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# A NOTE ON THE STABILITY OF AUTOTOPISM TRIANGLES IN TRANSLATION PLANES

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## 1 Introduction.

The finite nearfield planes admit a group fixing two infinite points and an affine point (an autotopism group) which is transitive on the remaining infinite points. Furthermore, the so-called autotopism triangle is invariant under the translation complement. Note that the infinite points of the triangle are interchanged but the triangle is left invariant.

There are other classes of translation planes which admit such a transitive autotopism group; namely the so-called $j$-planes [8] and the planes of Baker, Dover, Ebert and Weida [5] as well as some of the generalized twisted field planes.

In any translation plane with a transitive autotopism group, the question becomes whether it is possible to move the triangle. Some recent powerful results concerning various types of transitivity on translation planes may be applied to show that the autotopism triangle is almost never moved. Our main result indicates exactly the situation.

**Theorem 1** *Let $\pi$ be a finite translation plane of order $\neq 2^6$ that admits an autotopism group which is transitive on the remaining infinite points.*

*Then, one of the following occurs:*

*(1) the full translation complement leaves the autotopism triangle invariant, or*

*$\pi$ is one of the following types of planes:*

*(2) Desarguesian,*

*(3) Hall plane of order 9,*

*(3) Hering of order 27 or*

*(4) generalized twisted field.*

**Remark 2** *In a recent article, Biliotti, Jha and Johnson [2] have completely determined the set of generalized twisted field planes that admit an autotopism group acting transitively on a given side of the autotopism triangle.*

## 2 Background results.

**Theorem 3** *(Buekenhout, Doyen, Delantsheer, Kantor, Liebeck [3])*

*Let $\pi$ denote a finite affine plane which admits a nonsolvable flag-transitive collineation group.*

*Then $\pi$ is one of the following planes:*

*(1) Desarguesian,*

*(2) Hall of order 9,*

*(3) Hering of order 27,*

*(4) Lüneburg-Tits.*

**Theorem 4** *(see e.g. [1])*

*Let $S_z(2^{2a+1})$ denote a Suzuki group. Then the order of the outer automorphism group is* $2a+1$.

**Theorem 5** *(Lüneburg [10] (37.10), p. 192)*

*Let $p$ be a prime and let $\pi$ be an affine plane of order $p^r$. Assume that $u$ is a $p$-primitive divisor of $p^r-1$. If $G$ is a solvable flag-transitive collineation group of $\pi$ whose order is divisible by $u$, then $\pi$ is Desarguesian.*

**Theorem 6** *(Ganley, Jha [7], Cordero, Figueroa [4], Liebler [9]).*

*Let $\pi$ be a finite translation plane of order $p^r \neq 2^6$.*

*If $\pi$ admits a collineation group which fixes an infinite point and acts two-transitive on the remaining infinite points then $\pi$ is either Desarguesian or a generalized twisted field plane.*

## 3 The Proof.

Proof: Assume the full translation complement does not leave the autotopism triangle invariant. Note that we then assume that the two infinite points are not interchanged.

We have the following possibilities:

Case (1). One infinite point is fixed while the other is moved.

Case(2). Both infinite points are moved.

In Case (1), there is a fixed point, say $(\infty)$ such that there is a doubly transitive group on the line at infinity minus $(\infty)$. By the result of Ganley, Jha - Cordero, Figueroa - Liebler, the plane is a generalized twisted field plane.

Assume case (2). Hence, the plane is flag-transitive. Assume that the full group is nonsolvable. By the result of Buekenhout et al, the plane is either Desarguesian, Lüneburg-Tits, Hall of order 9 or Hering of order 27.

Assume that the plane is Lüneburg-Tits. Then the group generated by the elations is isomorphic to $S_z(2^{2a+1})$ (where $q=2^{2a+1}$ and the plane has order $q^2$) is normal in the full translation complement $G$. The order of $G$ is divisible by $(q^2-1)(q^2+1)$. The order of $S_z(q)$ is $(q^2+1)q^2(q-1)$. Hence, $G$ induces an automorphism group on $S_z(q)$ isomorphic to $G/C_G(S_z(q))$. The outer automorpism group of $S_z(q)$ has order $f=2a+1$.

Since $G$ is the full translation complement, it contains the kernel homology group of order $q-1$. Hence, the order of $G$ is divisible by $(q^2-1)(q^2+1)(q-1)$. So, the order of $C_G(S_z(q))$ is divisible by $(q+1)(q-1)/(q^2-1,f)$. Since the involutions of $S_z(q)$ are elations, it follows that each element in $C_G(S_z(q))$ is a kern homology. Hence, $(q+1)(q-1)/(q^2-1,f)=(q-1)$. So, if follows that $(q+1)=(q^2-1,f)=(2^{2f}-1,f)=(2^f+1)$. However, $f<2^f+1$.

(Another approach would be to realize that there must be a 3-transitive group containing $S_z(2^{2a+1})$.)

Now assume that the group is solvable in case (2). The group is divisible by $(q^2-1)$ where the order of the plane is $q$. Assume that there is a $p$-primitive divisor on $q-1$. By Lüneburg's result, this says that the plane is Desarguesian.

If there is not a $p$-primitive divisor either $q=2^6$ or $q=p^2$ for some prime $p$ such that $p+1=2^a$ for some integer $a$.

Hence, assume that the $q = p^2$ so that we have a flag-transitive plane of order $p^2$. By Ebert and Baker [6] all the planes are known and only the Desarguesian has the group that we obtain. Hence, this completes the proof of the main result stated in the introduction.

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