = E_n^{\sim}$ it is enough to prove that each band in $E^{\sim}$ is $\sigma(E^{\sim}, E)$-closed by Theorem 11.10 in [1]. Let B be an arbitrary band in $E^{\sim}$ and P be the band projection on B^d. Then $P \in Z(E^{\sim})$. By hypothesis there exists $P_1 \in Z(E)$ with $P_1^{\sim} = P$. As $P : (E^{\sim}, \sigma(E^{\sim}, E)) \to (E^{\sim}, \sigma(E^{\sim}, E))$ continuous, $B = P^{-1}(0)$ is $\sigma(E^{\sim}, E)$ closed. □

Let us recall that $Z^{\sim}(E)$ is the image of $Z(E)$ under the adjoint map $\psi : Z(E) \to Z(E^{\sim})$.

Proposition: Let E be a Riesz space with topologically full centre. Then $Z^{\sim}(E)_C =$ Orth $(E^{\sim})$.

Proof: $E^{\sim}$ has topologically full centre as it is Dedekind complete. Therefore $Z(E^{\sim})_C =$ Orth $(E^{\sim})$ by previous proposition. $Z^{\sim}(E) \subset Z(E^{\sim}) \subset$ Orth $(E^{\sim})$ implies that Orth $(E^{\sim}) \subset Z^{\sim}(E)_C$. Thus it is enough to show that $Z^{\sim}(E)_C \subset$ Orth $(E^{\sim})$. Let T be an order bounded operator on $E^{\sim}$ that commutes with every operator $\tilde{\pi}, \pi \in Z(E)$. Suppose $f \perp g$ in $E^{\sim}$. We claim $Tf \perp g$ in $E^{\sim}$.

Let us consider the bilinear map $\Psi : E^{\sim} \times E^{\sim\sim} \to Z(E^{\sim})'$ defined by $\Psi(f, F) = \mu_{f.F}$ where $\mu_{f.F}(\pi) = F(\pi f)$ where $F \in E^{\sim\sim}, f \in E^{\sim}$ and $\pi \in Z(E^{\sim})$. $E^{\sim}$ has topologically full centre and the bilinear map $\Psi : E^{\sim} \times E^{\sim\sim} \to Z(E^{\sim})'$ has similar properties to those of $\phi : E \times E^{\sim} \to Z(E)'$ defined earlier. In particular, it is a bi-lattice homomorphism, and if $x \to \hat{x}$ is the canonical embedding of E into $E^{\sim\sim}$, then $\mu_{x.f}(\pi) = \mu_{f.\hat{x}}(\tilde{\pi})$ for each $x \in E, f \in E^{\sim}$ and $\pi \in Z(E)$. Since $f \perp g$, we have $\mu_{f.F} \perp \mu_{g.G}$ for arbitrary F, G in $E^{\sim\sim}$. In particular, for $F = T^{\sim}\hat{x}$ and $G = \hat{x}$ we obtain $\mu_{f.T\hat{x}} \perp \mu_{g.\hat{x}}$ in $Z(E^{\sim})'$. We let $Z^{\sim}(E)^{\circ}$ be the polar of $Z^{\sim}(E)$ in $Z(E^{\sim})'$. Then $(Z^{\sim}(E)^{\circ})^d$ is a projection band in $Z(E^{\sim})'$. P will denote the band projection on $(Z^{\sim}(E)^{\circ})^d$. Let us consider $Z(E^{\sim})'|_{Z^{\sim}(E)}$ in $Z^{\sim}(E)'$.

$Z(E^{\sim})'|_{Z^{\sim}(E)}$ is the class of order bounded functionals on $Z^{\sim}(E)$ that have order bounded extentions to $Z(E^{\sim})$ and is an order ideal of $Z(E^{\sim})'$. Therefore $Z(E^{\sim})'|_{Z^{\sim}(E)}$ is a Riesz subspace of $Z(E^{\sim})'$. The map $P(\mu) \to \mu|_{Z^{\sim}(E)}$ is a Riesz isomorphism of $(Z^{\sim}(E)^{\circ})^d$ onto $Z(E^{\sim})'|_{Z^{\sim}(E)}$. By Theorem 7.3 in [1], it is enough to show that it is a positive bijection with positive inverse. Suppose $\mu_1|_{Z^{\sim}(E)} = \mu_2|_{Z^{\sim}(E)}$ for some $\mu_1, \mu_2 \in Z(E^{\sim})'$. Then $\mu_1 - \mu_2|_{Z^{\sim}(E)} = 0$ so that $(\mu_1 - \mu_2) \in Z^{\sim}(E)^{\circ}$. Hence $P(\mu_1 - \mu_2) = P(\mu_1) - P(\mu_2) = 0$ or $P(\mu_1) = P(\mu_2)$. Positivity of $P\ \mu \to \mu|_{Z^{\sim}(E)}$ is clear. Surjectivity of $\mu \to \mu|_{Z^{\sim}(E)}$ is also clear. Positivity of the inverse follows from Theorem 2.3. in [1]. Since a positive orthomorphism is a lattice homomorphism $P(\mu_{f.T\hat{x}}) \perp P(\mu_{g.\hat{x}})$ follows from $\mu_{f.T\hat{x}} \perp \mu_{g.\hat{x}}$. Therefore, we have $\mu_{f.T\hat{x}}|_{Z^{\sim}(E)} \perp \mu_{g.\hat{x}}|_{Z^{\sim}(E)}$. Since $T \in Z^{\sim}(E)_C$. $T\tilde{\pi} = \tilde{\pi}T$ for each $\pi \in Z(E)$ and we have $\mu_{f.T\hat{x}}|_{Z^{\sim}(E)} = \mu_{Tf.\hat{x}}|_{Z^{\sim}(E)}$. Therefore $\mu_{Tf.\hat{x}}|_{Z^{\sim}(E)} \perp \mu_{g.\hat{x}}|_{Z^{\sim}(E)}$. As $\mu_{Tf.\hat{x}}(\tilde{\pi}) = \mu_{x.Tf}(\pi)$ for each $\pi \in Z(E), f \in E^{\sim}$ and $x \in E$ it follows that $\mu_{x.Tf} \perp \mu_{x.g}$ in $Z(E)'$. $\phi : E \times E^{\sim} \to Z(E)'$ is a bi-lattice homomorphism implies that $\mu_{x.|Tf| \wedge |g|} = 0$ in $Z(E)'$ for each $x \in E_+$. Evaluating at $I \in Z(E)$ we see that $(|Tf| \wedge |g|)(x) = 0$ for each $x \in E_+$. Hence $|Tf| \wedge |g| = 0$ as $E^{\sim}$ separates the points of E. This shows $Tf \perp g$ in $E^{\sim}$ and $T \in$ Orth $(E^{\sim})$. □

Commutant of $Z^{\sim}(E)$ seems to be related to another subalgebra of $L_b(E^{\sim})$ which we now

define. Let $W(E^{\sim})$ be defined as $W(E^{\sim}) = \{T \in L_b(E^{\sim}) : T(I) \subseteq I$ for each $\sigma(E^{\sim}, E)$-closed ideal I in $E^{\sim}\}$.
When E has a topologically full centre, the class of $\sigma(E^{\sim}, E)$-closed ideals of $E^{\sim}$ coincide with the $\sigma(E^{\sim}, E)$ closed $Z(E)$-submodules of $E^{\sim}$. Thus $W(E^{\sim})$ can be described as the order bounded operators on $E^{\sim}$ that leave invariant each $\sigma(E^{\sim}, E)$-closed $Z(E)$-submodule of $E^{\sim}$.

The next result describes elements of $W(E^{\sim})$ in terms of the bilinear map $(x,f) \to \mu_{x,f}$ of $E \times E^{\sim} \to Z(E)'$.

Proposition: Let E be a Riesz space with separating order dual $E^{\sim}$ and topologically full centre. An operator $T \in L_b(E^{\sim})$ is in $W(E^{\sim})$ if and only if $\mu_{x,f} = 0$ implies $\mu_{x,Tf} = 0$ for $x \in E, f \in E^{\sim}$.

Proof: Let $T \in W(E^{\sim})$ and $\mu_{x,f} = 0$ for some $x \in E, f \in E^{\sim}$. We have $(Z(E)x)^0 = I_x^0$ for a Riesz space with topologically full centre. Hence, $f \in (I_x)^0$ which is a $\sigma(E^{\sim}, E)$-closed band of $E^{\sim}$. Therefore $Tf \in (Z(E)x)^0$. That is to say, $Tf(\pi x) = \mu_{x,Tf}(\pi) = 0$ for each $\pi \in Z(E)$ or $\mu_{x,Tf} = 0$ as claimed.

Let now $I \subset E^{\sim}$ be a $\sigma(E^{\sim}, E)$-closed ideal of $E^{\sim}$. $({}^0I)$, polar of I taken in E, is an ideal in E. We wish to show $T(I) \subseteq I$. Let us choose $f \in I_+$ and $x \in ({}^0I)_+$ and observe that $\mu_{x,f}(\pi) = 0$ for each $\pi \in Z(E)$. It follows that $\mu_{x,f}(\pi) = 0$ for each $x \in ({}^0I)$ and $f \in I$. Then, by hypothesis, $\mu_{x,Tf} = 0, x \in ({}^0I), f \in I$. Thus $Tf \in ({}^0I)^0$ and $Tf \in I$ by the Bipolar theorem. □

$\sigma(E^{\sim}, E)$-closed ideals of $E^{\sim}$ are bands (cf. Theorem 106.1 in [11]). Therefore for a Riesz space with topologically full centre, we have $Z^{\sim}(E)_C \subseteq W(E^{\sim})$. In general, an arbitrary band in $E^{\sim}$ may not be $\sigma(E^{\sim}, E)$-closed. A necessary and sufficient condition for this to happen is $E^{\sim} = E_n^{\sim}$. Thus, $W(E^{\sim}) = Orth(E^{\sim})$ whenever $E^{\sim} = E_n^{\sim}$.

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