A WEIERSTRASS TYPE REPRESENTATION FOR SURFACES IN HYPERBOLIC SPACE WITH MEAN CURVATURE ONE

CÉLIA CONTIN GÓES, M. ELISA, E.L. GALVÃO

Abstract. The subject of this paper is to give a Weierstrass type representation for mean curvature one surfaces in the hyperbolic space. This representation depends on the hyperbolic Gauss map. Some known examples are described and a new one, associated to the minimal Bonnet surface is constructed with this representation.

INTRODUCTION

A Weierstrass type formula for surfaces of prescribed mean curvature in \( \mathbb{R}^3 \) was given by Kenmotsu ([K]) in 1979. In 1987, R. Bryant ([B]) studied the surfaces of mean curvature one in hyperbolic space as local projections of null curves in the space of the \( 2 \times 2 \) Hermitian symmetric matrices with its Cartan-Killing metric. Recently, Umehara and Yamada ([UY-1], [UY-2], [RUY]) produced an explicit tool to construct examples of these surfaces. They described the null curves in terms of a meromorphic function \( g \) and a holomorphic 1-form \( \omega \) obtained as solutions of two ordinary differential equations.

The subject of this paper is to describe the surfaces in \( \mathbb{H}^3 \) with mean curvature one in a very similar manner as the minimal surfaces in \( \mathbb{R}^3 \). It is already well known that these surfaces have a hyperbolic holomorphic Gauss map ([B]); in our work, the function \( h \) describes the holomorphic Gauss map. Its properties will give us a Weierstrass type representation.

From the main theorem we have the immersion \( X : U \subset \mathbb{C} \rightarrow \mathbb{H}^3 \) as

\[
X(z) = \left( \frac{\phi_1(z) + \phi_2(z)}{2}, \Re \phi_3(z), \Im \phi_3(z), \frac{\phi_1(z) - \phi_2(z)}{2} \right)
\]

where \( \phi_j, j = 1, 2, 3 \) are solutions of the system:

\[
\begin{align*}
\phi_1 \phi_2 &= 1 + |\phi_3|^2 \\
\frac{\partial \phi_1}{\partial \bar{z}} &= h \frac{\partial \phi_3}{\partial \bar{z}} \\
\frac{\partial \phi_2}{\partial \bar{z}} &= \frac{1}{h} \frac{\partial \phi_3}{\partial \bar{z}}
\end{align*}
\]

whose integrability condition is that of

\[
\Im \{ \bar{h} \Delta \phi_3 \} = 0.
\]
We also have a local integral representation:

\[ X = \left( \Re \int_{x_0}^{x_1} \left( \frac{h}{z} \frac{d \Phi_3}{d z} + \frac{1}{h} \frac{d \Phi_3}{d z} \right) dz, \Re e \Phi_3, \Im m \Phi_3, \Re \int_{x_0}^{x_1} \left( \frac{h}{z} \frac{d \Phi_3}{d z} - \frac{1}{h} \frac{d \Phi_3}{d z} \right) dz \right) \]

In the last part of the paper we exhibit local solutions of this system for all functions \( h \).

**The hyperbolic Gauss map.**

We consider the Lorentz space \( \mathbb{L}^4 = \{ x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4 \} \) with the inner product

\[ \langle x, y \rangle = -x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3. \]

The Minkovskiy model for the hyperbolic space is the submanifold

\[ \mathbb{H}^3 = \{ x \in \mathbb{L}^4 \mid \langle x, x \rangle = -1, \ x_0 > 0 \}. \]

In \( \mathbb{H}^3 \) we will consider the induced orientation from \( \mathbb{L}^4 \) for which the vectors \( v_1, v_2, v_3 \) in \( T_p \mathbb{H}^3 \) form a positive oriented basis iff \( \{ p, v_1, v_2, v_3 \} \) forms a positive oriented basis of \( \mathbb{L}^4 \).

Let \( X : M \rightarrow \mathbb{H}^3 \) be an isometric immersion of an orientable Riemann surface \( M \) in the hyperbolic space and \( N(p) \) the oriented unitary normal vector at \( p \in M \). In local isothermical coordinates \( z = u + iv \) we have \( ||X_u|| = ||X_v|| = \lambda, \langle X_u, X_v \rangle = 0 \), and \( N \) is such that \( \{ X(p), \frac{1}{\lambda} X_u, \frac{1}{\lambda} X_v, N(p) \} \) is a positive basis of \( T_p \mathbb{L}^4 \).

We will consider the map

\[ \Phi : \mathbb{H}^3 \rightarrow D \]

\[ (x_0, x_1, x_2, x_3) \rightarrow \left( \frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0} \right) \]

and the vector \( \Phi_*(N(p)) \) where

\[ D = \{ (x_0, x_1, x_2, x_3) \mid x_0 = 1, x_1^2 + x_2^2 + x_3^2 < 1 \}. \]

This map is the natural isometry between \( \mathbb{H}^3 \) and the Klein model for the hyperbolic space given by unitary disc with the appropriated metric.

The boundary of \( D \) can be identified with the Riemann two sphere \( S^2 \).

**Definition.** The hyperbolic Gauss map of an immersion \( X : M \rightarrow \mathbb{H}^3 \) is

\[ n : M \rightarrow \partial D \]

given by

\[ n(p) = \Phi(X(p)) + t \Phi_*(N(p)) \]

where \( t > 0 \) and \( n(p) \in \partial D \).

It follows immediately:
Lemma 1. \( n = \frac{1}{x_0 + N_0} (X + N) \).

Proof. For \( X(p) = (x_0, x_1, x_2, x_3) \) and \( N = (N_0, N_1, N_2, N_3) \)

\[
\Phi_*(N) = -\frac{N_0}{x_0^2} X + \frac{1}{x_0} N.
\]

As \( n(p) = \Phi(X(p)) + t\Phi_*(N(p)) \) is in the cone \( -x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0 \) we have

\[
\langle n, n \rangle = -\frac{(x_0 - tN_0)^2}{x_0^4} + \frac{t^2}{x_0^2} = 0.
\]

The solution \( t \) with \( t > 0 \) is \( t = x_0 / (x_0 + N_0) \).

\[\square\]

Remarks. 1. Since the vector \( X + N \) is also in the cone there exists \( \psi : U \rightarrow \mathbb{R} \),

\[
\psi(p) = -\frac{1}{\langle n, X \rangle} = x_0 + N_0 = -\langle X + N, e_0 \rangle, \ e_0 = (1, 0, 0, 0)
\]

such that

\[
\psi(p) n(p) = X(p) + N(p), \ \forall p \in U
\]

and

\[
N = -\frac{1}{\langle n, X \rangle} n - X.
\]

2. The coefficients of the second fundamental form for the immersion \( X \) can be calculated as

\[
h_{ij} = -\langle \nabla_{e_i} N, e_j \rangle, \ i, j = 1, 2
\]

with \( e_1 = \frac{1}{\lambda} X_u \) and \( e_2 = \frac{1}{\lambda} X_v \). We have

\[
N_u = -h_{11} X_u - h_{12} X_v,
\]

\[
N_v = -h_{12} X_u - h_{22} X_v.
\]

The mean curvature in the choosen normal direction and the gaussian curvature have, respectively, the expressions

\[
H = \frac{1}{2} (h_{11} + h_{22}) \quad \text{and} \quad K = h_{11} h_{22} - h_{12}^2 - 1.
\]

3. In isothermical parameters

\[
\langle X_{zz}, N \rangle = \frac{1}{2} \lambda^2 H
\]
where
\[ \frac{1}{2} \lambda^2 = \langle X_z, X_{\bar{z}} \rangle. \]

The mean curvature \( H \) is equal to one if and only if
\[ \langle X_{\bar{z}}, n \rangle = \langle X_z, X_{\bar{z}} \rangle \]
or
\[ \langle X_{\bar{z}}, -\frac{1}{\langle n, X \rangle} n - X \rangle = \langle X_z, X_{\bar{z}} \rangle. \]

We will have \( H = 1 \) if and only if
\[ \langle X_z, n_{\bar{z}} \rangle = 0. \]

4. Taking \( z = u + iv \) isothermal parameters in \( U \subset \mathbb{C} \) we have the diagram:

\[
\begin{array}{ccc}
M & \xrightarrow{n} & \partial D \approx S^2 \\
\downarrow & & \downarrow \\
U \subset \mathbb{C} & \xrightarrow{h} & \mathbb{C}
\end{array}
\]

with \( \Pi \) the stereographic projection; then
\[
n(z) = \left( 1, \frac{2 \Re h}{|h|^2 + 1}, \frac{2 \Im h}{|h|^2 + 1}, \frac{|h|^2 - 1}{|h|^2 + 1} \right),
\]
and \( n \) is holomorphic if and only if \( h \) is holomorphic.

This hyperbolic Gauss map behaves as the classical Gauss map for minimal surfaces in an euclidean space, that is, we have the following theorem ([B]):

**Theorem 1.** Let \( n : M \rightarrow \partial D \) be the hyperbolic Gauss map of a surface \( X : M \rightarrow \mathbb{H}^3 \), \( n \) non constant. The map \( n : M \rightarrow \partial D \) is conformal iff the immersion \( X \) either has mean curvature \( H \) constant and equal to one (in which case \( n \) preserves the orientation) or \( X \) is totally umbilic (in which case \( n \) reverses the orientation).

**Proof.** From \( n = \frac{1}{x_0 + N_0} (X + N) \) we have
\[
n_u = \left( \frac{1}{x_0 + N_0} \right)_u (X + N) + \frac{1}{x_0 + N_0} (X_u + N_u) =
\]
\[= \left( \frac{1}{x_0 + N_0} \right)_u (X + N) + \frac{1}{x_0 + N_0} [(- h_{11}) X_u - h_{12} X_v] \]
and
\[
n_v = \left( \frac{1}{x_0 + N_0} \right)_v (X + N) + \frac{1}{x_0 + N_0} (X_v + N_v) =
\]
\[
= \left( \frac{1}{x_0 + N_0} \right) (X + N) + \frac{1}{x_0 + N_0} [-h_{12}X_u + (1 - h_{22})X_v].
\]

Consequently,

\[
||n_u||^2 = \frac{\lambda^2}{(x_0 + N_0)^2} [(1 - h_{11})^2 + h_{12}^2]
\]

\[
||n_v||^2 = \frac{\lambda^2}{(x_0 + N_0)^2} [(1 - h_{22})^2 + h_{12}^2]
\]

\[
<n_u, n_v> = \frac{\lambda^2}{(x_0 + N_0)^2} (h_{12})(2 - h_{11} - h_{22})
\]

and for \(H = 1\) or for umbilic immersions we will have \(||n_u||^2 = ||n_v||^2 \geq 0\) and \(<n_u, n_v> = 0.\)

We also observe that \(||n_u|| = ||n_v|| = 0\) if and only if the immersion is umbilical and \(H = 1;\) in this case we have a horosphere and the hyperbolic Gauss map \(n\) is constant.

Considering the complex differentiation

\[
n_{\bar{z}} = \left( \frac{1}{x_0 + N_0} \right) (X + N) + \frac{1}{x_0 + N_0} (X_{\bar{z}} + N_{\bar{z}})
\]

it follows that

(1) \[
<n_{\bar{z}}, n_{\bar{z}}> = \frac{\lambda^2}{2(x_0 + N_0)^2} (H - 1)[(h_{11} - h_{22}) + 2i h_{12}],
\]

from

\[
n(z) = (1, \frac{2Re h}{|h|^2 + 1}, \frac{2Im h}{|h|^2 + 1}, \frac{|h|^2 - 1}{|h|^2 + 1})
\]

we have

\[
n_{\bar{z}} = \left( \frac{1}{|h|^2 + 1} \right) \tilde{n} + \frac{1}{|h|^2 + 1} \tilde{n}_{\bar{z}}
\]

where

\[
\tilde{n} = (|h|^2 + 1, h + \bar{h}, -i (h - \bar{h}), |h|^2 - 1)
\]

and

\[
\tilde{n}_{\bar{z}} = h_{\bar{z}} (\bar{h}, 1, -i, \bar{h}) + \bar{h}_{\bar{z}} (h, 1, i, h).
\]

With these calculations we conclude that

(2) \[
<n_{\bar{z}}, n_{\bar{z}}> = \frac{4h_{\bar{z}} \bar{n}_{\bar{z}}}{(|h|^2 + 1)^2}
\]

From (1) and (2)

\[
\frac{4h_{\bar{z}} \bar{n}_{\bar{z}}}{(|h|^2 + 1)^2} = \frac{\lambda^2}{2(x_0 + N_0)^2} (H - 1)[(h_{11} - h_{22}) + 2i h_{12}]
\]
and the hyperbolic Gauss map is conformal iff either $H = 1$ or the immersion is umbilical. In both cases the induced metric is given by

$$<dn,dn> = <n_z dz + n_{\bar{z}} d\bar{z}, n_z dz + n_{\bar{z}} d\bar{z}> = 2 <n_z,n_{\bar{z}}> |dz|^2$$

or

$$<dn,dn> = \frac{\lambda^2}{(x_0 + N_0)^2} [(2H - 1) - K] |dz|^2.$$

When $H = 1$ we have

$$<dn,dn> = \frac{\lambda^2}{(x_0 + N_0)^2} (-K) |dz|^2,$$

if the immersed surface is different from a horosphere $2H(H - 1) - K > 0$ and $-K > 0$.

Finally we compare the orientations of $X : M \rightarrow \mathbb{H}^3 \subset \mathbb{L}^4$ and $n : M \rightarrow \partial D \subset \mathbb{L}^4$, when the immersion $X$ is distinct from the horosphere.

The stereographic projection $\Pi : \partial D \approx S^2 \rightarrow \{1\} \times \mathbb{R}^3 \subset \mathbb{L}^4$ induces a positive orientation in $S^2$ in which the normal vector is the internal one.

Let $\{X(p), \frac{1}{\lambda} X_u, \frac{1}{\lambda} X_v, N\}$ and $\{e_0, n_u, n_v, e_0 - n\}$, $e_0 = (1,0,0,0)$ be orthogonal frames adapted to $X(M)$ and $n(M)$, respectively.

These frames are related by the matrix

$$\begin{pmatrix}
  x_0 & \left(\frac{1}{x_0 + N_0}\right)_u & \left(\frac{1}{x_0 + N_0}\right)_v & x_0 - \frac{1}{x_0 + N_0} \\
  \frac{1}{\lambda} <X_u,e_0> & \frac{\lambda(1 - h_{11})}{x_0 + N_0} & -\frac{\lambda h_{12}}{x_0 + N_0} & \frac{1}{\lambda} <X_u,e_0> \\
  \frac{1}{\lambda} <X_v,e_0> & -\frac{\lambda h_{12}}{x_0 + N_0} & \frac{\lambda(1 - h_{22})}{x_0 + N_0} & \frac{1}{\lambda} <X_v,e_0> \\
  -N_0 & \left(\frac{1}{x_0 + N_0}\right)_u & \left(\frac{1}{x_0 + N_0}\right)_v & -N_0 - \frac{1}{x_0 + N_0}
\end{pmatrix}$$

whose determinant is

$$\begin{vmatrix}
  x_0 & \left(\frac{1}{x_0 + N_0}\right)_u & \left(\frac{1}{x_0 + N_0}\right)_v & x_0 - \frac{1}{x_0 + N_0} \\
  \frac{1}{\lambda} <X_u,e_0> & \frac{\lambda(1 - h_{11})}{x_0 + N_0} & -\frac{\lambda h_{12}}{x_0 + N_0} & 0 \\
  \frac{1}{\lambda} <X_v,e_0> & -\frac{\lambda h_{12}}{x_0 + N_0} & \frac{\lambda(1 - h_{22})}{x_0 + N_0} & 0 \\
  -(x_0 + N_0) & 0 & 0 & 0
\end{vmatrix} = -\frac{\lambda^2}{(x_0 + N_0)^2} [(1 - h_{11})(1 - h_{22}) - h_{12}^2] = \frac{\lambda^2}{(x_0 + N_0)^2} [-K + 2(H - 1)].$$

It is easy to see that the determinant is positive if $H = 1$ in which case $n$ preserves the orientation (that is, $n$ is holomorphic); in the umbilic case the determinant is negative, $n$ reverses the orientation and is antiholomorphic.
Remark. We observe that

\[
\langle n_z, n_{\bar{z}} \rangle = \frac{2}{(1 + |h|^2)^2} \left[ |h_z|^2 + |h_{\bar{z}}|^2 \right].
\]

When \( H = 1 \)

\[
\langle d n, d n \rangle = 2 \langle n_z, n_{\bar{z}} \rangle |d z|^2 = \frac{4|h_z|^2}{(|h|^2 + 1)^2} |d z|^2 = -K \frac{\lambda^2}{(x_0 + N_0)^2} |d z|^2,
\]

and

\[
-K = \frac{4|h_z|^2}{(|h|^2 + 1)^2} \left( \frac{\lambda^2}{(x_0 + N_0)^2} \right)^{-1}.
\]

A Representation Theorem.

Working with a holomorphic hyperbolic Gauss map, that is, with surfaces with constant mean curvature equal to one, we have a local representation theorem similar to the Weierstrass representation for minimal surfaces in the euclidean space.

**Theorem 2.** Let \( X : M \to H^3 \) be a non-umbilic immersion in \( H^3 \) with mean curvature one and

\[
n(z) = \left( 1, \frac{2 \text{Re} h}{1 + |h|^2}, \frac{2 \text{Im} h}{1 + |h|^2}, \frac{|h|^2 - 1}{|h|^2 + 1} \right)
\]

its hyperbolic Gauss map. Denoting \( X(z) = (x_0(z), x_1(z), x_2(z), x_3(z)) \), the real functions \( \phi_1(z) = x_0(z) + x_3(z) \) and \( \phi_2(z) = x_0(z) - x_3(z) \) and the complex function \( \phi_3(z) = x_1(z) + i x_2(z) \) satisfy

\[
\begin{cases}
\phi_1 \phi_2 = 1 + |\phi_3|^2 \\
\frac{\partial \phi_1}{\partial z} = h \frac{\partial \phi_3}{\partial z} \\
\frac{\partial \phi_2}{\partial z} = \frac{1}{h} \frac{\partial \phi_3}{\partial z}
\end{cases}
\]

\((\ast)\)

Conversely, given a holomorphic non-constant function \( h : U \subset \mathbb{C} \to \mathbb{C} \), two real functions \( \phi_1 \) and \( \phi_2 \) (\( \phi_2 > 0 \)) and a complex function \( \phi_3 \) satisfying \((\ast)\) in the simply connected domain \( U \), then

\[
X(z) = \left( \frac{\phi_1(z) + \phi_2(z)}{2}, \text{Re} \phi_3(z), \text{Im} \phi_3(z), \frac{\phi_1(z) - \phi_2(z)}{2} \right)
\]
defines a conformal immersion in \( H^3 \) with constant mean curvature one and hyperbolic Gauss map \( n \) given by \( h \) as above.
Proof. First of all we observe that

\[ X(z) = (x_0, x_1, x_2, x_3) \in \mathbb{H}^3 \iff -x_0^2 + x_1^2 + x_2^2 + x_3^2 = -1 \iff \Phi_1 \Phi_2 = 1 + |\Phi_3|^2, \]

from the first equivalence it also follows that if \( \Phi_2 = x_0 - x_3 \) then \( \Phi_2 > 0 \).

Given \( \Phi_1, \Phi_2, \Phi_3 \) as above we have

\[
X(z) = \left( \frac{\Phi_1(z) + \Phi_2(z)}{2}, \Re \Phi_3(z), \Im \Phi_3(z), \frac{\Phi_1(z) - \Phi_2(z)}{2} \right)
\]

and \( <X_z, n> = 0 \) if and only if

\[
\frac{1}{2} \left(1 - \frac{|h|^2 - 1}{|h|^2 + 1} \right) \frac{\partial \Phi_1}{\partial z} + \frac{1}{2} \left(1 + \frac{|h|^2 - 1}{|h|^2 + 1} \right) \frac{\partial \Phi_2}{\partial z} + \frac{1}{1 + |h|^2} \left( \Re h \left( \frac{\partial \Phi_3}{\partial z} + \frac{\partial \Phi_3}{\partial \overline{z}} \right) + i \Im h \left( \frac{\partial \Phi_3}{\partial z} - \frac{\partial \Phi_3}{\partial \overline{z}} \right) \right) = 0
\]

or

\[
\frac{\partial \Phi_1}{\partial z} + |h|^2 \frac{\partial \Phi_2}{\partial z} - h \frac{\partial \Phi_3}{\partial z} - \overline{h} \frac{\partial \Phi_3}{\partial \overline{z}} = 0
\]

(4)

The assumption on the mean curvature gives us

\[ H = 1 \iff <X_z, n_z> = <X_z, \overline{n_z}> = 0 \]

where

\[ n(z) = \left( \frac{2 \Re h}{1 + |h|^2}, \frac{2 \Im h}{1 + |h|^2}, \frac{|h|^2 - 1}{1 + |h|^2} \right) \]

and

\[ \overline{n} = (1 + |h|^2, h + \overline{h}, -i(h - \overline{h}), |h|^2 - 1) \]

We have in this case \( h \) holomorphic and therefore

\[ \overline{n_z} = (h \overline{h_z}, \overline{h_z}, i \overline{h_z}, h \overline{h_z}), \]

as \( h \) is nonconstant \( (h_z \neq 0) \) it follows

\[ H = 1 \iff <X_z, n_z> = 0 \iff
\]

\[ -h \left( \frac{\partial \Phi_1}{\partial z} + \frac{\partial \Phi_2}{\partial z} \right) + \left( \frac{\partial \Phi_3}{\partial z} + \frac{\partial \Phi_3}{\partial \overline{z}} \right) + h \left( \frac{\partial \Phi_1}{\partial z} - \frac{\partial \Phi_2}{\partial \overline{z}} \right) = 0 \iff
\]

(5)

\[ \frac{\partial \Phi_3}{\partial z} = h \frac{\partial \Phi_2}{\partial z}. \]
Returning with this last equation in (4), finally we have

$$\frac{\partial \Phi_1}{\partial z} = h \frac{\partial \Phi_3}{\partial z}. $$

Let $p \in M$ be a zero of $h$ with order 1; we have from (5) that $p$ is a zero of $\frac{\partial \Phi_3}{\partial z}$ whose order is greater or equal to 1 and we can write

$$\frac{\partial \Phi_2}{\partial z} = \frac{1}{h} \frac{\partial \Phi_3}{\partial z}. $$

Let now be

$$X(z) = \left( \frac{\phi_1(z) + \phi_2(z)}{2}, \Re \phi_3(z), \Im \phi_3(z), \frac{\phi_1(z) - \phi_2(z)}{2} \right)$$

with $\phi_1, \phi_2, \phi_3$ verifying (*). It is easy to see that

$$X_z = \frac{1}{2} \left[ \frac{\partial \Phi_3}{\partial z} \left( \frac{1}{h}, 1, -i, -\frac{1}{h} \right) + \frac{\partial \Phi_3}{\partial z} (h, 1, i, h) \right] \tag{6}$$

From the fact that $\langle X_z, X_z \rangle = 0$ it follows that we have isothermical parameters.

Let now consider

$$\hat{n}(z) = \left( 1, \frac{2 \Re h}{1 + |h|^2}, \frac{2 \Im h}{1 + |h|^2}, \frac{|h|^2 - 1}{|h|^2 + 1} \right)$$

with $h$ the holomorphic function from (*). The vector

$$\hat{N} = -\frac{1}{\langle \hat{n}, X \rangle} \hat{n} - X$$

has norm equal to one, verifies $\langle X_z, \hat{N} \rangle = 0$, $\langle X, \hat{N} \rangle = 0$ and

$$-\frac{1}{\langle \hat{n}, X \rangle} \hat{n} = X + \hat{N}$$

therefore $\hat{N}$ is exactly the normal vector $N$ and $\hat{n}$ the hyperbolic Gauss map $n$ of the immersion $X$. With some calculations we obtain

$$\langle n_\xi, X_\xi \rangle = \frac{h_\xi}{(1 + |h|^2)} \left[ \frac{\partial \Phi_3}{\partial z} - \frac{\partial \phi_3 h}{\partial z \ h} \right].$$

From the fact that $h$ is holomorphic it follows that $\langle n_\xi, X_\xi \rangle = 0$ which implies $H = 1$; $h$ non-constant gives us a non-umbilic immersion.
Remarks.
1. The compatibility condition for the two partial differential equations in (\(*\)) is the same and writes

\( (7) \quad \Im m \{ \overline{h} \Delta \phi_3 \} = 0. \)

This follows from the fact that each differential equation of (\(*\)) is as

\[ \frac{\partial \Phi}{\partial z} = F(z) \]

or as

\[ \begin{align*}
\frac{\partial \Phi}{\partial u} &= 2F_1(u, v) \\
\frac{\partial \Phi}{\partial v} &= -2F_2(u, v)
\end{align*} \]

with \( z = u + iv, F(z) = F_1(u, v) + iF_2(u, v), \partial / \partial z = \frac{1}{2} (\partial / \partial u - i \partial / \partial v). \)

The integrability condition for this system is:

\[ \frac{\partial F_1}{\partial v} = -\frac{\partial F_2}{\partial u} \iff \Im m \{ \frac{\partial F}{\partial z} \} = 0 \]

Returning to the system (\(*\)), each equation will have its integrability condition respectively given by:

\[ \Im m \{ h \frac{\partial^2 \Phi_3}{\partial z \partial z} \} = 0 \]

and

\[ \Im m \{ \frac{1}{|h|^2} \frac{\partial^2 \Phi_3}{\partial z \partial z} \} = -\frac{1}{|h|^2} \Im m \{ h \frac{\partial^2 \Phi_3}{\partial z \partial z} \} = 0 \]

Consequently, the two compatibility conditions are verified if and only if, locally,

\[ \Im m \{ \overline{h} \Delta \phi_3 \} = 0 \]

2. Choosing \( h \) and \( \phi_3 \) such as to verify (7) we will have \( \phi_1 \) and \( \phi_2 \) given locally by

\[ \phi_1 = 2 \Re e \int_{z_0}^{z} h \frac{\partial \Phi_3}{\partial z} \, dz \]

and

\[ \phi_2 = 2 \Re e \int_{z_0}^{z} \frac{1}{h} \frac{\partial \Phi_3}{\partial z} \, dz. \]

3. An integral formula can be written from (6):

(1) \( X = \left( \Re e \int_{z_0}^{z} \left( h \frac{\partial \Phi_3}{\partial z} + \frac{1}{h} \frac{\partial \Phi_3}{\partial z} \right) \, dz, \Re e \phi_3, \Im m \phi_3, \Re e \int_{z_0}^{z} \left( h \frac{\partial \Phi_3}{\partial z} - \frac{1}{h} \frac{\partial \Phi_3}{\partial z} \right) \, dz \right). \)
4. The metric \( ds^2 = \lambda^2 |dz|^2 \) is such that \( \lambda^2 = 2 < X, \bar{X} > \), from (6) we have

\[
\lambda^2 = \left| \frac{\partial \Phi_3}{\partial z} \right|^2 + \left| \frac{\partial \Phi_3}{\partial \bar{z}} \right|^2 - 2 \Re \left( \frac{\bar{h}}{h} \frac{\partial \Phi_3}{\partial z} \frac{\partial \Phi_3}{\partial \bar{z}} \right)
\]

and from this last expression we can conclude that \( p \) is a regular point if the derivatives \( \frac{\partial \Phi_3}{\partial z} \) and \( \frac{\partial \Phi_3}{\partial \bar{z}} \) do not vanish simultaneously at \( p \).

We also can write:

\[
\lambda^2 = \left| \frac{\partial}{\partial \bar{z}} \left( \Phi_3 - \bar{h}\Phi_2 \right) \right|^2 = \left| \frac{\partial}{\partial \bar{z}} \left( \Phi_3 - h\Phi_2 \right) \right|^2
\]

5. From Lemma 1 we have:

\[
- \frac{1}{< n, X >} = x_0 + N_0 = - < X + N, e_0 >,
\]

some calculations give us:

\[1 + |\Phi_3 - h\Phi_2|^2 = \Phi_2(\Phi_1 + |h|^2 \Phi_2 - \bar{h}\Phi_3 - h\Phi_3)\]

and

\[< n, X > = \frac{1}{|h|^2 + 1} (-\Phi_1 - |h|^2 \Phi_2 + \bar{h}\Phi_3 + h\Phi_3) = - \frac{1 + |\Phi_3 - h\Phi_2|^2}{\Phi_2 (|h|^2 + 1)}\]

The total curvature is

\[c = \int_M K dA\]

and from (3) it follows that

\[K = - \frac{4|h|^2}{(|h|^2 + 1)^2} \left( \frac{\lambda^2}{(x_0 + N_0)^2} \right)^{-1}.\]

In local coordinates

\[
c = - \int \frac{4|h|^2}{(\Phi_1 + |h|^2 \Phi_2 - \bar{h}\Phi_3 - h\Phi_3)^2} \frac{i}{2} dz \wedge \overline{dz} = \]

\[\int \frac{4|h|^2 \Phi_2^2}{(1 + |\Phi_3 - h\Phi_2|^2)^2} \frac{i}{2} dz \wedge \overline{dz} = \int \frac{4 |\frac{\partial}{\partial \bar{z}}(\Phi_3 - h\Phi_2)|^2}{(1 + |\Phi_3 - h\Phi_2|^2)^2} \frac{i}{2} dz \wedge \overline{dz}\]

6. Given the immersion \( X: U \subset \mathbb{C} \rightarrow \mathbb{H}^3 \) in isothermal coordinates, calling \( F = \lambda^2 / 2 \), the gaussian curvature can also be calculated as

\[K = - \frac{\partial \overline{\partial} \log F}{F}.\]
If we denote 
\[ \psi = \frac{1}{2} \left( (h_{11} - h_{22}) - 2i h_{12} \right) = \frac{2}{\lambda^2} < X_{zz}, N > \]
the Gauss equation can be written as
\[ |\psi|^2 = -K - \frac{4}{\lambda^2} < X_{zz}, X_{zz} > = -K + H^2 - 1. \]

By using (*) we get
\[ |\psi|^2 = \frac{4|h_2|^2 \phi_2}{\lambda^2 (1 + |\phi_3 - h \phi_2|^2)^2} = -K \]
that means, we have the Gauss’ equation verified.

We will call the Hopf’s form ([H]) the quadratic form
\[ \Psi = \psi \lambda^2 dz^2. \]

As in ([H]) the Codazzi equations can be written in a complex form and we have
\[ \frac{\partial (\lambda^2 \psi)}{\partial \bar{z}} = \lambda^2 \frac{\partial H}{\partial z} \]
With some calculations we can show that (*) implies that \( \lambda^2 \psi \) is holomorphic (Proposition 2 in [B]) and the Codazzi equations are also verified.

**Examples.**

To exhibit some examples we need to get two real functions \( \phi_1 \) and \( \phi_2, \phi_2 > 0 \) and a complex function \( \phi_3 \), solutions of the the system:

\[
\begin{cases}
\phi_1 \phi_2 = 1 + |\phi_3|^2 \\
\frac{\partial \phi_1}{\partial z} = h \frac{\partial \phi_3}{\partial z} \\
\frac{\partial \phi_2}{\partial z} = \frac{1}{h} \frac{\partial \phi_3}{\partial z}
\end{cases}
\]

(∗)

To find solutions, we begin with some important remarks.

1. First of all we will analyse the solutions that correspond to \( \phi_3 \) holomorphic (or antiholomorphic); we will have that \( \phi_1 \) (resp. \( \phi_2 \)) is constant. As \( \phi_1 \phi_2 = 1 + |\phi_3|^2 \) the constant cannot be zero; it is easy to see that \( \phi_1 \) (resp. \( \phi_2 \)) constant implies that the surface is umbilical and \( x_0 + x_3 \) (resp. \( x_0 - x_3 \)) is constant; in this case the functions \( x_1 \) and \( x_2 \) will be harmonical conjugates.

2. Given the function \( h \) we can search solutions as
\[ \phi_3 = h(z)F(|z|^2), \]
with $F$ a one real variable differentiable function.

Since

$$
\overline{h} \Delta \phi_3 = z\overline{h}h_z F'(|z|^2) + |h|^2 F'(|z|^2) + |z|^2 |h|^2 F''(|z|^2)
$$

the compatibility condition is

$$
\Im \{ \overline{h} \Delta \phi_3 \} = 0 \iff \Im \{ z\overline{h}h' \} = 0.
$$

The last condition is satisfied by all the functions $h(z) = z^\alpha$, for real $\alpha$.

3. If $\phi_3 = h(z)F(|z|^2)$ then metric (6) will be

$$
\lambda^2 = |h_z|^2 F^2(|z|^2).
$$

**Example 1.** We have an immersion with constant mean curvature one

$$
X : \mathbb{C} - \{0\} \longrightarrow \mathbb{H}^3
$$

solving $(\ast)$ with $h(z) = z$, $F_\alpha(t) = t^\alpha$ and

$$
\phi_3(z) = h(z) \left[ AF_\alpha(|z|^2) + BF_{\beta}(|z|^2) \right].
$$

Now, the integrability condition is satisfied (remark 2) and the solutions $\phi_1$ and $\phi_2$ are:

$$
\phi_1(z) = \frac{\alpha}{\alpha + 1} A |z|^{2\alpha + 2} + \frac{\beta}{\beta + 1} B |z|^{2\beta + 2}
$$

and

$$
\phi_2(z) = \frac{\alpha + 1}{\alpha} A |z|^{2\alpha} + \frac{\beta + 1}{\beta} B |z|^{2\beta}
$$

with $\alpha$ and $\beta$ both distinct from zero and $-1$.

The condition $\phi_1 \phi_2 = 1 + |\phi_3|^2$ is verified under the restrictions:

$$
\alpha + \beta = -1
$$

and

$$
AB \left( \frac{2\alpha + 1}{\alpha(\alpha + 1)} \right)^2 = 1
$$

therefore

$$
2\alpha + 1 \neq 0.
$$
In this case the solutions of (*) are:

\[
\begin{align*}
\Phi_1(z) &= \frac{\alpha}{\alpha + 1} A |z|^{2\alpha + 2} + \frac{\alpha + 1}{\alpha} B |z|^{-2\alpha} \\
\Phi_2(z) &= \frac{\alpha + 1}{\alpha} A |z|^{2\alpha} + \frac{\alpha}{\alpha + 1} B |z|^{-2\alpha - 2} \\
\Phi_3 &= z[A |z|^{2\alpha} + B |z|^{-2\alpha - 2}]
\end{align*}
\]

(12)

Writing \( z = re^{i\theta} \):

\[
\begin{align*}
\Phi_1(r, \theta) &= \frac{\alpha}{\alpha + 1} A r^{2\alpha + 2} + \frac{\alpha + 1}{\alpha} B r^{-2\alpha} = f_1(r) \\
\Phi_2(r, \theta) &= \frac{\alpha + 1}{\alpha} A r^{2\alpha} + \frac{\alpha}{\alpha + 1} B r^{-2\alpha - 2} = f_2(r) \\
\Phi_3(r, \theta) &= r(\cos \theta + i \sin \theta)(A r^{2\alpha} + B r^{-2\alpha - 2}) = f_3(r) e^{i\theta}
\end{align*}
\]

Now it is easy to see that all these surfaces are rotational surfaces generated by the curve

\[
C(r) = (c_0(r), c_1(r), 0, c_3(r)) = \left( \frac{f_1(r) + f_2(r)}{2}, f_3(r), 0, \frac{f_1(r) - f_2(r)}{2} \right),
\]

\( C(r) \subset H^3 P^3 \), with \( P^3 = [e_0, e_1, e_3] \). We have a spherical rotation and

\[
X(r, \theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left[ \begin{array}{c} c_0 \\ c_1 \\ c_3 \end{array} \right] = \begin{bmatrix} c_0 \\ c_1 \cos \theta \\ c_1 \sin \theta \\ c_3 \end{bmatrix}
\]

Using (9) and (11) we can have the total curvature

\[
c = \int_{C \setminus \{0\}} \frac{4\alpha^2(\alpha + 1)^2 |z|^{4\alpha + 2}}{(A|z|^{4\alpha + 2} + B)^2} \frac{i}{2} dz \overline{dz} = -4(2\alpha + 1)\pi
\]
As in (10) we have

\[ ds^2 = (A|z|^{2\alpha} + B|z|^{-2\alpha-2})^2|dz|^2 \]

and the surface is complete.

The minimal catenoid in the euclidean space has a Weierstrass representation given by

\[ g(z) = \mu z \quad \eta = f(z) dz = \nu z^{-2} dz. \]

The \( k \)-recovering of a minimal catenoid has Weierstrass data

\[ g(z) = \mu z^k \quad \eta = f(z) dz = \nu z^{-k-1} dz \]

and its induced metric is given by

\[ ds^2 = \frac{1}{4}[\nu|z|^{-k-1} + \nu\mu^2|z|^{k-1}]^2|dz|^2. \]

Given a rotational hyperbolic mean curvature one surface (consequently, given \( A \), \( B \) and \( \alpha \)), there exists a minimal catenoid isometric to this surface.

Choosing \( \mu \) and \( \nu \) such that \( A = \nu \mu^2 / 2 \), \( B = \nu / 2 \) the hyperbolic rotational surface is isometric to the \( k = (2\alpha + 1) \)-recovering of the minimal catenoid.

Conversely, it is easy to see that a \( k \)-recovering of a minimal catenoid is isometric to one immersion in the family of rotational surfaces exhibited above. This immersion is such that \( A = \nu \mu^2 / 2 \), \( B = \nu / 2 \),

\[ \left( \frac{2\alpha + 1}{\alpha(\alpha + 1)} \right)^2 = \frac{4}{\nu^2 \mu^2} \]

and \( k = 2\alpha + 1 \).

The rotational hyperbolic mean curvature one surfaces are called "the catenoid cousins".

**Example 2.** The system \((\ast)\) also admits solutions as

\[ \phi_3(z) = F(z) \overline{G(z)} \]

with \( F \) and \( G \) holomorphic functions. In this case if

\[ (13) \quad F'(z) = h(z)G'(z) \]

the integrability condition (7) is verified.

The two last equations in \((\ast)\) can be integrated and the solutions are

\[ \phi_1(z) = |F(z)|^2 \]

and

\[ \phi_2(z) = |G(z)|^2. \]
We will modify these solutions to have the first equation satisfied; in this way, we will take $F_1, G_1, F_2, G_2$ as in (13), $A$ and $B$ real constants such that

\[
\begin{align*}
\phi_1 &= A |F_1|^2 + B |F_2|^2 \\
\phi_2 &= A |G_1|^2 + B |G_2|^2 \\
\phi_3 &= A F_1 \bar{G}_1 + B F_2 \bar{G}_2
\end{align*}
\]

with

(14) \[ AB(\bar{F}_1 \bar{G}_2 - \bar{F}_2 \bar{G}_1)(F_1 G_2 - F_2 G_1) = 1. \]

The surfaces called "Enneper Cousins" are corresponding to

\[
h(z) = \tanh \lambda z,
\]
\[
G_1'(z) = \cosh \lambda z
\]
\[
G_2'(z) = z \cosh \lambda z,
\]

consequently, by (13) and (14),

\[
F_1(z) = \frac{1}{\lambda} \cosh \lambda z
\]
\[
F_2(z) = \frac{1}{\lambda} \left( z \cosh \lambda z - \frac{1}{\lambda} \sinh \lambda z \right)
\]
\[
G_1(z) = \frac{1}{\lambda} \sinh \lambda z
\]
\[
G_2(z) = \frac{1}{\lambda} \left( z \sinh \lambda z - \frac{1}{\lambda} \cosh \lambda z \right)
\]

and

\[
AB = |\lambda|^6, \quad \lambda \in \mathbb{C}.
\]

The total curvature can be calculated by (9), observing that

\[
\Phi_1 + |h|^2 \Phi_2 - \bar{h} \Phi_3 - h \bar{\Phi}_3 = A |F_1 - h G_1|^2 + B |F_2 - h G_2|^2 = \frac{(A + B |z|^2)}{|\lambda|^2 |\cosh z|^2}
\]

and

\[
K = -\int \frac{4|\lambda|^6}{A^2 \left(1 + \frac{B}{A} |z|^2\right)^2} \frac{i}{2} dz \wedge \bar{dz} = -\int \frac{4}{(1 + |w|^2)^2} \frac{i}{2} dw \wedge \bar{dw} = -4\pi.
\]
It is also easy to see that the metric

$$ds^2 = \left( \frac{A}{\lambda} + \frac{B}{\lambda} |z|^{2} \right)^2 |dz|^2$$

is complete.

The classical Enneper surface is given by the Weierstrass data

$$g(z) = \mu z \quad f(z)dz = \nu dz$$

and has the metric:

$$ds^2 = \frac{1}{4} [\nu + \nu \mu^2 |z|^2]^2 |dz|^2.$$  

The corresponding isometric Enneper cousin will be given by $\lambda, A$ and $B$ such that $\lambda^2 = \nu \mu / 2$, $A = \lambda \nu / 2$ and $B = \lambda \nu \mu^2 / 2$.

**Example 3.** By taking

$$h(z) = \tanh \left( \frac{\sqrt{5}}{2} z \right) = \frac{\sinh(\alpha_1 z) + \sinh(\alpha_2 z)}{\cosh(\alpha_1 z) + \cosh(\alpha_2 z)},$$

with $\alpha_1 = \frac{\sqrt{5} - 1}{2}$ and $\alpha_2 = \frac{\sqrt{5} + 1}{2}$ and

$$\phi_3 = AF_1 \overline{G}_1 + BF_2 \overline{G}_2,$$

we can obtain the “Bonnet Cousins” ([GN]) corresponding to the solutions:

$$F_1(z) = \frac{1}{\alpha_1} \cosh(\alpha_1 z) + \frac{1}{\alpha_2} \cosh(\alpha_2 z)$$

$$G_1(z) = \frac{1}{\alpha_1} \sinh(\alpha_1 z) + \frac{1}{\alpha_2} \sinh(\alpha_2 z)$$

$$F_2(z) = \frac{1}{\alpha_1} \sinh(\alpha_1 z) - \frac{1}{\alpha_2} \sinh(\alpha_2 z)$$

$$G_2(z) = \frac{1}{\alpha_1} \cosh(\alpha_1 z) - \frac{1}{\alpha_2} \cosh(\alpha_2 z),$$

$$AB = \frac{1}{(\alpha_2^2 - \alpha_1^2)^2} = \frac{1}{(\alpha_1 + \alpha_2)^2}.$$  

The metric in this case is given by

$$ds^2 = 4 \left[ A(\alpha_1 + \alpha_2) |\cosh z / 2|^2 + B(\alpha_1 + \alpha_2) |\sinh z / 2|^2 \right]^2 |dz|^2 =$$
\[
= 4 \left[ A (\alpha_1 + \alpha_2) \left| \cosh z / 2 \right|^2 + \frac{1}{A(\alpha_1 + \alpha_2)} \left| \sinh z / 2 \right|^2 \right]^2 |dz|^2
\]

and the surface is regular, complete and isometric to a homothety of the classical Bonnet minimal surface given by the Weierstrass data

\[g(z) = -i \alpha \tanh \left( \frac{z}{2} \right) \quad \text{and} \quad f(z)dz = \frac{2i}{\alpha} \cosh^2 \left( \frac{z}{2} \right) dz, \quad z \in \mathbb{C}, \quad \alpha = \sqrt{\frac{1+a}{1-a}}, \quad 0 < a < 1\]

and metric

\[
ds^2 = \frac{1}{4} \left[ \frac{1}{|\cosh \frac{z}{2}|^2 + \alpha |\sinh \frac{z}{2}|^2} \right]^2 |dz|^2.
\]

To get new examples we have to find solutions of

\[\Im \{ \bar{h} \Delta \phi_3 \} = 0\]

and a linear combination of this solutions in order to have

\[\phi_1 \phi_2 = 1 + |\phi_3|^2\]

that is, in order to have the corresponding immersion in \( \mathbb{L}^4 \) contained in \( \mathbb{H}^3 \).

The classification of these immersions depends on the description of all the solutions of this problem.
REFERENCES


Celia C. Góes (goes@ime.usp.br)
M. Elisa E.L. Galvão (elisa@ime.usp.br)
INSTITUTO DE MATEMÁTICA E ESTÁTICA
UNIVERSIDADE DE SÃO PAULO
Rua do Matão 1010
Cidade Universitária
Caixa Postal 66281 - Ag. Cidade de São Paulo
CEP 05889-970
SÃO PAULO - BRASIL