THE SUBMANIFOLDS $X_m$ OF THE MANIFOLD $^*g-MEX_n$

I. THE INDUCED CONNECTION ON $X_m$ OF $^*g-MEX_n$

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Abstract. An Einstein’s connection which takes the form (2.33) is called an $^*g$-ME-connection. Recently, Chung and et al ([15],1993) introduced a new manifold, called an n-dimensional $^*g$-ME-manifold (denoted by $^*g$-MEX$_n$). The manifold $^*g$-MEX$_n$ is a generalized n-dimensional Riemannian manifold $X_n$ on which the differential geometric structure is imposed by the unified field tensor $^*g^\lambda\mu$ satisfying certain conditions through the $^*g$-ME-connection. In the following series of two papers, we investigate the submanifold $X_m$ of $^*g$-MEX$_n$:

I. The induced connection on $X_m$ of $^*g$-MEX$_n$

II. The generalized fundamental equations on $X_m$ of $^*g$-MEX$_n$

In this paper, Part I of the series, we present a brief introduction of n-dimensional $^*g$-unified field theory, the C-nonholonomic frame of reference in $X_n$ at points of $X_m$, and the manifold $^*g$-MEX$_n$. And then, we introduce the generalized coefficients of the second fundamental form of $X_m$ and prove a necessary and sufficient condition for the induced connection on $X_m$ of $^*g$-MEX$_n$ to be a $^*g$-ME-connection. Our subsequent paper, Part II of the series, deals with the generalized fundamental equations on $X_m$ of $^*g$-MEX$_n$, such as the generalized Gauss formulae, the generalized Weingarten equations, and the Gauss-Codazzi equations.

1 Introduction

In Appendix II to his last book Einstein ([18],1950) proposed a new unified field theory that would include both gravitation and electromagnetism. Although the intent of this theory is physical, its exposition is mainly geometrical. It may be characterized as a set of geometrical postulates in $X_4$, Hlavatý ([19],1957) gave its mathematical foundation for the first time. Since then Hlavatý and number of mathematicians contributed for the development of this theory and obtained many geometrical consequences of these postulates.

Generalizing $X_4$ to n-dimensional generalized Riemannian manifold $X_n$, n-dimensional generalization of this theory, so called Einstein’s n-dimensional unified field theory (denoted by n-g-UFT hereafter), had been attempted by Wrede ([23],1958) and Mishra ([21], 1959). On the other hand, corresponding to n − g-UFT, Chung ([1],1963) introduced a new unified field theory, called Einstein’s n-dimensional $^*g$-unified field theory (denoted by n − $^*g$-UFT hereafter). This theory is more useful than n − g-UFT in some physical aspects. Chung and et al obtained many results concerning this theory ([2]-[5], 1968-1983; [4],1981; [9],1988; [16][17];1998), particularly proving that n − $^*g$-UFT is equivalent to n − g-UFT so far as the classes and indices of inertia are concerned ([6],1985).

Recently, Chung and et al ([7],1987) introduced a very interesting manifold, called n-dimensional SE-manifold (denoted by SE$_n$ hereafter), imposing the semi-symmetric condition to the Einstein’s connection of $X_n$, and displayed a unique representation of the n-
dimensional Einstein’s connection in a beautiful and surveyable form in terms of $g_{\lambda \mu}$. Many results concerning $\text{SEX}_n$ have been obtained since then ([8],1988; [10]-[14],1989-1991).

An Einstein’s connection which takes the form (2.33) is called a $^*g$-ME-connection. Recently, Chung and et al ([15],1993) introduced a new manifold, called an $n$-dimensional $^*g$-ME-manifold (denoted by $^*g$-MEX$_n$). The manifold $^*g$-MEX$_n$ is a generalization of $n$-dimensional Riemannian manifold $X_n$, on which the differential geometric structure is imposed by the unified field tensor $^*g^{\lambda \nu}$ satisfying the present conditions through the $^*g$- MEX$_n$-connection (see above Definition (2.11) for the words “the present conditions”). In the following series of two papers, we investigate the submanifolds $X_m$ of $^*g$-MEX$_n$:

I. The induced connection on $X_m$ of $^*g$-MEX$_n$

II. The generalized fundamental equations on $X_m$ of $^*g$-MEX$_n$

In this paper, Part I of the series, we present a brief introduction of $n$-dimensional $^*g$-unified field theory, the C-nonholonomic frame of reference in $X_n$, at points of $X_m$, and the manifold $^*g$-MEX$_n$. And then, we introduce the generalized coefficients of the second fundamental form of $X_m$ and prove a necessary and sufficient condition for the induced connection on $X_m$ of $^*g$-MEX$_n$ to be a $^*g$-ME-connection. Our subsequent paper, Part II of the series, deals with the generalized fundamental equations on $X_m$ of $^*g$-MEX$_n$, such as the generalized Gauss formulae, the generalized Weingarten equations, and the Gauss-Codazzi equations.

2 Preliminaries

This section is a brief collection of basic concepts, notations, and results, which are needed in our subsequent considerations. It consists of three subsections; the first subsection (a) is mostly due to [1], the second subsection (b) due to [10], and the third subsection (c) due to [15].

(a) $n$-dimensional $^*g$-unified field theory. Corresponding to the Einstein’s $n$-$g$-UFT\(^1\), our $n$-$^*g$-UFT, initiated by Chung ([1], 1963), is based on the following three principles.

**Principle A.** Let $X_n$ be an $n$-dimensional generalized Riemannian manifold referred to a real coordinate system $x^\nu$, which obeys the coordinate transformation $x^\nu \rightarrow x'^\nu$ for which

\[ \det(\frac{\partial x^\nu}{\partial x^\mu}) \neq 0 \]  

2.1

In $n$-$g$-UFT the manifold $X_n$ is endowed with a real nonsymmetric tensor $g_{\lambda \mu}$, which may be decomposed into its symmetric part $h_{\lambda \mu}$ and skew-symmetric part $k_{\lambda \mu}$\(^2\)

\[ g_{\lambda \mu} = h_{\lambda \mu} + k_{\lambda \mu} \]  

2.2a

where

\[ g = \det(g_{\lambda \mu}) \neq 0, \quad h = \det(h_{\lambda \mu}) \neq 0 \]  

2.2b

\(^1\)Hlavatý characterized Einstein’s 4-dimensional unified field theory 4-$g$-UFT as a set of geometrical postulates in $X_4$ for the first time [19] and gave its mathematical foundation.

\(^2\)Throughout the present paper, Greek indices are used for the holonomic components of tensors in $X_n$. They take the values 1, 2, …, and follow the summation convention. We also assume that $n > 1$ in this paper.
In \( n^* g \)-UFT the algebraic structure on \( X_n \) is imposed by the basic real tensor \(*g^{\lambda\nu}\) defined by
\[
 g_{\lambda\mu} * g^{\lambda\nu} \overset{\text{def}}{=} g_{\mu\lambda} * g^{\nu\lambda} = \delta^\nu_\mu
\]
2.3

It may be also decomposed into its symmetric part \(*h^{\lambda\nu}\) and skew-symmetric part \(*k^{\lambda\nu}\) :
\[
 *g^{\lambda\nu} = *h^{\lambda\nu} + *k^{\lambda\nu}
\]
2.4

Since \( \text{det}(*h^{\lambda\nu}) \neq 0 \), we may define a unique tensor \(*h_{\lambda\mu}\) by
\[
 *h_{\lambda\mu} * h^{\lambda\nu} \overset{\text{def}}{=} \delta^\nu_\mu
\]
2.5

In \( n^* g \)-UFT we use both \(*h^{\lambda\nu}\) and \(*h_{\lambda\mu}\) as tensors for raising and/or lowering indices of all tensors defined in \( X_n \) in the usual manner. We then have
\[
 *k_{\lambda\mu} = *k^{\rho\sigma} * h_{\lambda\rho} * h_{\mu\sigma}, \quad *g_{\lambda\mu} = *g^{\rho\sigma} * h_{\lambda\rho} * h_{\mu\sigma}
\]
2.6a

so that
\[
 *g_{\lambda\mu} = *h_{\lambda\mu} + *k_{\lambda\mu}
\]
2.6b

**Principle B.** The differential geometric structure on \( X_n \) is imposed by the tensor \(*g^{\lambda\nu}\) by means of a connection \( \Gamma^\nu_{\lambda\mu} \) defined by a system of equations
\[
 D_\omega * g^{\lambda\mu} = -2S_{\omega\alpha\mu} * g^{\lambda\alpha}
\]
2.7a

Here \( D_\omega \) denotes the symbol of the covariant derivative with respect to \( \Gamma^\nu_{\lambda\mu} \) and \( S_{\lambda\alpha\nu} \) is the torsion tensor of \( \Gamma^\nu_{\lambda\mu} \). Under certain conditions the system (2.7) admits a unique solution \( \Gamma^\nu_{\lambda\mu} \). A connection satisfying (2.7a) is called an Einstein’s connection in \( n^* g \)-UFT.

**Principle C.** In order to obtain \(*g^{\lambda\nu}\) involved in the solution for \( \Gamma^\nu_{\lambda\mu} \) certain conditions are imposed. These conditions may be condensed to
\[
 S_{\lambda} \overset{\text{def}}{=} S_{\lambda\alpha} = 0, \quad R_{[\mu\lambda]} = \partial_{[\mu} Y_{\lambda]}, \quad R_{(\mu\lambda)} = 0
\]
2.8

where \( Y_\lambda \) is an arbitrary vector, and \( R_{\alpha\mu\lambda\nu} \) together with \( R_{\mu\lambda\nu} \) and \( V_{\omega\mu} \) are the curvature tensors of \( X_n \) defined by
\[
 R_{\alpha\mu\lambda\nu} \overset{\text{def}}{=} 2(\partial_{[\mu} \Gamma^\nu_{\lambda|\omega]} + \Gamma^\alpha_{[\mu} \Gamma^\nu_{\lambda] |\omega])
\]
2.9

\[
 R_{\mu\lambda} \overset{\text{def}}{=} R_{\alpha\mu\lambda\alpha}, \quad V_{\omega\mu} \overset{\text{def}}{=} R_{\alpha\omega\mu\alpha}
\]
2.10

In the following remark, we summarize the main differences between \( n-g \)-UFT and \( n^* g \)-UFT.

\[\text{Hlavatý ([19]) proved that system (2.7a) is equivalent to}\]
\[
 D_\omega g_{\mu\lambda} = 2S_{\omega\alpha\mu} g_{\lambda\alpha}
\]
2.7b

which is also equivalent to the original Einstein’s equations
\[
 \partial_\omega g_{\lambda\mu} - \Gamma^\alpha_{\lambda\omega} g_{\alpha\mu} - \Gamma^\alpha_{\omega\mu} g_{\alpha\lambda} = 0
\]
2.7c
Remark 1 In \( \begin{cases} n - g - UFT \\ n - *g - UFT \end{cases} \), the algebraic structure on \( X_n \) is imposed by the tensor \( g_{\lambda\mu} \) and its inverse tensor \( h^{\lambda\nu} \), and \( \begin{cases} *g_{\lambda\nu} \\ *g^{\lambda\nu} \end{cases} \). The tensor \( *h^{\lambda\nu} \) and its inverse tensor \( *h_{\lambda\mu} \) are used for raising and/or lowering the indices of tensors in \( X_n \). On the other hand, the differential geometric structure on \( X_n \) is imposed by \( \begin{cases} g_{\lambda\mu} \text{ in } n - g - UFT \\ *g^{\lambda\nu} \text{ in } n - *g - UFT \end{cases} \) through the Einstein's connection \( \Gamma_{\lambda\mu}^{\nu} \) satisfying \( \begin{cases} (2.7b) \\ (2.7a) \end{cases} \). Therefore, if the system \( \begin{cases} (2.7b) \\ (2.7a) \end{cases} \) admits a unique solution, the connection \( \Gamma_{\lambda\mu}^{\nu} \) will be expressed in terms of \( \begin{cases} g_{\lambda\mu} \text{ in } n - g - UFT \\ *g^{\lambda\nu} \text{ in } n - *g - UFT \end{cases} \) in virtue of \( \begin{cases} (2.7b) \\ (2.7a) \end{cases} \).

The following quantities are frequently used in our further considerations:

\[
* g = \text{det}(g_{\lambda\mu}), \quad *h = \text{det}(h_{\lambda\mu}), \quad *t = \text{det}(t_{\lambda\mu})
\]

\[
* g = \frac{*g}{*h}, \quad *k = \frac{*t}{*h}
\]

\[
\sigma = \begin{cases} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases}
\]

\[
K_p = *k_{[\alpha_1 \alpha_2 \ldots \alpha_p]}, \quad (p = 0, 1, 2, \ldots)
\]

\[
(0)^*k_{\lambda\nu} = \delta_{\lambda\nu}^{\nu}, \quad (p)^*k_{\lambda\nu} = *k_{\lambda\nu}^{\alpha} (p-1)^*k_{\alpha\nu}^{\nu}, \quad (p = 1, 2, \ldots)
\]

Using these notations we may prove the following two theorems.

**Theorem 2** The following relations hold in \( X_n \) (4):

\[
(p)^*k_{\lambda\mu} = (-1)^p (p)^*k_{\mu\lambda}, \quad (p = 0, 1, 2, \ldots)
\]

\[
K_0 = 1, \quad K_n = *k \text{ if } n \text{ is even, and } K_p = 0 \text{ if } p \text{ is odd}
\]

\[
* g = \sum_{s=0}^{n-\sigma} K_s
\]

\[
\sum_{s=0}^{n-\sigma} K_s (n-s)^*k_{\lambda\mu} = 0
\]

Here and in what follows, the index \( s \) is assumed to take the values 0, 2, 4, \ldots in the specified range.
The submanifolds $X_m$ of the manifold $\ast g \rightarrow MEX_n$

Theorem 3 If the system (2.7) admits a solution $\Gamma^y_{\alpha \mu}$, it must be of the form (11)

$$\Gamma^y_{\alpha \mu} = \ast \{ \nu \} \gamma^\nu_{\alpha \mu} + S^y_{\alpha \mu} \ast U^y_{\alpha \mu}$$

where $\ast \{ \nu \} \gamma^\nu_{\alpha \mu}$ are the Christoffel symbols defined by $\ast h_{\alpha \mu}$ and

$$\ast U^y_{\alpha \mu} = S^y_{\beta (\lambda \nu} k_{\mu \rho)} \ast S^v_{\beta (\nu k_{\lambda \mu})} = S^\beta_{\alpha \mu} \ast k^\nu_{\beta}$$

(b) The C-nonholonomic frame of reference in $n\ast g$-UFT.

This subsection deals with a brief introduction of the concept, the C-nonholonomic frame of reference in $X_m$ at points of its submanifold $X_m$, $m < n$, in $n\ast g$-UFT. It is based on the symbols and results of [15].

Agreement 4 In our further considerations in the present paper, we use the following types of indices:

(a) Small Greek indices $\alpha, \beta, \gamma, \cdot \cdot \cdot$, running from 1 to $n$ and used for the holonomic components of tensors in $X_m$.

(b) Capital Roman indices $A, B, C, \cdot \cdot \cdot$, running from 1 to $n$ and used for the C-nonholonomic components of tensors in $X_m$ at points of $X_m$.

(c) Small Roman italic indices $i, j, k, \cdot \cdot \cdot$ with the exception of $x, y$ and $z$, running from 1 to $m(\cdot n)$. 

(d) Small Roman italic indices $x, y$ and $z$, running from $m + 1$ to $n$.

The summation convention is operative with respect to each set of the above indices within their range, with the exception of $x, y$ and $z$.

Let $X_m$ be a submanifold of $X_n$ defined by a system of sufficiently differentiable equations

$$y^\nu = y^\nu (x^1, \cdot \cdot \cdot , x^m)$$

where the matrix of derivatives $B^\nu_i = \frac{\partial y^\nu}{\partial x^i}$ is of rank $m$. At each point of $X_m$ there exists the first set $\{B^\nu_i, N^\nu_x\}$ of $n$ linearly independent non-null vectors. The $m$ vectors $B^\nu_i$ are tangential to $X_m$, and $n - m$ vectors $N^\nu_x$ are normals to $X_m$ and mutually orthogonal. That is,

$$\ast h_{\alpha \beta} B^\nu_i N^\beta_x = 0, \quad \ast h_{\alpha \beta} N^\nu_x N^\beta_y = 0 \quad \text{for} \quad x \neq y$$

The process of determining the set $\{N^\nu_x\}$ is not unique unless $m = n - 1$. However, we may choose their magnitudes such that

$$\ast h_{\alpha \beta} N^\alpha_x N^\beta_x = \epsilon_x$$

where $\epsilon_x = +1$ or $-1$ according as the left-hand sides of (2.16b) is positive or negative. Put

$$E^\nu_A = \begin{cases} B^\nu_i, & \text{if} \quad A = i = 1, \cdot \cdot \cdot , m \\ N^\nu_x, & \text{if} \quad A = x = m + 1, \cdot \cdot \cdot , n \end{cases}$$
Corresponding to the first set \( \{ E^v_A \} \) of \( n \) linearly independent vectors, there exists a unique second set \( \{ E^\alpha_B \} \) of linearly independent vectors at points of \( X_m \) such that
\[
E^v_A E^v_A = \delta^v_v, \quad E^\alpha_B E^\alpha_B = \delta^\alpha_\alpha
\]  
2.18

Putting
\[
E^\alpha_A = \begin{cases} 
B^i_{\lambda}, & \text{if } A = i = 1, \ldots, m \\
N^x_{\lambda}, & \text{if } A = x = m + 1, \ldots, n
\end{cases}
\]  
2.19
we note that the vectors \( B^i_{\lambda} \) and \( N^x_{\lambda} \) are also tangential and normal respectively to \( X_m \) in virtue of Theorem (2.6).

Now, we are ready to introduce the following concept of C-nonholonomic frame of reference and induced tensors.

**Definition 5** The set \( E^v_A \) and \( E^\alpha_B \) will be referred to as the C-nonholonomic frame of reference in \( X_n \) at points of \( X_m \). This frame gives rise to C-nonholonomic components of tensors in \( X_n \): if \( T^v_{\lambda::} \) are holonomic components of a tensor in \( X_n \), then at points of \( X_m \) its C-nonholonomic components \( T^A_{B::} \) are defined by
\[
T^A_{B::} = T^v_{B::} E^A_v \cdots E^B_B \cdots
\]  
2.20a
In particular, the quantities
\[
T^i_{j::} = T^\alpha_{B::} B^i_{\alpha} \cdots B^j_B \cdots
\]  
2.20b
are components of a tensor in \( X_m \) and are called the components of the induced tensor of \( T^v_{\lambda::} \) on \( X_m \) of \( X_n \).

In virtue of (2.18), an easy inspection shows that
\[
T^v_{\lambda::} = T^A_{B::} E^v_A \cdots E^B_B \cdots
\]  
2.21

The following theorems are consequences of the powerful C-nonholonomic frame of reference.

**Theorem 6** The tensors \( B^v_i, B^i_{\lambda}, N^v_x, N^x_{\lambda} \) and
\[
B^v_{\lambda} = B^i_{\lambda} B^v_i
\]  
2.22
are involved in the following identities:
\[
B^i_{\alpha} B^\alpha_j = \delta^i_j, \quad N^x_{\alpha} N^\alpha_y = \delta^x_y, \quad B^i_{\alpha} N^\alpha_x = N^x_{\alpha} B^\alpha_i = 0
\]  
2.23
\[
B^i_{\lambda} = B^j_{\lambda} h_{\lambda \alpha} * h^{ij}
\]  
2.24a
\[
B^v_{\lambda} = B^j_{\lambda} h^v_{\alpha} * h_{ij}
\]  
2.24b
The submanifolds $X_n$ of the manifold $\ast g-MEX_n$

\[ \ast h^{\alpha \beta} B^i_{\alpha} = \ast h^{ij} B^i_j, \quad \ast h_{\lambda \alpha} B^\alpha_i = \ast h_{ij} B^i_j \]
\[ B^\lambda_i = \delta^\lambda_i - \sum_x N^x_N^\lambda N^i_x \]
\[ B^\alpha_i N^i_\alpha = B^\alpha_i N^i_\alpha = 0 \]
\[ B^\alpha_i B^i_\alpha = B^i_i, \quad B^\alpha_i B^\alpha_i = B^\gamma_i, \quad B^\alpha_i B^\alpha_i = B^\gamma_i \]

2.25

2.26a

2.26b

2.26c

**Theorem 7** At each points of $X_m$, a vector $X_\lambda$ of $X_n$ may be expressed as the sum of two vectors $X_i B^i_\lambda$ and $\sum_x X_x N^x_\lambda$, the former tangential to $X_m$ and the latter normal to $X_m$. That is

\[ X_\lambda = X_i B^i_\lambda + \sum_x X_x N^x_\lambda \]

2.27a

or equivalently

\[ X^\gamma = X^i B^i_\gamma + \sum_x X^x N^x_\gamma \]

2.27b

where

\[ X_i = X_\alpha B^\alpha_i, \quad X_x = X_\alpha N^\alpha_x, \quad \epsilon_x X^x \]

2.28a

\[ X^i = X^\alpha B^\alpha_i, \quad X^x = X^\alpha N^\alpha_x \]

2.28b

Furthermore, $X_i(X^i)$ are components of a tangent vector relative to the transformations of $X_m$, while $X_x(X^x)$ is invariant relative to the transformations of $X_m$ and $X_n$.

**Theorem 8** The induced tensor $\ast g_{ij}$ of $\ast g_{\lambda \mu}$ may be given by

\[ \ast g_{ij} = \ast g_{\alpha \beta} B^\alpha_i B^\beta_j \]

2.29a

where its symmetric part $\ast h_{ij}$ and skew-symmetric part $\ast k_{ij}$ are

\[ \ast h_{ij} = \ast h_{\alpha \beta} B^\alpha_i B^\beta_j, \quad \ast k_{ij} = \ast k_{\alpha \beta} B^\alpha_i B^\beta_j \]

2.29b

so that

\[ \ast g_{ij} = \ast h_{ij} + \ast k_{ij} \]

2.30

In this paper, we restrict our considerations to submanifolds for which the following condition holds:

\[ \text{Det}(\ast h_{ij}) \neq 0 \]

2.31

In virtue of the condition (2.31), we may define a unique inverse tensor $\ast \bar{h}^{ik}$ of $\ast h_{ij}$ by

\[ \ast h_{ij} \ast \bar{h}^{ik} = \delta^k_j \]

2.32

It has been shown that $\ast \bar{h}^{ik}$ is the induced tensor $\ast \bar{h}^{ik}$ of $\ast \bar{h}^{ij}$. That is, $\ast \bar{h}^{ik} = \ast \bar{h}^{ik}$. Therefore, the tensors $\ast h_{ij}$ and $\ast h^{ij}$ may be used for raising and/or lowering indices of the induced tensors in $X_m$ in the usual manner.

(c) **The manifold $\ast g-MEX_n$ in $n-\ast g$-UFT.** All results and symbols in this subsection are based on [15].
**Definition 9**  An Einstein's connection $\Gamma^\nu_{\lambda\mu}$ of the form

\[ \Gamma^\nu_{\lambda\mu} = \{ A^\nu_{\lambda\mu} \} + 2d^\nu_\lambda X_\mu - 2g^\nu_{\lambda\mu} X^\nu \]  

for a non-null vector $X_\lambda$ is called a *$g$-ME-connection in n-*$g$-UFT, and $X_\lambda$ the corresponding *$g$-ME-vector.

In the following theorem, we need the tensor $A^\nu_{\lambda\mu}$ defined by

\[ A^\nu_{\lambda\mu} \overset{\text{def}}{=} -n^*g^\nu_{\lambda\mu} + g^\nu_{\mu\lambda}. \]

Since this tensor is of rank $n$, there exists a unique tensor $B^{\lambda\nu}$ satisfying

\[ A^\nu_{\lambda\mu} B^{\lambda\nu} = A_{\mu\lambda} B^{\nu\lambda} = d^\nu_\mu \]

**Theorem 10** (a) If $X_\mu$ admits a *$g$-ME-connection $\Gamma^\nu_{\lambda\mu}$, it must be of the form (2.13), where

\[ S^\nu_{\lambda\mu} = 2d^\nu_\lambda X_\mu - 2k^\nu_{\lambda\mu} X^\nu, \quad U^\nu_{\lambda\mu} = 2d^\nu_{\lambda\mu} X_\mu - 2h^\nu_{\lambda\mu} X^\nu \]

(b) A necessary and sufficient condition for the system (2.7a) to admit exactly one *$g$-ME-connection $\Gamma^\nu_{\lambda\mu}$ of the form (2.33) is that the tensor field *$g^{\lambda\nu}$ satisfies the following condition

\[ \nabla_\alpha^*k^\lambda_{\mu} = 2h^\alpha_{00[{\lambda}^*}h_{0\beta]} - h^\alpha_{0\lambda} k^\lambda_{\mu}\}C_{\alpha}B^{\alpha\beta} \]

If this condition is satisfied, then

\[ X^\nu = C_{\alpha}B^{\alpha\nu} \]

where

\[ C_{\lambda} = \nabla^*k^\lambda_\alpha \]

Hence, if (2.37) is satisfied, there always exists a unique *$g$-ME-connection $\Gamma^\nu_{\lambda\mu}$ in our n-*$g$-UFT. In virtue of (2.33) and (2.38), this connection may be written as

\[ \Gamma^\nu_{\lambda\mu} = \{ A^\nu_{\lambda\mu} \} + 2(d^\nu_\lambda h_{\mu\beta} - g^\nu_{\lambda\mu} d^\nu_\beta)C_{\alpha}B^{\alpha\beta} \]

The situation that the conditions (2.31) and (2.37) are imposed on the unified field tensor *$g^{\lambda\nu}$ are described in this paper by the words "under the present conditions".

**Definition 11**  An n-dimensional generalized Riemannian manifold $X_n$, on which the differential geometric structure is imposed by the tensor *$g^{\lambda\nu}$ under the present conditions by means of the unique *$g$-ME-connection given by (2.40), is called an n-dimensional *$g$-ME-manifold and denoted by *$g$-MEX$_n$. 
3 The induced connection on $X_m$ of $^g\text{MEX}_n$

This section is devoted to the investigations of the induced connection of the $^g\text{-ME-con}
nection$ imposed on a submanifold $X_m$ of $^g\text{-MEX}_n$ together with the generalized coef
icients $\Omega_{ij}^\gamma$ of the second fundamental form of $X_m$ with emphasis on the proof of Theorem 17, in
which we prove a necessary and sufficient condition for the induced connection of $X_m$ in
$^g\text{MEX}_n$ to be a $^g\text{ME-connection}$. The convenient and powerful $C$-nonholonomic
frame of reference in $^g\text{MEX}_n$ at points of $X_m$ will be employed throughout the present
section. Particularly, we note in virtue of Definition 11 that under the present conditions the
$^g\text{ME-connection}$ of a given $^g\text{MEX}_n$ is unique.

**Definition 12** If $\Gamma_{ij}^\gamma_{\lambda\mu}$ is a connection on a general $X_n$, the connection $\Gamma_{ij}^k$ defined by

$$
\Gamma_{ij}^k = B_k^\gamma(B_{ij}^\gamma + \Gamma_{ij}^\gamma_{\alpha\beta}B_i^\alpha B_j^\beta), \quad B_{ij}^\gamma = \frac{\partial B_i^\gamma}{\partial x^j} = \frac{\partial^2 h_i^\gamma}{\partial x^j \partial y^\gamma} \tag{3.1}
$$

is called the induced connection of $\Gamma_{ij}^\gamma$ on $X_m$ of $X_n$.

The following Theorem is an immediate consequence of Definition (3.1).

**Theorem 13** (a) The torsion tensor $S_{ij}^k$ of the induced connection $\Gamma_{ij}^k$ is the induced tensor of the torsion tensor $S_{ij}^\gamma$ of $\Gamma_{ij}^\gamma$. That is

$$
S_{ij}^k = S_{ij}^\gamma B_i^\gamma B_j^\gamma B_k^\gamma \tag{3.2}
$$

(b) The induced connection $^*\{^k_{ij}\}$ of $^*\{^\gamma_{ij}\}$ is the Christoffel symbols defined by $^*h_{ij}$. That is,

$$
^*\{^k_{ij}\} = \frac{1}{2} h^{kp}(\partial_i^*h_{ip} + \partial_j^*h_{ip} - \partial_p^*h_{ij}) \tag{3.3}
$$

**Proof.** The statement (a) is a direct consequence of (3.1). Using (2.5), (2.21), (2.23), (2.25),
and (2.27b), the statement (b) may be proved as in the following way:

The right-hand side of (3.3)=

$$
= \frac{1}{2} h^{kp}(B_i^\gamma \partial_\alpha(*h_{\beta\epsilon} B_j^\beta B_p^\epsilon) + B_j^\beta \partial_\beta(*h_{\alpha\epsilon} B_i^\alpha B_p^\epsilon) - B_p^\epsilon \partial_\epsilon(*h_{\alpha\beta} B_i^\alpha B_j^\beta))
$$

$$
= \frac{1}{2} (h^{kp} B_p^\epsilon)(\partial_\alpha^*h_{\beta\epsilon} + \partial_\beta^*h_{\alpha\epsilon} - \partial_\epsilon^*h_{\alpha\beta}^*) B_i^\alpha B_j^\beta + *h_{\alpha\epsilon} \partial_\epsilon^* B_i^\alpha B_j^\beta (B_p^\epsilon *) h^{kp})
$$

$$
= B_i^\gamma (* \{ \epsilon_{\alpha\beta} \}) B_i^\alpha B_j^\beta (B_k^\gamma) = * \{^k_{ij}\}
$$

**Theorem 14** The vector $D_j^\gamma B_i^\alpha$ in $X_n$ is normal to $X_m$ and may be given by

$$
D_j^\gamma B_i^\alpha = - \sum_x \Omega_{ij}^\gamma N_x^\alpha \tag{3.4}
$$

where $D_j^\gamma$ is the symbolic vector of the generalized covariant derivative with respect to $x's$. Hence

$$
\Omega_{ij}^\gamma = -(D_j^\gamma B_i^\alpha) N_x^\alpha \tag{3.5}
$$
Theorem 15 The coefficients $\Omega^\alpha_{ij}$ have the following representations:

(a) The tensor $\Omega^\alpha_{ij}$ is the induced tensor of $D^\beta_{\beta} N^\alpha_{\alpha}$ on $X_m$ of $X_n$. That is,

$$\Omega^\alpha_{ij} = (D^\beta_{\beta} N^\alpha_{\alpha}) B^\alpha_i B_j^\beta$$

(b) On $X_m$ of *g-MEX, the coefficients $\Omega^\alpha_{ij}$ may be given by

$$\Omega^\alpha_{ij} = \Lambda^\alpha_{ij} - 2 \varepsilon_x X_x^* g_{ij}$$

where

$$\Lambda^\alpha_{ij} = (\nabla^\beta N^\alpha_{\alpha}) B^\alpha_i B_j^\beta$$

are the generalized coefficients of the second fundamental form with respect to *{$\lambda^\alpha_{\alpha}$}. Here $\nabla^\beta$ denotes the symbolic vector of the covariant derivative with respect to *{$\lambda^\alpha_{\alpha}$}.

Proof. In virtue of (2.23), we first note that

$$0 = \partial_j (B^\alpha_i N^\alpha_{\alpha}) = B^\alpha_i N^\alpha_{\alpha} + (\partial_\beta N^\alpha_{\alpha}) B^\alpha_i B_j^\beta$$

Using (3.5), (3.6) and (3.10), our assertion (3.7) follows as in the following way:

$$\Omega^\alpha_{ij} = -(B^\alpha_i + \Gamma^\alpha_{\beta\gamma} B^{\gamma}_i - \Gamma^\alpha_{ij} B^\beta_k) N^\alpha_{\alpha}$$

$$\Omega^\alpha_{ij} = -(B^\alpha_i N^\alpha_{\alpha} - \Gamma^\alpha_{\beta\gamma} N^\alpha_{\alpha} B^\beta_i B_j^\gamma)$$

$$\Omega^\alpha_{ij} = (D^\beta_{\beta} N^\alpha_{\alpha}) B^\alpha_i B_j^\beta$$

On the other hand, making use of (2.33), (3.9), (2.28a) and (2.29a), the representation (3.8) may be obtained from (3.7) as:

$$\Omega^\alpha_{ij} = (\partial_\beta N^\alpha_{\alpha} + \Gamma^\alpha_{\alpha \beta} N^\alpha_{\alpha}) B^\alpha_i B_j^\beta$$

$$\Omega^\alpha_{ij} = \left[\partial_\beta N^\alpha_{\alpha} + (\{\frac{\gamma}{\gamma}\} + 2 \delta^\gamma_{\alpha} X_{\beta} - 2 g_{\alpha\beta} X^\gamma N^\alpha_{\alpha}) B^\alpha_i B_j^\beta\right]$$

$$\Omega^\alpha_{ij} = \Lambda^\alpha_{ij} - 2 \varepsilon_x X_x^* g_{ij}$$

In virtue of Theorem 15, we note that the coefficients $\Lambda^\alpha_{ij}$ are symmetric, while the coefficients $\Omega^\alpha_{ij}$ are not.

Now, we are ready to prove the following two important theorems.
Theorem 16  On $X_m$ of $^g\text{-}\text{MEX}_n$, the induced connection $\Gamma^k_{ij}$ of the $^g\text{-}\text{ME-connection} \Gamma^\gamma_{\lambda\mu}$ is of the form

$$\Gamma^k_{ij} = \{ \{ i \} \} + 2\delta^k_i X_j - 2^g_{gij} X^k$$

where $X_i$ is the induced vector of $X_N$.

Proof.  Substituting (2.33) into (3.1) and making use of (2.21), the representation (3.11) may be obtained as in the following way:

$$\Gamma^k_{ij} = B^k_{ij} \left[ B^l_{ij} + \{ \{ i \} \} + 2\delta^l_i X_j - 2^g_{gij} X^l \right]$$

$$= B^k_{ij} \left[ B^l_{ij} + \{ \{ i \} \} + 2\delta^l_i X_j - 2^g_{gij} X^l \right] + 2\delta^i_j B^k_{ij} (X^l B^l_j) - 2^g_{gij} B^k_{ij} (X^l B^l_j)$$

$$\Gamma^k_{ij} = \{ \{ i \} \} + 2\delta^k_i X_j - 2^g_{gij} X^k$$

Theorem 17  On $X_m$ of $^g\text{-}\text{MEX}_n$, the induced connection $\Gamma^k_{ij}$ of the unique $^g\text{-}\text{ME-connection} \Gamma^\gamma_{\lambda\mu}$ is a $^g\text{-}\text{ME-connection}$ if and only if the following conditions hold:

$$\delta^i_j X^i - 2^h_{ijk} X_k - 2^g_{gij} X^h - 2^g_{gij} (X^i) = 0$$

$$\nabla^i_k \ast k^j = 2\delta^i_j X_k - 2^g_{gij} X^i + \delta^i_j X_k$$

Proof.  In virtue of Theorem 16, we first note that on $X_m$ of $^g\text{-}\text{MEX}_n$, the induced connection of the $^g\text{-}\text{ME-connection}$ is of the form (3.11). Suppose that it is an Einstein's connection on $X_m$. Then, in virtue of (2.7)a, we have

$$D^*_k g^{ij} = -2S_{kh^j i} g^{ih}$$

Substituting (3.11) into the left-hand side of (3.13), we have

$$D^*_k g^{ij} = \partial^*_k g^{ij} + \Gamma^*_{hk} g^{hj} + \Gamma^*_{hi} g^{ij}$$

$$= \partial^*_k g^{ij} + \{ \{ i \} \} + 2\delta^i_j X_k - 2^g_{gkh} X^i g^{hj} + \delta^i_j X_k - 2^g_{gik} X^j$$

$$= \nabla^i_k \ast k^j + 4^g_{gik} X_k - 4\delta^i_j X^j + 4^g_{gik} X^j - 4^g_{gik} (X^j)$$

In virtue of (2.36) and Theorem 13(a), the right-hand side of (3.13) may be written as

$$-2S_{kh^j i} g^{ih} = -2(2\delta^i_k X_k - 2^g_{gkh} X^i) \ast (h^j + k^ih)$$

$$= 2\delta^i_k (g^{j^i i} - X^i) + 2^g_{gij} X_k + 4^g_{gik} (X^j)$$

Consequently, substitution of (3.14) into (3.13) gives

$$\nabla^i_k \ast k^j = 2 \left[ \delta^i_k X_j - 2^g_{gij} X^i - 2^g_{gik} (X^j) \right]$$

$$= 2(\delta^i_k X^j - 2^g_{gij} X^i + \delta^i_j X_k - 2^g_{gik} X^j)$$

The conditions (3.12) immediately follow from (3.15). Conversely, suppose that the conditions (3.12) hold. Then, since $\nabla^i_k \ast k^j = 0$, we have (3.13) in virtue of (3.14) and (3.15). Hence, the induced connection $\Gamma^k_{ij}$ is Einstein.
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