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ON THE RELAXATION OF SOME TYPES OF DIRICHLET MINIMUM PROBLEMS FOR UNBOUNDED FUNCTIONAL

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Abstract. *In this paper, considered a Borel function g on $\mathbf{R}^n$ taking its values in $[0,+\infty]$, verifying some weak hypothesis of continuity, such that $(\mathrm{dom}g)^{\circ} = \emptyset$ and domg is convex, we obtain an integral representation result for the lower semicontinuous envelope in the $L^1(\Omega)$-topology of the integral functional $G^0(u_0,\Omega,u) = \int_\Omega g(\nabla u)dx$, where $u \in W^{1,\infty}_{loc}(\mathbf{R}^n)$, $u = u_0$ only on suitable parts of the boundary of Ω that lie, for example, on affine spaces orthogonal to aff(domg), for boundary values u_0 satisfying suitable compatibility conditions and Ω is geometrically well situated respect to domg. Then we apply this result to Dirichlet minimum problems.*

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1 Introduction

Let g be a Borel function on $\mathbf{R}^n$ taking its values in $[0,+\infty]$, $\mathrm{dom}g = \{z \in \mathbf{R}^n : g(z) < +\infty\}$ be the effective domain of g, g^{**} be the bipolar of g and $(g^{**})^{\infty}$ be the recession function of g^{**} (see (5) and (6)).

Moreover, if C is a subset of $\mathbf{R}^n$, let aff(C) be the affine hull of C, and, in the case in which C is also convex, ri(C) be the relative interior of C (see §2.1).

Let us suppose that:

$$\begin{array}{l} -\mathrm{dom}\, g \text{ is convex,} \\ -g \text{ is locally bounded on ri } (\mathrm{dom}\, g), \\ -\text{for every bounded subset } L \text{ of dom } g \text{ there exists} \\ \quad z_L \in \mathrm{ri}(\mathrm{dom}g) \text{ such that the function } t \in [0,1] \mapsto g((1-t)z_L + tz) \\ \quad \text{is upper semicontinuous at } t=1 \text{ uniformly as } z \text{ varies in } L. \end{array} \tag{1}$$

For every $u_0 \in W^{1,\infty}_{loc}(\mathbf{R}^n)$ and every bounded open set Ω, let us consider the integral functional

$$G_0(u_0,\Omega,\cdot) : u \in u_0 + W^{1,\infty}_0(\Omega) \mapsto \int_\Omega g(\nabla u)\,dx,$$

the class of admissible boundary values:

$$\begin{array}{rl} T(g,\Omega) = \{u_0 \in W^{1,\infty}_{loc}(\mathbf{R}^n) : & \text{there exist } x_0 \in \Omega \text{ and a compact set} \\ & K \subseteq \mathrm{ri}(\mathrm{dom}\, g) \text{ such that } u_0(x+x_0) - u_0(x_0) \text{ is positively} \\ & 1-\text{homogeneous and } \nabla u_0(x) \in K \text{ for a.e. } x \in \mathbf{R}^n\}. \end{array}$$

and the lower semicontinuous envelope of $G_0(u_0,\Omega,\cdot)$ in the $L^1(\Omega)$-topology given by

$$\overline{G}_0(u_0,\Omega,\cdot) \quad : \quad u \in L^1(\Omega) \mapsto$$
$$\inf\left\{\liminf_h \int_\Omega g(\nabla u_h)\,dx : u_h \in u_0 + W_0^{1,\infty}(\Omega),\ u_h \to u \text{ in } L^1(\Omega)\right\}.$$

In [6] the following representation result for $\overline{G}_0$ was obtained in the case in which $(\mathrm{dom}g)^o \neq \emptyset$: for every convex bounded open set Ω and $u_0 \in T(g,\Omega)$

$$\overline{G}_0(u_0,\Omega,u) =$$
$$= \int_\Omega g^{**}(\nabla u)\,dx + \int_\Omega (g^{**})^\infty \frac{dD^s u}{d|D^s u|} d|D^s u| + \int_{\partial\Omega} (g^{**})^\infty ((u_0-u)\mathbf{n}_\Omega)\,dH^{n-1},$$
$$\text{for every } u \in BV(\Omega),$$

$BV(\Omega)$ being the set of the functions in $L^1(\Omega)$ having distributional partial derivatives that are Borel measures on Ω, ∇u the density of the absolutely continuous part of the vector measure Du with respect to Lebesgue measure, $D^s u$ its singular part and $\frac{dD^s u}{d|D^s u|}$ the Radon-Nikodym derivative of $D^s u$ with respect to the total variation $|D^s u|$ of $D^s u$, $\mathbf{n}_\Omega$ the unit outward vector normal to $\partial\Omega$ and H^{n-1} is the $(n-1)$-dimensional Hausdorff measure.

If we suppose that

$$(\mathrm{dom}g)^o = \emptyset, \tag{2}$$

again in [6] the following representation formula has been proved:

$$\overline{G}_0(u_0,\Omega,u) = \begin{cases} \int_\Omega g^{**}(\nabla u_0)\,dx & \text{if } u = u_0, \\ +\infty & \text{otherwise,} \end{cases}$$

$(u_0 \in T(g,\Omega))$.

This shows that when u assumes the value u_0 on the whole boundary, we have a not much significant result. On the other hand, we are expecting that by assigning values only on suitable parts of the boundary of Ω that lie, for example, on affine spaces orthogonal to aff(dom$g$), it is possible to get more interesting representation results.

The study of such cases is realized in the present paper. More precisely, let us consider an affine transformation $E : \mathbf{R}^n \to \mathbf{R}^n$ such that, denoting by M_E the orthogonal matrix associated to the linear part of E, $E^{-1}(\mathrm{aff}(\mathrm{dom}g)) = \mathbf{R}^k \times \{0_{n-k}\}$ if $k > 0$ or $E^{-1}(\mathrm{aff}(\mathrm{dom}g)) = 0$ if $k = 0$ ($k \in \{0,1,...,n-1\}$ is the dimension of aff(dom$g$)), and the class of sets

$$\mathcal{A} = \left\{E(A \times B) : A \subseteq \mathbf{R}^k \text{and } B \subseteq \mathbf{R}^{n-k} \text{ convex bounded open sets}\right\}.$$

We denote with $\partial_p\Omega$ the part of the boundary of Ω such that $\partial_p\Omega = E((\partial A) \times B)$, for any $\Omega \in \mathcal{A}$.

Then we consider the relaxed functional in the $L^1(\Omega)$-topology

$$\overline{G}^0(u_0,\Omega,u) =$$
$$= \inf\left\{\liminf_h \int_\Omega g(\nabla u_h)\,dx : u_h \in W^{1,\infty}_{loc}(\mathbf{R}^n),\ u_h \to u \text{ in } L^1(\Omega),\quad u_h = u_0 \text{ on } \partial_p\Omega\right\}$$

of the integral functional

$$G^0(u_0,\Omega,\cdot) : u \in W \to \int_\Omega g(\nabla u)dx, \quad \text{with } W = \left\{u \in W^{1,\infty}_{loc}(\mathbf{R}^n) : u = u_0 \text{ on } \partial_p\Omega\right\}.$$

We obtain the following result

$$\overline{G}^0(u_0,\Omega,u) =$$
$$= \int_\Omega g^{**}(\nabla u)\,dx + \int_\Omega (g^{**})^\infty \left(\frac{dD^s u}{d|D^s u|}\right) d|D^s u| + \int_{\partial_p\Omega} (g^{**})^\infty ((u_0 - u)\,\mathbf{n}_\Omega)\,dH^{n-1},$$

for every $\Omega \in \mathcal{A}$, $u_0 \in T(g,\Omega)$, $u \in BV(\Omega)$.

Then we apply this result to Dirichlet minimum problems.
Let g be a Borel function verifying (1), (2) and the coerciveness condition

$$|z| \le g(z) \qquad \text{foreveryz} \in \mathbf{R}^n.$$

If $\Omega \in \mathcal{A}$, $\beta \in L^\infty(\Omega)$, $\lambda > ||\beta||_{L^\infty(\Omega)}$ and $u_0 \in T(g,\Omega)$, then:

$$\inf\left\{\int_\Omega g(\nabla u)\,dx + \int_\Omega \beta u dx + \lambda \int_\Omega |u|\,dx : u \in W^{1,\infty}(\Omega) \text{ such that } u = u_0 \text{ su } \partial_p\Omega\right\} =$$
$$= \min\left\{\int_\Omega g^{**}(\nabla u)\,dx + \int_\Omega (g^{**})^\infty\left(\frac{dD^s u}{d|D^s u|}\right) d|D^s u| + \int_{\partial_p\Omega}(g^{**})^\infty((u_0 - u)\,\mathbf{n}_\Omega)\,dH^{n-1} + \right.$$
$$\left. + \int_\Omega \beta u dx + \lambda \int_\Omega |u|\,dx : u \in BV(\Omega)\right\}. \tag{3}$$

the minimizing sequences of the left hand-side of (3) are compact in $L^1(\Omega)$ and the converging subsequences converge to solutions of the right hand-side of (3).

2 General Notations and Preliminary Results.

2.1 Some results about bipolar and recession functions

Let g be a Borel function with

$$g : z \in \mathbf{R}^n \mapsto g(z) \in [0,+\infty]. \tag{4}$$

Let g^{**} the bipolar of g , i.e. the function defined by (cf. Prop. 4.1, p. 18 in [8])

$$g^{**} : z \in \mathbf{R}^n \mapsto \sup\{\phi(z) : \phi\text{affine},\phi \le g\text{on}\mathbf{R}^n\}. \tag{5}$$

Obviously g^{**} is convex, lower semicontinuous and we have:

$$g^{**}(z) = \sup\{\phi(z) : \phi\text{convex, lowersemicontinuous},\phi \le g\text{on}\mathbf{R}^n\}$$

for every $z \in \mathbf{R}^n$.

Let g^∞ the recession function of a convex function g, i.e.

$$g^\infty : z \in \mathbf{R}^n \longmapsto \lim_{t \to +\infty} \frac{1}{t} g(z_0 + tz), \tag{6}$$

with $z_0 \in$dom$g$. This definition is independent on the choice of z_0 (cf. [13]).

If C is a subset of $\mathbf{R}^n$, aff(C) is the affine hull of C, i.e. the intersection of all affine subsets of $\mathbf{R}^n$ containing C; moreover if C is convex, ri(C) denote the relative interior of C, i.e. the set of the interior points of C in the topology of aff(C).

Remark 2.1. Let g be a Borel function as in (4) such that dom$g$ is convex. By Corollary 7.4.1 of [13] it results

$$\mathrm{ri}(\mathrm{dom}g) = \mathrm{ri}(\mathrm{dom}g^{**}).$$

Moreover, since for any convex subset C of $\mathbf{R}^n$, we have

$$\mathrm{aff}(C) = \mathrm{aff}(\mathrm{ri}C) = \mathrm{aff}(\overline{C}),$$

by the convexity of dom$g$, we obtain

$$\mathrm{aff}(\mathrm{dom}g) = \mathrm{aff}(\mathrm{dom}g^{**}).$$

□

Let g_p be the function

$$g_p : (z_1,...,z_k) \in \mathbf{R}^k \mapsto g(z_1,...,z_k,0_{n-k}) \in [0,+\infty] \tag{7}$$

Lemma 2.2. *Let g be a Borel function as in (4) such that aff(domg)=* $\mathbf{R}^k \times \{0_{n-k}\}$ *and* g_p *as in (7). Then*

$$g_p^{**}(z_1,...,z_k) = g^{**}(z_1,...,z_k,0_{n-k}), \tag{8}$$

$$\left(g_p^{**}\right)^{\infty}(z_1,...,z_k) = \left(g^{**}\right)^{\infty}(z_1,...,z_k,0_{n-k}). \tag{9}$$

Proof. By definition of g^{**} and g_p^{**}, we easily get the inequality

$$g_p^{**}(z_1,...,z_k) \geq g^{**}(z_1,...,z_k,0_{n-k}).$$

Let us prove the opposite inequality. Let $\varphi(z_1,..,z_k)$ a convex and lower semicontinuous function such that

$$\varphi(z_1,..,z_k) \leq g_p(z_1,..,z_k).$$

If we consider the function

$$\overline{\varphi}(z_1,..,z_n) = \begin{cases} \varphi(z_1,..,z_k) & \text{if } z_{k+1} = ... = z_n = 0, \\ +\infty & \text{otherwise,} \end{cases}$$

it results that $\overline{\varphi}$ is convex, lower semicontinuous and $\overline{\varphi}(z_1,..,z_n) \leq g(z_1,..,z_n)$ for every $(z_1,..,z_n) \in \mathbf{R}^n$. So

$$g_p^{**}(z_1,...,z_k) \leq g^{**}(z_1,...,z_k,0_{n-k}).$$

By (8) and definition of the recession function we get (9). □

Let $E : \mathbf{R}^n \to \mathbf{R}^n$ be an affine transformation such that

$$E(z) = M_E z + z_0, \quad \text{with } z_0 \in \mathbf{R}^n, \tag{10}$$

where M_E is a $n \times n$ orthogonal matrix. We will denote with M_E^T the transpose matrix and recall that, obviously, $M_E^{-1} = M_E^T$, $\det M_E = 1$ and

$$||M_E x|| = ||x|| \quad \text{for every } x \in \mathbf{R}^n, \tag{11}$$

where $||\cdot||$ is the euclidean norm.

Let us set

$$g_E : z \in \mathbf{R}^n \to g(E(z)). \tag{12}$$

We have

Lemma 2.3. *Let g be a Borel function as in (4) and g_E the function as in (12). Then*

$$g_E^{**}(z) = g^{**}(E(z)), \quad \forall z \in \mathbf{R}^n, \tag{13}$$

$$(g_E^{**})^\infty(z) = (g^{**})^\infty(M_E z), \quad \forall z \in \mathbf{R}^n. \tag{14}$$

Proof. In fact, by definition of bipolar function, we have

$$g^{**}(E(z)) = \sup\{\phi(E(z)) : \phi \text{ convex, lower semicontinuous and } \phi \leq g \text{ on } \mathbf{R}^n\}.$$

Then, fixed $\varepsilon > 0$ and $z \in \mathbf{R}^n$, there exists a function convex and lower semicontinuous ϕ such that

$$g^{**}(E(z)) - \varepsilon \leq \phi(E(z)) = \psi(z).$$

We know that $\phi \leq g$, and so

$$\psi(z) = \phi(E(z)) \leq g(E(z)) = g_E(z)$$

from which

$$\psi(z) \leq g_E^{**}(z).$$

Then

$$g^{**}(E(z)) \leq g_E^{**}(z).$$

In the same way we obtain the opposite inequality, and so (13).

By (13) and definition of the recession function, with $\bar{z} \in \mathrm{dom} g_E$ and so $E(\bar{z}) \in \mathrm{dom} g$, we have

$$\begin{aligned}(g_E^{**})^\infty(z) &= \lim_{t\to+\infty} \frac{1}{t}(g_E^{**})(tz+\bar{z}) = \lim_{t\to+\infty}\frac{1}{t}(g^{**})(t(M_E z)+E(\bar{z})) = \\ &= (g^{**})^\infty(M_E z).\end{aligned}$$

□

2.2 Some notations and recalls about measure theory and BV functions

We denote the Lebesgue measure on $\mathbf{R}^n$ by $(dx)_n$ and, for every measurable subset E of $\mathbf{R}^n$, the n-dimensional Lebesgue measure of E by $|E|_n$ and its characteristic function by χ_E.

In the following we will adopt the definition of a vector measure μ on an open subset Ω of $\mathbf{R}^n$ given in [2] (cf. Def. 1.1). Then, for every Borel subset B of Ω, the total variation of μ on B is defined by

$$|\mu|(B) = \sup\left\{\sum_{i\in\mathbf{N}} ||\mu(B_i)|| : B = \bigcup_{i\in\mathbf{N}} B_i,\, B_i \text{ disjoint}\right\}. \tag{15}$$

We recall that, by Hahn decomposition Theorem, for a scalar measure μ, we have (cf. Corollary at p. 37 in [15])

$$|\mu|(B) = \sup\left\{\int_B \varphi d\mu : \varphi\ \mu\text{−measurable},\ |\varphi| \leq 1\right\}. \tag{16}$$

If we have two measures μ and ν we denote, as usual, the product measure by $\mu \otimes \nu$.

Remark 2.4. Let $A \subseteq \mathbf{R}^k$ and $B \subseteq \mathbf{R}^{n-k}$ be open subsets. If μ and ν are scalar measures respectively on A and B such that $\nu \geq 0$, then

$$|\mu \otimes \nu| = |\mu| \otimes \nu.$$

Indeed, let $C_1 \subseteq \mathbf{R}^k$, $C_2 \subseteq \mathbf{R}^{n-k}$ be open subsets, and φ a $\mu\otimes\nu$-measurable function such that $|\varphi| \leq 1$. Then

$$\left|\int_{C_1\times C_2} \varphi(x) d\,(\mu\otimes\nu)\right| = \left|\int_{C_2}\left(\int_{C_1}\varphi(x_1,\dots,x_n)d\mu\right)d\nu\right|.$$

Now let us pose

$$\phi(x_{k+1},\dots,x_n) = \frac{1}{\mu(C_1)}\int_{C_1}\varphi(x_1,\dots,x_n)d\mu.$$

So ϕ is a ν-measurable function and $|\phi| \leq 1$. Then

$$\left|\int_{C_2}\left(\int_{C_1}\varphi(x_1,\dots,x_n)d\mu\right)d\nu\right| \leq |\mu|\,(C_1)\left|\int_{C_2}\phi d\nu\right| \leq |\mu|\,(C_1)\nu(C_2).$$

By (16)

$$|\mu\otimes\nu|\,(C_1\times C_2) \leq |\mu|\,(C_1)\otimes\nu(C_2),$$

and so

$$|\mu\otimes\nu| \leq |\mu|\otimes\nu.$$

On the other hand, by (16), there exists a function φ_1 μ-measurable such that $|\varphi_1| \leq 1$ and

$$\int_{C_1}\varphi_1(x_1,\dots,x_k)d\mu \geq |\mu|\,(C_1) - \varepsilon.$$

Since $\nu \geq 0$, there exists a positive ν-measurable function φ_2 such that $|\varphi_2| \leq 1$ and

$$\int_{C_2}\varphi_2(x_{k+1},\dots,x_n)d\nu \geq \nu(C_2) - \varepsilon.$$

If we set $\varphi(x_1,...,x_n) = \varphi_1(x_1,...,x_k)\varphi_2(x_{k+1},...,x_n)$, we have that φ is $(\mu\otimes\nu)$-measurable and $|\varphi| \leq 1$. Then

$$\begin{aligned}\int_{C_1\times C_2}\varphi(x)d(\mu\otimes\nu) &= \int_{C_2}\left(\int_{C_1}\varphi_1(x_1,...,x_k)d\mu\right)\varphi_2(x_{k+1},...,x_n)d\nu \geq \\ &\geq (|\mu|(C_1)-\varepsilon)(\nu(C_2)-\varepsilon).\end{aligned}$$

By (16)

$$|\mu\otimes\nu|(C_1\times C_2) \geq |\mu|(C_1)\nu(C_2),$$

and so

$$|\mu\otimes\nu| \geq |\mu|\otimes\nu.$$

□

If a measure μ is absolutely continuous with respect to a measure ν, we will write $\mu \ll \nu$. If $\mu \ll \nu$, we recall that, by Radon-Nycodim Theorem, there exists a function $\frac{d\mu}{d\nu} \in L^1(\nu)$ such that, for every Borel set A,

$$\mu(A) = \int_A \frac{d\mu}{d\nu}d\nu.$$

Moreover (cf. Theorem 1.13 in [2])

$$\frac{d\mu}{d\nu}(x) = \lim_{\rho\to 0}\frac{\mu(C_\rho(x))}{\nu(C_\rho(x))} \quad \text{for } \nu\text{-}a.e.x. \tag{17}$$

where $C_\rho(x) = (x_1-\rho, x_1+\rho)\times\cdots\times(x_n-\rho, x_n+\rho)$.

We recall that if $\lambda \ll \mu$ and $\mu \ll \nu$, then (cf. [7], p. 138)

$$\frac{d\lambda}{d\nu} = \frac{d\lambda}{d\mu}\frac{d\mu}{d\nu} \qquad \nu\text{-}a.e. \tag{18}$$

Eventually, if $\mu \ll \nu$ and $\nu \ll \mu$, then we write $\mu \simeq \nu$.

For every open set Ω we denote by $BV(\Omega)$ the set of the functions in $L^1(\Omega)$ having distributional partial derivatives that are measures on Ω (cf. Def. 1.41 in [2]) and by $BV_{loc}(\Omega)$ the set of the functions in $L^1_{loc}(\Omega)$ that are in $BV(A)$ for every open set A such that $A \subset\subset \Omega$.

If u is in $BV(\Omega)$, where Ω is an open set, by Lebesgue decomposition theorem, we have $Du = D^a u + D^s u = \int \nabla u\, dx + D^s u$, where $D^a u$ is the absolutely continuous part with respect to Lebesgue measure of Du and $D^s u$ is its singular part; we also denote by $|Du|$ and $|D^s u|$ the total variations of the $\mathbf{R}^n$-valued measures Du and $D^s u$. $BV(\Omega)$ is a Banach space with norm $||u||_{BV(\Omega)} = \int_\Omega |u|\, dx + |Du|(\Omega)$. By $|\nabla u|$ we will denote the total variation of the part absolutely continuous of u, i.e. $|\nabla u|(A) = \int_A ||\nabla u||\, dx$.

We recall that

$$|Du|(\Omega) = \sup\left\{\int_\Omega u\, div\varphi dx : \varphi \in C_0^1(\Omega;\mathbf{R}^n),\ ||\varphi|| \leq 1\right\}.$$

Moreover, if $z \in \mathbf{R}^n$, we will denote with zdx the derivative measure of the function $z\cdot x$, i.e. the measure $A \to \int_A zdx$.

If Ω is a bounded open set with Lipschitz boundary and $u \in BV(\Omega)$, then (cf. [12], [16]) there exists a function in $L^1(\partial\Omega)$, called the trace of u on $\partial\Omega$ and still denoted by u, such that for $H^{n-1}-a.e.\ x \in \partial\Omega$

$$\lim_{\rho\to 0} \rho^{-n} \int_{B_\rho(x)\cap\Omega} |u(z)-u(x)|dz = 0.$$

We recall that (cf. Remark 2.13 in [12]) if Ω is a bounded open set with Lipschitz boundary, Ω' is a bounded open set with $\Omega \subset\subset \Omega'$, $u \in BV(\Omega)$, $v \in BV(\Omega' \setminus \overline{\Omega})$, then the function w defined by

$$w : x \in \mathbf{R}^n \to \begin{cases} u(x) \text{ if } x \in \Omega, \\ v(x) \text{ if } x \in \Omega' \setminus \overline{\Omega} \end{cases}$$

belongs to $BV(\Omega')$ and

$$D^s w(B) = \int_B (v-u)\,\mathbf{n}\, dH^{n-1} \qquad \text{for every Borel subset } B \text{ of } \partial\Omega. \tag{19}$$

where $\mathbf{n}_\Omega$ is the unit outward normal to $\partial\Omega$ and H^{n-1} is the Hausdorff measure.

Given $u_0 \in L^1(\partial\Omega)$, if $u \in BV(\Omega)$ has trace u_0 on $\partial\Omega$ we say that $u = u_0$ on $\partial\Omega$.

If $\{u_h\}$ is a sequence in $BV(\Omega)$ and $u \in BV(\Omega)$, we say that $\{u_h\}$ converges to u in $w^* - BV(\Omega)$, and write $u_h \to u$ in $w^* - BV(\Omega)$, if $u_h \to u$ in $L^1_{loc}(\Omega)$ and the sequence $\{|Du_h|(\Omega)\}$ is bounded. Given a functional F on $BV(\Omega)$ we say that F is sequentially $w^* - BV(\Omega)$-lower semicontinuous if for every sequence $\{u_h\} \subseteq BV(\Omega)$, $u \in BV(\Omega)$ such that $\{u_h\} \to u$ in $w^* - BV(\Omega)$ it results $F(u) \leq \liminf_h F(u_h)$.

For a deeper study of BV-functions we refer to [1], [2], [10], [11], [12], and [16].

For every bounded open set Ω and $\varepsilon > 0$ we define the sets Ω_ε^- and Ω_ε^+ as $\Omega_\varepsilon^- = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$ and $\Omega_\varepsilon^+ = \{x \in \mathbf{R}^n : \operatorname{dist}(x, \partial\Omega) < \varepsilon\}$.

For every $r > 0$ and $x_0 \in \mathbf{R}^n$ let $Q_r(x_0)$ the open cube of $\mathbf{R}^n$ with faces parallel to the coordinate planes centered in x_0 and with sidelenght r and set $Q_r = Q_r(0)$.

Let α be a mollifier, i.e. a nonnegative function in $C_0^\infty(Q_1)$ such that $\int_{\mathbf{R}^n} \alpha(y)\,dy = 1$ and set, for every $\varepsilon > 0$, $\alpha^{(\varepsilon)} : y \in \mathbf{R}^n \mapsto \frac{1}{\varepsilon^n}\alpha\left(\frac{y}{\varepsilon}\right)$.

Let Ω be an open set. For every $u \in L^1(\Omega)$, $\varepsilon > 0$ and $x \in \Omega_\varepsilon^-$ we define the regularization u_ε of u at x by

$$u_\varepsilon(x) = \left(\alpha^{(\varepsilon)} * u\right)(x) = \int_{\mathbf{R}^n} \alpha^{(\varepsilon)}(x-y)\,u(y)\,dy. \tag{20}$$

Remark 2.5. If $u \in L^1(\Omega)$, then, as $\varepsilon \to 0$ (cf. p. 144, note 2 in [14])

$$\nabla u_\varepsilon(x) \to \nabla u(x) \quad L^n - a.e. \text{ in } \Omega.$$

□

Let Ω be a bounded open set and $p \in [1, +\infty]$. If $\{u_h\}$ is a sequence in $W^{1,p}(\Omega)$, $\{u_h\}$ is said to converge to u in $w - W^{1,p}(\Omega)$ ($w^* - W^{1,\infty}(\Omega)$ if $p = +\infty$), if, and only if, $\{u_h\}$ converges weakly to u in $L^p(\Omega)$ (weakly* in $L^\infty(\Omega)$ if $p = +\infty$) and $\{\nabla u_h\}$ converges weakly to ∇u in $(L^p(\Omega))^n$ (weakly* in $(L^\infty(\Omega))^n$ if $p = +\infty$). In this case we write $u_h \to u$ in

$w - W^{1,p}(\Omega)$ ($w^* - W^{1,\infty}(\Omega)$ if $p = +\infty$). Moreover, given a functional F on $W^{1,p}(\Omega)$, we say that F is sequentially $w - W^{1,p}(\Omega)$ ($w^* - W^{1,\infty}(\Omega)$ if $p = +\infty$)-lower semicontinuous if $F(u) \leq \liminf_h F(u_h)$, for every sequence $\{u_h\} \subseteq W^{1,p}(\Omega)$, $u \in W^{1,p}(\Omega)$ such that $u_h \to u$ in $w - W^{1,p}(\Omega)$ ($w^* - W^{1,\infty}(\Omega)$ if $p = +\infty$).

2.3 Some recalls about relaxation

Let g be a Borel function as in (4).
Let us suppose that

$$\begin{aligned} &-\mathrm{dom}\, g \text{ is convex}, \\ &-g \text{ is locally bounded on } \mathrm{ri}(\mathrm{dom}\, g), \\ &-\text{for every bounded subset } L \text{ of dom } g \text{ there exists} z_L \in \mathrm{ri}(\mathrm{dom}\, g) \text{ such that the} \\ &\quad \text{function } t \in [0,1] \mapsto g((1-t)z_L + tz) \text{ is upper semicontinuous at } t = 1 \\ &\quad \text{uniformly as } z \text{ varies in } L, \text{i.e. for every } \varepsilon > 0 \text{ there exists } t_\varepsilon < 1 \text{ such that} \\ &\quad g((1-t)z_L + tz) \leq g(z) + \varepsilon \text{ for every } t \in]t_\varepsilon, 1] \text{ and } z \in L. \end{aligned} \tag{21}$$

For every $u_0 \in W^{1,\infty}_{loc}(\mathbf{R}^n)$ and every bounded open set Ω let us consider the integral functional

$$G_0(u_0, \Omega, \cdot) : u \in u_0 + W^{1,\infty}_0(\Omega) \mapsto \int_\Omega g(\nabla u)\, dx$$

and its lower semicontinuous envelope in the $L^1(\Omega)$-topology given by

$$\begin{aligned} \overline{G}_0(u_0, \Omega, \cdot) \quad &: \quad u \in L^1(\Omega) \mapsto \\ &\to \quad \inf\left\{\liminf_h \int_\Omega g(\nabla u_h)\, dx : u_h \in u_0 + W^{1,\infty}_0(\Omega),\ u_h \to u \text{in} L^1(\Omega)\right\}. \end{aligned} \tag{22}$$

Obviously for every $u_0 \in W^{1,\infty}_{loc}(\mathbf{R}^n)$ and every bounded open set Ω, $\overline{G}_0(u_0, \Omega, \cdot)$ is $L^1(\Omega)$-lower semicontinuous, and

$$\begin{aligned} \overline{G}_0(u_0, \Omega, u) \quad = \quad \min\Big\{\liminf_h \int_\Omega g(\nabla u_h)\, dx : u_h \in u_0 + W^{1,\infty}_0(\Omega), \nabla u_h(x) \in \mathrm{dom}\, g \\ \text{for a.e. } x \in \Omega,\ u_h \to u \text{ in } L^1(\Omega)\Big\}. \end{aligned}$$

for every $u_0 \in W^{1,\infty}_{loc}(\mathbf{R}^n)$, every bounded open set Ω, $u \in L^1(\Omega)$.

Now let us define a classe of admissible boundary values:

$$\begin{aligned} T(g, \Omega) = \{u_0 \in W^{1,\infty}_{loc}(\mathbf{R}^n) : \text{there exist } x_0 \in \Omega \text{ and a compact set} \\ K \subseteq \mathrm{ri}(\mathrm{dom}\, g) \text{ such that } T[x_0]u_0 - u_0(x_0) \text{ is} \\ \text{positively } 1-\text{homogeneous and } \nabla u_0(x) \in K \text{ for a.e. } x \in \mathbf{R}^n\}. \end{aligned} \tag{23}$$

where $T[x_0]u(x) = u(x + x_0)$.

Now we give an integral representation result for relaxed functionals of integrals defined only on functions with fixed boundary data and an application to minimum problems that was proved in Theorem 3.4 of [6] in the case in which

$$(\mathrm{dom} g)^{\circ} \neq \emptyset. \tag{24}$$

Theorem 2.6. *Let g be a Borel function as in (4) verifying (21), (24), $\overline{G}_0$ be given by (22) and $T(g,\cdot)$ by (23), then*

$$\overline{G}_0(u_0,\Omega,u) = \int_\Omega g^{**}(\nabla u)\,dx + \int_\Omega (g^{**})^\infty\left(\frac{dD^s u}{d|D^s u|}\right) d|D^s u| + \int_{\partial\Omega}(g^{**})^\infty((u_0-u)\mathbf{n}_\Omega)\,dH^{n-1},$$

for every convex bounded open set Ω, $u_0 \in T(g,\Omega)$, $u \in BV(\Omega)$. □

Remark 2.7. If Ω has Lipschitz boundary and u is a function in $BV(\Omega)$, let us denote by $\overline{u}$ the extension of u to $\mathbf{R}^n$ defined by $\overline{u} = u_0$ in $\mathbf{R}^n \setminus \Omega$. Clearly $\overline{u}$ is in $BV_{loc}(\mathbf{R}^n)$.

By (19) we can write the result of Theorem 2.6 in the following way:

$$\overline{G}_0(u_0,\Omega,u) = \int_\Omega g^{**}(\nabla u)\,dx + \int_{\overline{\Omega}} (g^{**})^\infty\left(\frac{dD^s \overline{u}}{d|D^s \overline{u}|}\right) d|D^s \overline{u}|.$$

□

3 Some Observations About BV-Functions.

Let $C_1 \subseteq \mathbf{R}^k$ and $C_2 \subseteq \mathbf{R}^{n-k}$ be bounded open sets and $v \in L^1(C_1)$. Let us introduce the function

$$\widetilde{v}\,(\text{or } v^\sim) : x = (x_1,\ldots,x_n) \in C_1 \times C_2 \mapsto v(x_1,\ldots,x_k). \tag{25}$$

Lemma 3.1. *Let $C_1 \subseteq \mathbf{R}^k$ and $C_2 \subseteq \mathbf{R}^{n-k}$ be bounded open sets and $v \in BV(C_1)$, then, considered $C = C_1 \times C_2$, $\widetilde{v} \in BV(C)$ and*

$$\nabla\widetilde{v}(x_1,\ldots,x_n) = (\nabla v(x_1,\ldots,x_k), 0_{n-k}) \quad for\; L^n\text{-}a.e.(x_1,\ldots,x_n) \in C, \tag{26}$$

$$D^s\widetilde{v} = ((D^s v \otimes (dx)_{n-k}), 0_{n-k}), \tag{27}$$

$$|D^s\widetilde{v}| \simeq (|D^s v| \otimes (dx)_{n-k}), \tag{28}$$

$$\left(\left(\frac{dD^s v}{d|D^s v|}\right)^{\sim}(x_1,\ldots,x_n), 0_{n-k}\right) = \frac{dD^s\widetilde{v}}{d(|D^s v| \otimes (dx)_{n-k})}(x_1,\ldots,x_n), \tag{29}$$

$$|D^s v| \otimes (dx)_{n-k} - a.e.in C.$$

Proof. Let us first prove that $\widetilde{v} \in BV(C)$. Obviously $\widetilde{v} \in L^1(C)$. Let $\varphi = (\varphi_1,..,\varphi_n) \in C_0^1(C;\mathbf{R}^n)$ such that $||\varphi|| \leq 1$. It results

$$\int_C \widetilde{v}(x_1,..,x_n)\, div\varphi\, dx_1 \cdot\cdot dx_n = \int_{C_1} v(x_1,..,x_k) dx_1 \cdot\cdot dx_k \int_{C_2} \sum_{i=1}^n \frac{\partial\varphi_i}{\partial x_i} dx_{k+1} \cdot\cdot dx_n.$$

But, being $\int_{C_2} \sum_{i=k+1}^{n} \frac{\partial \varphi_i}{\partial x_i} dx_{k+1} \cdot \cdot dx_n = 0$, we have

$$\int_C \widetilde{v}(x_1,..,x_n)\, div\varphi\, dx_1 \cdot \cdot dx_n = \int_{C_1} v(x_1,..,x_k) dx_1 \cdot \cdot dx_k \sum_{i=1}^{k} \frac{\partial}{\partial x_i} \int_{C_2} \varphi_i(x_1,..,x_n) dx_{k+1} \cdot \cdot dx_n.$$

The functions, for $i = 1,..,k$,

$$\phi_i(x_1,..,x_k) = \frac{1}{|C_2|_{n-k}} \int_{C_2} \varphi_i(x_1,..,x_n) dx_{k+1} \cdot \cdot dx_n$$

belong to $C_0^1(C_1)$ and, since $\left| \int_{C_2} \varphi_i(x_1,..,x_n) dx_{k+1} \cdot \cdot dx_n \right| \le |C_2|_{n-k}$, we have that $\phi = (\phi_1,..,\phi_k) \in C_0^1(C_1;\mathbf{R}^k)$ and $||\phi|| \le 1$; so

$$\begin{aligned}\int_C \widetilde{v}(x_1,..,x_n)\, div\varphi\, dx_1 \cdot \cdot dx_n &= |C_2|_{n-k} \int_{C_1} v(x_1,..,x_k) \sum_{i=1}^{k} \frac{\partial \phi_i}{\partial x_i} dx_1 \cdot \cdot dx_k = \\ &= |C_2|_{n-k} \int_{C_1} v(x_1,..,x_k) div\phi\, dx_1 \cdot \cdot dx_k.\end{aligned}$$

Since $v \in BV(C_1)$, it results that $\widetilde{v} \in BV(C)$.

Let us prove (26).

Let $\widetilde{v}_\varepsilon$ be the regularization of $\widetilde{v}$. Let $\alpha_1 \in C_0^\infty(\mathbf{R}^k)$ and $\alpha_2 \in C_0^\infty(\mathbf{R}^{n-k})$ such that $\alpha_1 > 0$, $\alpha_2 > 0$, $\int_{\mathbf{R}^k} \alpha_1(x_1,...,x_k) dx_1 \cdots dx_k = 1$ and $\int_{\mathbf{R}^{n-k}} \alpha_2(x_{k+1},...,x_n) dx_{k+1} \cdots dx_n = 1$. Let us take $\alpha^{(\varepsilon)}(x) = \alpha_1^{(\varepsilon)}(x_1,...,x_k)\alpha_2^{(\varepsilon)}(x_{k+1},...,x_n)$.

Then for $j \le k$, for the derivative of $\widetilde{v}_\varepsilon$ respect to x_j, we have

$$\nabla_j \widetilde{v}_\varepsilon(x) = \int_{\mathbf{R}^n} \left(\nabla_j \alpha^{(\varepsilon)}(x-y) \right) \widetilde{v}(y) dy \qquad \forall x \in C_\varepsilon^-.$$

Then

$$\begin{aligned}\nabla_j \widetilde{v}_\varepsilon(x_1,...,x_n) &= \\ &= \int_{\mathbf{R}^k} \left(\nabla_j \frac{\alpha_1}{\varepsilon^k} \left(\frac{(x_1-y_1,...,x_k-y_k)}{\varepsilon} \right) \right) v(y_1,..,y_k) dy_1 \cdot \cdot dy_k \\ &\qquad \int_{\mathbf{R}^{n-k}} \frac{\alpha_2}{\varepsilon^{n-k}} \left(\frac{(x_{k+1}-y_{k+1},...,x_n-y_n)}{\varepsilon} \right) dy_{k+1} \cdot \cdot dy_n \\ &= \int_{\mathbf{R}^k} \frac{\alpha_1}{\varepsilon^k} \left(\frac{(x_1-y_1,...,x_k-y_k)}{\varepsilon} \right) dD_j v = (\nabla_j v_\varepsilon)(x_1,...,x_k).\end{aligned}$$

For $\varepsilon \to 0$, we have, by Remark 2.5, $\nabla_j \widetilde{v}_\varepsilon(x) \to \nabla_j \widetilde{v}(x)$ and $\nabla_j v_\varepsilon(x) \to \nabla_j v(x)$ L^n-*a.e.* in C; so $\nabla_j \widetilde{v}(x_1,...,x_n) = \nabla_j v(x_1,...,x_k)$ L^n-*a.e.* in C, for $j \le k$.

In the same way, for $j > k$, we have $\nabla_j \widetilde{v}_\varepsilon(x) = 0$ L^n-*a.e.* in C_ε^-, and so (26).

Let us prove (27).

Let $D_1 \subset C_1$ and $D_2 \subset C_2$ be open sets, and let us pose $D = D_1 \times D_2$. We know that there exist two sequences $\{\varphi_h^1\}_h$ in $C_0^1(D_1)$ and $\{\varphi_h^2\}_h$ in $C_0^1(D_2)$ such that $\varphi_h^1 \to \chi_{D_1}$ and $\varphi_h^2 \to \chi_{D_2}$

everywhere. Taking $\varphi_h(x) = \varphi_h^1(x_1,...,x_k)\varphi_h^2(x_{k+1},...,x_n)$ we have for $j \le k$

$$\begin{aligned}\int_D \varphi_h dD_j\widetilde{v} &= \int_D \varphi_h^1(x_1,...,x_k)\varphi_h^2(x_{k+1},...,x_n)dD_j\widetilde{v} = \\ &= -\int_D \widetilde{v}\nabla_j\varphi_h^1(x_1,...,x_k)\varphi_h^2(x_{k+1},...,x_n)dx = \\ &= -\int_{D_1} v\nabla_j\varphi_h^1(x_1,...,x_k)dx_1\cdots dx_k\int_{D_2}\varphi_h^2(x_{k+1},...,x_n)dx_{k+1}\cdots dx_n = \\ &= \int_{D_1}\varphi_h^1(x_1,...,x_k)dD_jv\int_{D_2}\varphi_h^2(x_{k+1},...,x_n)dx_{k+1}\cdots dx_n.\end{aligned}$$

Passing to the limit on h, we obtain

$$D_j\widetilde{v}(D) = |D_2|_{n-k}\cdot D_jv(D_1).$$

For $j > k$

$$\begin{aligned}\int_D \varphi_h dD_j\widetilde{v} &= \int_D \varphi_h^1(x_1,...,x_k)\varphi_h^2(x_{k+1},...,x_n)dD_j\widetilde{v} = \\ &= -\int_{D_1} v\varphi_h^1(x_1,...,x_k)dx_1\cdots dx_k\int_{D_2}\nabla_j\varphi_h^2(x_{k+1},...,x_n)\,dx_{k+1}\cdots dx_n = 0\end{aligned}$$

and so

$$D\widetilde{v}(D) = \left(\left(Dv\otimes(dx)_{n-k}\right)(D), 0_{n-k}\right). \tag{30}$$

By (26) and (30) we obtain (27).

Let us prove (28).

By definition (15), we have that

$$|D^s\widetilde{v}| \simeq \sum_{i=1}^n |D_i^s\widetilde{v}| \tag{31}$$

and

$$\left(|D^sv|\otimes(dx)_{n-k}\right) \simeq \sum_{i=1}^n \left(|D_i^sv|\otimes(dx)_{n-k}\right). \tag{32}$$

Moreover, by Remark 2.4, we have

$$|D_i^s\widetilde{v}| \simeq \left(|D_i^sv|\otimes(dx)_{n-k}\right). \tag{33}$$

By (31), (32) and (33) we obtain (28).

By definition (17), (27) and (28) we get (29). □

Remark 3.2. Let $A \subseteq \mathbf{R}^k$, $B \subseteq \mathbf{R}^{n-k}$ be convex bounded open sets, $v_0 \in W^{1,\infty}_{loc}(\mathbf{R}^k)$ and $v \in BV(A\times B)$ such that $D_jv(C) = 0$, $\forall j > k$, $\forall C \subset A\times B$ and $v = \widetilde{v}_0$ on $(\partial A)\times B$. Let us set

$$\overset{\vee}{v}(x_1,..,x_k) = \begin{cases} v(x_1,..,x_k,c_{k+1},..,c_n) & \text{if}(x_1,..,x_k)\in A, \\ v_0 & \text{in } \mathbf{R}^k - A, \end{cases}$$

where $(c_{k+1},..,c_n)$ is a fixed vector constant of B. Then

$$\begin{aligned} &\overset{\vee}{v} \in BV_{loc}(\mathbf{R}^k), \\ &\overset{\vee}{v} = v_0 \text{ on } \mathbf{R}^k - A, \\ &\overset{\vee}{\tilde{v}} = v \text{ on } A \times B. \end{aligned} \tag{34}$$

Moreover if $v \in W^{1,\infty}_{loc}(\mathbf{R}^n)$, then $\overset{\vee}{v} \in W^{1,\infty}_{loc}(\mathbf{R}^k)$. □

Remark 3.3. Let μ and ν two measure on $\mathbf{R}^n$ such that ν is absolutely continuous respect to $|\mu|$, and M a $n \times n$-matrix. Then $M\nu$ is absolutely continuous respect to $|\mu|$ and

$$\frac{dM\nu}{d|\mu|} = M\frac{d\nu}{d|\mu|}.$$

□

Given an affine transformation E as in (10) and a function $u \in BV_{loc}(\mathbf{R}^n)$, let us set

$$u^E : y \in \mathbf{R}^n \to u(E(y)) \tag{35}$$

Lemma 3.4. *Let E be an affine transformation as in (10). Then for any $u \in BV_{loc}(\mathbf{R}^n)$, it results that $u^E \in BV_{loc}(\mathbf{R}^n)$ and for any Borel set B of $\mathbf{R}^n$*

$$Du^E(B) = M_E^T Du(E(B)), \tag{36}$$

$$\nabla_y u^E(y) = M_E^T \nabla_x u(E(y)), \quad \forall y \in \mathbf{R}^n, \tag{37}$$

$$D^s u^E(B) = M_E^T D^s u(E(B)), \tag{38}$$

where u^E is defined as in (35).

Proof. Let us consider $\varphi \in C_0^1(\mathbf{R}^n;\mathbf{R}^n)$ such that $||\varphi|| \leq 1$. We know that

$$(D_y\varphi)(E^{-1}(x)) = M_E^T D_x \varphi^{E^{-1}}(x). \tag{39}$$

Then by (39), we obtain

$$\begin{aligned} \int_{\mathbf{R}^n} \varphi(y) dDu^E &= -\int_{\mathbf{R}^n} u^E(y) div\varphi(y)dy = -\int_{\mathbf{R}^n} u(E(y))div\varphi(y)dy = \\ &= -\int_{\mathbf{R}^n} u(x)(div\varphi)(E^{-1}(x))dx = -\int_{\mathbf{R}^n} u(x) M_E^T (div\varphi^{E^{-1}})(x)dx = \\ &= -M_E^T \int_{\mathbf{R}^n} u(x)(div\varphi^{E^{-1}})(x)dx = M_E^T \int_{\mathbf{R}^n} \varphi^{E^{-1}}(x)dDu \end{aligned} \tag{40}$$

We have that, for any $\varphi \in C_0^1(Q;\mathbf{R}^n)$ such that $||\varphi|| \leq 1$, by (11)

$$\begin{aligned} \left\| \int_Q u^E div\varphi(y)dy \right\| &= \left\| -M_E^T \int_{E(Q)} \varphi^{E^{-1}}(x)dDu \right\| = \\ &= \left\| \varphi^{E^{-1}} \right\|_\infty ||Du(E(Q))|| \leq \left\| \varphi^{E^{-1}} \right\|_\infty |Du|(E(Q)) < +\infty, \end{aligned}$$

where Q is a cube in $\mathbf{R}^n$. Then u^E is in $BV_{loc}(\mathbf{R}^n)$.

Since $u^E \in BV_{loc}(\mathbf{R}^n)$, it is sufficient to prove (36) when B is an open set of $\mathbf{R}^n$. We know that there exists a sequence $(\varphi_h)_h$ of functions of $C_0^1(\mathbf{R}^n;\mathbf{R}^n)$, with $\|\varphi_h\| \leq 1$, such that $\varphi_h^i \to \chi_B$ everywhere, for $i = 1,..,n$. By (40) we have that

$$\int_{\mathbf{R}^n} \varphi_h^i(y) dDu^E = \int_{\mathbf{R}^n} (\varphi_h^i)^{E^{-1}}(x)\, dM_E^T Du. \tag{41}$$

But $(\varphi_h^i)^{E^{-1}}(x) = \varphi_h^i(E^{-1}(x))$ and $\varphi_h^i(E^{-1}(x)) \to \chi_B(E^{-1}(x))$ everywhere. Obviously

$$\chi_B(E^{-1}(x)) = \chi_{E(B)}(x). \tag{42}$$

Passing to the limit on h in (41)

$$Du^E(B) = M_E^T Du(E(B)),$$

for every Borel set B of $\mathbf{R}^n$.

Let us prove (37).

By Remark 2.5, we have

$$\nabla_y u^E(y) = \lim_{\varepsilon\to 0} \nabla_y (u^E)_\varepsilon(y).$$

It results

$$(u^E)_\varepsilon(y) = (u_\varepsilon)^E(y). \tag{43}$$

In fact, taking $x = E(z)$ and $\beta(x) = \alpha(M_E^{-1}x)$, we have that β is still a mollifier and so

$$\begin{aligned}
(u^E)_\varepsilon(y) &= \int_{\mathbf{R}^n} \alpha_\varepsilon(y-z) u(E(z))dz = \int_{\mathbf{R}^n} \alpha_\varepsilon(y - E^{-1}(x))u(x)dx = \\
&= \int_{\mathbf{R}^n} \alpha_\varepsilon(E^{-1}(E(y)) - E^{-1}(x))u(x)dx = \int_{\mathbf{R}^n} \alpha_\varepsilon(M_E^{-1}(E(y)-x)u(x)dx = \\
&= \frac{1}{\varepsilon^n}\int_{\mathbf{R}^n} \alpha\left(\frac{M_E^{-1}(E(y)-x)}{\varepsilon}\right) u(x)dx = \frac{1}{\varepsilon^n}\int_{\mathbf{R}^n} \alpha\left(M_E^{-1}\left(\frac{E(y)-x}{\varepsilon}\right)\right) u(x)dx = \\
&= \frac{1}{\varepsilon^n}\int_{\mathbf{R}^n} \beta\left(\frac{E(y)-x}{\varepsilon}\right) u(x)dx = \int_{\mathbf{R}^n} \beta_\varepsilon(E(y)-x)u(x)dx = \\
&= (u_\varepsilon)(E(y)) = (u_\varepsilon)^E(y).
\end{aligned}$$

Then, since $u^E \in BV_{loc}(\mathbf{R}^n)$ and by (43)

$$\nabla_y (u^E)_\varepsilon(y) = \nabla_y u_\varepsilon(E(y)) = M_E^T \nabla_x u_\varepsilon(E(y)). \tag{44}$$

So, by (44), we have:

$$\nabla_y u^E(y) = \lim_{\varepsilon\to 0} \nabla_y (u^E)_\varepsilon(y) = M_E^T \lim_{\varepsilon\to 0} \nabla_x u_\varepsilon(E(y)) = M_E^T \nabla_x u(E(y)).$$

Let us prove (38).

By Lemma 2.3 and Lemma 3.4 we have that

$$
\begin{aligned}
D^s u^E(B) &= Du^E(B) - \int_B \nabla u^E(y)dy = \\
&= M_E^T Du(E(B)) - M_E^T \int_{E(B)} \nabla u(x)dx = M_E^T \left(Du(E(B)) - \int_{E(B)} \nabla u(x)dx \right) = \\
&= M_E^T D^s u(E(B)).
\end{aligned}
$$

□

Lemma 3.5. *Let E be an affine transformation as in (10), μ a measure on $\mathbf{R}^n$. Then for any Borel subset B of $\mathbf{R}^n$*

$$|\mu|(E(B)) = \left|M_E^T \mu\right|(E(B)),$$

Proof. Let B a Borel subset of $\mathbf{R}^n$. By (11) we have

$$\|\mu(E(B))\| = \left\|M_E^T \mu(E(B))\right\|.$$

By definition (15) we get the thesis. □

Corollary 3.6. *Let E be an affine transformation as in (10). Then for any $u \in BV_{loc}(\mathbf{R}^n)$ and for any Borel subset B of $\mathbf{R}^n$*

$$\left|Du^E\right|(B) = |Du|(E(B)),$$

$$\left\|\nabla_y u^E\right\|(y) = \|\nabla_x u\|(E(y)),$$

$$\left|D^s u^E\right|(B) = |D^s u|(E(B)),$$

where u^E is defined as in (35).

Proof. By Lemma 3.4 and Lemma 3.5, taking $\mu = Du$, we have (45).
By Lemma 3.4 and Lemma 3.5, taking $\mu = \int \nabla u dx$, we have

$$|\nabla u|(A) = \int_A \|\nabla u\| dx, \qquad \text{for every open set } A \subseteq \mathbf{R}^n.$$

Then by definition of Lebesgue points, we get (46).

By Lemma 3.4 and Lemma 3.5, taking $\mu = D^s u$, we have (47). □

Lemma 3.7. *Let E an affine transformation as in (10), μ a non negative measure on $\mathbf{R}^n$. If $\mu^E(B) = \mu(E(B))$ for any Borel subset B of $\mathbf{R}^n$ and for every μ^E-measurable positive function ψ, then*

$$\int_{\mathbf{R}^n} \psi(y) d\mu^E = \int_{\mathbf{R}^n} \psi(E^{-1}(x)) d\mu.$$

Proof. Let $\psi(y) = \sum_{i=1}^N \alpha_i \chi_{B_i}(y)$ be a simple function, where $\alpha_1, ..., \alpha_N$ are the values of ψ and $B_i = \{y : \psi(y) = \alpha_i\}$, for $i = 1, ..., N$. By (42) we have

$$\begin{aligned}
\int_{\mathbf{R}^n} \psi(y) d\mu^E &= \sum_{i=1}^{N} \alpha_i \int_{\mathbf{R}^n} \chi_{B_i}(y) d\mu^E = \\
&= \sum_{i=1}^{N} \alpha_i \mu^E(B_i) = \sum_{i=1}^{N} \alpha_i \mu(E(B_i)) = \\
&= \sum_{i=1}^{N} \alpha_i \int_{\mathbf{R}^n} \chi_{E(B_i)}(x) d\mu = \sum_{i=1}^{N} \alpha_i \int_{\mathbf{R}^n} \chi_{B_i}(E^{-1}(x)) d\mu = \\
&= \int_{\mathbf{R}^n} \psi(E^{-1}(x)) d\mu.
\end{aligned}$$

Let now ψ a μ^E-measurable positive function. So there exists an increasing sequence $(\psi_h)_h$ of positive simple functions such that $\psi_h \to \psi \; \mu^E - a.e.$ in $\mathbf{R}^n$. Then taking $y = E^{-1}(x)$

$$\int_{\mathbf{R}^n} \psi(y) d\mu^E = \lim_{h \to +\infty} \int_{\mathbf{R}^n} \psi_h(y) d\mu^E = \lim_{h \to +\infty} \int_{\mathbf{R}^n} \psi_h(E^{-1}(x)) d\mu.$$

By B. Levi theorem, since $\psi_h \to \psi \; \mu^E - a.e.$ in $\mathbf{R}^n$, then

$$\int_{\mathbf{R}^n} \psi(y) d\mu^E = \int_{\mathbf{R}^n} \psi(E^{-1}(x)) d\mu.$$

□

Lemma 3.8. *Let E an affine transformation as in (10) and $u^E \in BV_{loc}(\mathbf{R}^n)$ as in (35). Then for every $\left|D^s u^E\right|$ −measurable positive function ψ it results*

$$\int_{\mathbf{R}^n} \psi(y) d\left|Du^E\right| = \int_{\mathbf{R}^n} \psi(E^{-1}(x)) d\left|Du\right|,$$

$$\int_{\mathbf{R}^n} \psi(y) \left\|\nabla_y uE\right\| dy = \int_{\mathbf{R}^n} \psi(E^{-1}(x)) \left\|\nabla_x u\right\| dx,$$

$$\int_{\mathbf{R}^n} \psi(y) d\left|D^s u^E\right| = \int_{\mathbf{R}^n} \psi(E^{-1}(x)) d\left|D^s u\right|.$$

Proof. By Corollary 3.6 and Lemma 3.7, we obtain the thesis. □

Let us define the function

$$u_T^E : y \in \mathbf{R}^n \to u(E(y)) - (M_E^T z_0) y. \tag{48}$$

Corollary 3.9. *Let E be an affine transformation as in (10). Then for any $u \in BV_{loc}(\mathbf{R}^n)$, it results that $u_T^E \in BV_{loc}(\mathbf{R}^n)$ and for any Borel set B of $\mathbf{R}^n$*

$$Du_T^E(B) = M_E^T Du(E(B)) - (M_E^T z_0) \left|B\right|_n, \tag{49}$$

$$\nabla_y u_T^E(y) = M_E^T \nabla_x u(E(y)) - M_E^{-1} z_0, \quad \forall y \in \mathbf{R}^n, \tag{50}$$

$$D^s u_T^E(B) = M_E^T D^s u(E(B)), \tag{51}$$

$$\left|Du_T^E\right|(B) = |Du - z_0 dx|(E(B)), \tag{52}$$

$$\left\|\nabla_y u_T^E(y)\right\| = \|\nabla_x u - z_0\|(E(y)), \quad \forall y \in \mathbf{R}^n, \tag{53}$$

$$\left|D^s u_T^E\right|(B) = |D^s u|(E(B)), \tag{54}$$

$$\frac{dD^s u_T^E}{d\left|D^s u_T^E\right|}(y) = M_E^T \frac{dD^s u}{d|D^s u|}(E(y)), \quad for \left|D^s u_T^E\right| - a.e.\ y \in \mathbf{R}^n, \tag{55}$$

$$\int_{\mathbf{R}^n} \psi(y) d\left|D^s u_T^E\right| = \int_{\mathbf{R}^n} \psi(E^{-1}(x)) d|D^s u|, \tag{56}$$

$$\int_{\mathbf{R}^n} \psi(y) d\left|Du_T^E\right| = \int_{\mathbf{R}^n} \psi(E^{-1}(x)) d|Du| + |z_0| \int_{\mathbf{R}^n} \psi(E^{-1}(x)) dx, \tag{57}$$

where u_T^E is defined as in (48).

Proof. By Lemma 3.4, Corollary 3.6, Remark 3.3 and Lemma 3.8, it follows the thesis. □

4 A Representation Result in the Case of Fixed Boundary Datum

Let g be a Borel function as in (4).
For every convex bounded open set $A \times B$, such that $A \subseteq \mathbf{R}^k$, $B \subseteq \mathbf{R}^{n-k}$, and for every $u_0 \in W_{loc}^{1,\infty}(\mathbf{R}^n)$ let us consider the integral functional

$$G^0(u_0, A \times B, \cdot) : u \in V \mapsto \int_{A \times B} g(\nabla u)\, dx,$$

with $V = \left\{u \in W_{loc}^{1,\infty}(\mathbf{R}^n) : u = u_0 \text{ on } (\partial A) \times B\right\}$ and its lower semicontinuous envelope in the $L^1(A \times B)$-topology given by

$$\overline{G}^0(u_0, A \times B, \cdot) \quad : \quad u \in L^1(A \times B) \mapsto \inf\left\{\liminf_h \int_{A \times B} g(\nabla u_h)\, dx : u_h \in W_{loc}^{1,\infty}(\mathbf{R}^n), \right. \tag{58}$$
$$\left. u_h \to u \text{ in } L^1(A \times B), \quad u_h = u_0 \text{ on } (\partial A) \times B\right\}.$$

We want to represent the functional $\overline{G}^0(u_0, A \times B, \cdot)$ in the $L^1(A \times B)$-topology, for some bounded convex open set $A \times B$ and boundary values u_0 satisfying compatibility conditions in the case in which

$$(\text{dom} g)^\circ = \emptyset \tag{59}$$

and

$$\text{aff}(\text{dom} g) = \mathbf{R}^k \times \{0_{n-k}\}. \tag{60}$$

Remark 4.1. Given a set Ω with Lipschitz boundary and a function $u \in BV(\Omega)$, let $\overline{u}$ be a function as in Remark 2.7 and $S \subseteq \partial\Omega$; it results by (19)

$$\int_\Omega g^{**}(\nabla u)\,dx + \int_\Omega (g^{**})^\infty \left(\frac{dD^s u}{d|D^s u|}\right) d|D^s u| + \int_S (g^{**})^\infty ((u_0 - u)\mathbf{n})\,dH^{n-1} =$$

$$= \int_\Omega g^{**}(\nabla u)\,dx + \int_{\Omega\cup S} (g^{**})^\infty \left(\frac{dD^s\overline{u}}{d|D^s\overline{u}|}\right) d|D^s\overline{u}|.$$

□

Let us state the following theorem

Theorem 4.2. *Let g be a Borel function as in (4) verifying (21), (59) and (60), $\overline{G}_0$ be given by (58), $A \subseteq \mathbf{R}^k$, $B \subseteq \mathbf{R}^{n-k}$ be convex bounded open sets and $T(g,\cdot)$ by (23), then for every $u_0 \in T(g, A\times B)$, it results*

$$\overline{G}^0(u_0, A\times B, u) = \int_{A\times B} g^{**}(\nabla u)\,dx + \int_{\overline{A}\times B} (g^{**})^\infty \left(\frac{dD^s\overline{u}}{d|D^s\overline{u}|}\right) d|D^s\overline{u}|;$$

for every $u \in BV(A\times B)$.

Proof. Now we first prove that

$$\overline{G}^0(u_0, A\times B, u) \geq \int_{A\times B} g^{**}(\nabla u)\,dx + \int_{\overline{A}\times B} (g^{**})^\infty \left(\frac{dD^s\overline{u}}{d|D^s\overline{u}|}\right) d|D^s\overline{u}|. \tag{61}$$

We can suppose that $\overrightarrow{G}^0(u_0, A\times B, u) < +\infty$. Then there exists $u_h \in W^{1,\infty}_{loc}(\mathbf{R}^n)$, such that $u_h \to u$ in $L^1(A\times B)$, $\nabla u_h \in$dom$g$ and $\overrightarrow{G}^0(u_0, A\times B, u) = \lim_h \int_{A\times B} g(\nabla u_h)\,dx$.

Moreover $u_0 \in T(g, A\times B)$, and so $\nabla u_0(x) \in K \subseteq$ri(dom$g$). Then u_0 depends effectively only on its first k variables in $A\times B$. Then we denote with v_0 the function from A to $\mathbf{R}$ such that $\widetilde{v}_0 = u_0$.

We know that $\nabla u_h \in$dom$g$ and aff(dom$g$) $= \mathbf{R}^k \times \{0_{n-k}\}$. Then $\nabla_{k+1} u_h = \ldots = \nabla_n u_h = 0$ in $A\times B$. So, by the convexity of $A\times B$, u_h depends effectively only on its first k variables in $A\times B$, for h enough large.

Moreover, since $u_h \to u$ in $L^1(A\times B)$, then $|D_j u| = 0$ for $j > k$. So by Remark 3.2 we get that the functions $\overset{\vee}{u} \in BV_{loc}(\mathbf{R}^k)$ and $\overset{\vee}{u}_h \in W^{1,\infty}_{loc}(\mathbf{R}^k)$ satisfying (34). Obviously $\overset{\vee}{u}_h \to \overset{\vee}{u}$ in $L^1(A)$.

Let us set

$$\hat{u}(x) = \widetilde{\overset{\vee}{u}}(x) \text{ in } \mathbf{R}^n. \tag{62}$$

Let g_p the function as in (7). Let us define the functional

$$G^0_p(v_0, A, \cdot) : v \in v_0 + W^{1,\infty}_0(A) \mapsto \int_A g_p(\nabla v)\,dy$$

and

$$\overrightarrow{G}^0_p(v_0, A, \cdot) : v \in L^1(A) \mapsto \inf\{\liminf_h \textstyle\int_A g(\nabla v_h)\,dy :$$
$$: v_h \in v_0 + W^{1,\infty}_0(A), v_h \to v \text{ in } L^1(A)\}.$$

We have that $(\text{dom}g_p)^{\circ} \neq \emptyset$ and ∇v_0 belongs to a compact contained in $\text{ri}(\text{dom}g_p)$.

Then by Theorem 2.6 and Remark 2.7:

$$\overline{G}_p^0(v_0,A,v) = \int_A g_p^{**}(\nabla v)\,dy + \int_{\overline{A}} \left(g_p^{**}\right)^{\infty}\left(\frac{dD^s\overline{v}}{d\left|D^s\overline{v}\right|}\right) d\left|D^s\overline{v}\right|,$$

$\forall v \in BV(A)$.

So we have that

$$\begin{aligned}\overline{G}^0(u_0,A\times B,u) &= \lim_h \int_{A\times B} g\left(\nabla_1 u_h,\dots,\nabla_k u_h,0_{n-k}\right)dx = \\ &= \lim_h |B|_{n-k}\int_A g_p\left(\nabla \overset{\vee}{u}_h\right)dy \geq |B|_{n-k}\overline{G}_p^0\left(v_0,A,\overset{\vee}{u}\right).\end{aligned}$$

We observe that $\overline{\overset{\vee}{u}} = \overset{\vee}{u}$ by (34); then

$$|B|_{n-k}\overline{G}_p^0\left(v_0,A,\overset{\vee}{u}\right) = |B|_{n-k}\int_A g_p^{**}\left(\nabla \overset{\vee}{u}\right)dy + |B|_{n-k}\int_{\overline{A}}\left(g_p^{**}\right)^{\infty}\left(\frac{dD^s\overset{\vee}{u}}{d\left|D^s\overset{\vee}{u}\right|}\right)d\left|D^s\overset{\vee}{u}\right|,$$

and so

$$\overline{G}^0(u_0,A\times B,u) \geq |B|_{n-k}\int_A g_p^{**}\left(\nabla \overset{\vee}{u}\right)dy + |B|_{n-k}\int_{\overline{A}}\left(g_p^{**}\right)^{\infty}\left(\frac{dD^s\overset{\vee}{u}}{d\left|D^s\overset{\vee}{u}\right|}\right)d\left|D^s\overset{\vee}{u}\right|. \quad (63)$$

By Lemma 3.1 we have

$$\nabla \hat{u}(x_1,..,x_n) = \left(\nabla \overset{\vee}{u}(x_1,..,x_k),0_{n-k}\right) \quad \text{for } L^n\text{-}a.e.(x_1,\dots,x_n) \in A\times B.$$

Then by Lemma 2.2 and Fubini Theorem it results

$$\begin{aligned}&|B|_{n-k}\textstyle\int_A g_p^{**}\left(\nabla \overset{\vee}{u}\right)dy + |B|_{n-k}\int_{\overline{A}}\left(g_p^{**}\right)^{\infty}\left(\frac{dD^s u}{d|D^s u|}\right)d\left|D^s\overset{\vee}{u}\right| = \\ &= \textstyle\int_{A\times B} g^{**}\left(\nabla \hat{u}\right)dx + \int_{\overline{A}\times B}\left(\left(g_p^{**}\right)^{\infty}\right)^{\sim}\left(\left(\frac{dD^s u}{d|D^s u|}\right)^{\sim},0_{n-k}\right)d\left(\left|D^s\overset{\vee}{u}\right|\otimes(dx)_{n-k}\right).\end{aligned} \quad (64)$$

By (9) and (25) we have

$$\left(\left(g_p^{**}\right)^{\infty}\right)^{\sim}(z_1,..,z_n) = \left(g^{**}\right)^{\infty}(z_1,..,z_k,0_{n-k}). \quad (65)$$

By (29), (28) and (18), it results

$$\left(\left(\frac{dD^s\overset{\vee}{u}}{d\left|D^s\overset{\vee}{u}\right|}\right)^{\sim},0_{n-k}\right) = \frac{dD^s\widetilde{\overset{\vee}{u}}}{d\left(\left|D^s\overset{\vee}{u}\right|\otimes(dx)_{n-k}\right)} = \frac{dD^s\widetilde{\overset{\vee}{u}}}{d\left|D^s\widetilde{\overset{\vee}{u}}\right|}\;\frac{d\left|D^s\widetilde{\overset{\vee}{u}}\right|}{d\left(\left|D^s\overset{\vee}{u}\right|\otimes(dx)_{n-k}\right)}. \quad (66)$$

By (65), (66), the identity $\hat{u}(x) = \tilde{\overset{\vee}{u}}$ and positively 1-homogenity of the recession function, we obtain

$$\int_{A\times B} g^{**}\left(\nabla \hat{u}\right) dx + \int_{\overline{A}\times B} \left((g_p^{**})^{\infty}\right)^{\sim} \left(\left(\frac{dD^s \overset{\vee}{u}}{d\left|D^s \overset{\vee}{u}\right|}\right)^{\sim}, 0_{n-k}\right) d\left(\left|D^s \overset{\vee}{u}\right| \otimes (dx)_{n-k}\right) = \\ = \int_{A\times B} g^{**}\left(\nabla \hat{u}\right) dx + \int_{\overline{A}\times B} (g^{**})^{\infty} \left(\frac{dD^s \hat{u}}{d\left|D^s \hat{u}\right|}\right) d\left|D^s \hat{u}\right|. \tag{67}$$

By Remark 4.1 applied to $\hat{u}$ and $\overline{u}$ and observing that $\hat{u} = \overline{u}$ on $\mathbf{R}^k \times B$, we obtain

$$\int_{A\times B} g^{**}\left(\nabla \hat{u}\right) dx + \int_{\overline{A}\times B} (g^{**})^{\infty} \left(\frac{dD^s \hat{u}}{d\left|D^s \hat{u}\right|}\right) d\left|D^s \hat{u}\right| = \\ = \int_{A\times B} g^{**}(\nabla u)\, dx + \int_{\overline{A}\times B} (g^{**})^{\infty} \left(\frac{dD^s \overline{u}}{d|D^s \overline{u}|}\right) d|D^s \overline{u}|. \tag{68}$$

Then by (63), (64), (67) and (68) we obtain (61).

Now let us prove the opposite inequality. Let us suppose that

$$\int_{A\times B} g^{**}(\nabla u)\, dx + \int_{\overline{A}\times B} (g^{**})^{\infty} \left(\frac{dD^s \overline{u}}{d|D^s \overline{u}|}\right) d|D^s \overline{u}| < +\infty.$$

Then $\nabla u \in \mathrm{dom} g^{**}$ $L^n - a.e.$ in $A\times B$ and by (60) we get that $\frac{dD^s u}{d|D^s u|} \in \mathrm{aff}(\mathrm{dom} g^{**})$ $|D^s u| - a.e.$ in $A\times B$.

By (60) and Remark 2.1, we obtain that $\nabla_{k+1} u = \ldots = \nabla_n u = 0$ $L^n - a.e.$ in $A\times B$ and

$$\frac{dD^s_{k+1} u}{d\left|D^s_{k+1} u\right|} = \ldots = \frac{dD^s_n u}{d\left|D^s_n u\right|} = 0 \qquad |D^s u| - a.e. \text{ in } A\times B.$$

Then we obtain that $|D_{k+1} u|(A\times B) = \cdots = |D_n u|(A\times B) = 0$.

Let u_ε be the regularization of u given by

$$u_\varepsilon(x) = \int_{\mathbf{R}^n} \alpha^{(\varepsilon)}(x-y)\, u(y)\, dy, \ \varepsilon > 0, \\ \forall x \in (A\times B)^-_\varepsilon = \{x \in A\times B : \mathrm{dist}(x, \partial(A\times B)) > \varepsilon\}.$$

We know that $u_\varepsilon \to u$ in $L^1(A\times B)$. Let us consider $\nabla_j u_\varepsilon(x) = \int_{\mathbf{R}^n} \alpha_\varepsilon(x-y)\, dD_j u$. Since $|D_{k+1} u|(A\times B) = \cdots = |D_n u|(A\times B) = 0$, we have $\nabla_{k+1} u_\varepsilon = \ldots = \nabla_n u_\varepsilon = 0$ $L^n - a.e.$ in $(A\times B)^-_\eta$, where $\eta > \varepsilon$. So u_ε depends effectively only on its first k variables in $(A\times B)^-_\eta$. Since, for $\varepsilon \to 0$, $u_\varepsilon \to u$ in $L^1((A\times B)^-_\eta)$, we obtain that u depends effectively only on its first k variables in $(A\times B)^-_\eta$. Now passing to the limit for $\eta \to 0$, the sets $(A\times B)^-_\eta$ tends to cover all $A\times B$ and so u depends effectively only on its first k variables in $A\times B$.

Moreover $u_0 \in T(g, A\times B)$, and so $\nabla u_0(x) \in K \subseteq \mathrm{ri}(\mathrm{dom} g)$. Then u_0 depends effectively only on its first k variables in $A\times B$. Let $v_0 \in W^{1,\infty}_{loc}\left(\mathbf{R}^k\right)$ such that $u_0 = \widetilde{v}_0$ in $A\times B$.

By Remark 3.2, let us consider $\overset{\vee}{u}$ and $\hat{u}$ defined in (62).

Then (68) and (64) hold. So

$$\int_A g_p^{**}\left(\nabla \overset{\vee}{u}\right) dy + \int_{\overline{A}} (g_p^{**})^{\infty} \left(\frac{dD^s \overset{\vee}{u}}{d\left|D^s \overset{\vee}{u}\right|}\right) d\left|D^s \overset{\vee}{u}\right| < +\infty.$$

But

$$\int_A g_p^{**}\left(\nabla \overset{\vee}{u}\right) dy + \int_{\overline{A}} (g_p^{**})^{\infty} \left(\frac{dD^s \overset{\vee}{u}}{d\left|D^s \overset{\vee}{u}\right|}\right) d\left|D^s \overset{\vee}{u}\right| = \overline{G}_p^0\left(v_0, A, \overset{\vee}{u}\right).$$

Then there exists $v_h \in v_0 + W_0^{1,\infty}(A)$ such that $\nabla v_h(y) \in \mathrm{dom} g_p$ $L^n - a.e.$ in A, $v_h \to \overset{\vee}{u}$ in $L^1(A)$ and

$$\int_A g_p^{**}\left(\nabla \overset{\vee}{u}\right) dy + \int_{\overline{A}} (g_p^{**})^{\infty} \left(\frac{dD^s \overset{\vee}{u}}{d\left|D^s \overset{\vee}{u}\right|}\right) d\left|D^s \overset{\vee}{u}\right| = \liminf_h \int_A g_p(\nabla v_h)\, dy. \tag{69}$$

Obviously $\widetilde{v}_h \to \hat{u}$ in $L^1(A \times B)$ and $\nabla \widetilde{v}_h \in \mathrm{dom} g$. So $\widetilde{v}_h \to u$ in $L^1(A \times B)$ and $\nabla \widetilde{v}_h \in \mathrm{dom} g$ $L^n - a.e.$ in $A \times B$. By (64), (67), (68), (69) and Remark 2.1 we have:

$$\begin{aligned}
\overline{G}^0(u_0, A \times B, u) &\leq \liminf_h \int_{A \times B} g(\nabla \widetilde{v}_h)\, dx = \liminf_h |B|_{n-k} \int_A g_p(\nabla v_h)\, dy = \\
&= |B|_{n-k} \int_A g_p^{**}\left(\nabla \overset{\vee}{u}\right) dy + |B|_{n-k} \int_{\overline{A}} (g_p^{**})^{\infty} \left(\frac{dD^s \overset{\vee}{u}}{d\left|D^s \overset{\vee}{u}\right|}\right) d\left|D^s \overset{\vee}{u}\right| = \\
&= \int_{A \times B} g^{**}(\nabla u)\, dx + \int_{\overline{A} \times B} (g^{**})^{\infty} \left(\frac{dD^s \overline{u}}{d|D^s \overline{u}|}\right) d|D^s \overline{u}|.
\end{aligned}$$

□

We now treat the case in which aff(dom$g$) is just an affine subspace of $\mathbf{R}^n$. Let E be an affine transformation satisfying (10) such that

$$\begin{array}{ll} E^{-1}(\mathrm{aff}(\mathrm{dom} g) = \mathbf{R}^k \times \{0_{n-k}\}, & \text{if } k > 0, \text{or} \\ E^{-1}(\mathrm{aff}(\mathrm{dom} g)) = 0, & \text{if } k = 0, \end{array} \tag{70}$$

where $k \in \{0, 1, ..., n-1\}$ is the dimension of aff(dom$g$). Obviously

$$E^{-1}(y_1) - E^{-1}(y_2) = M_E^{-1}(y_1 - y_2).$$

For every affine transformation E as in (10) and verifying (70), let us consider the class

$$\mathcal{A} = \left\{E(A \times B) : A \subseteq \mathbf{R}^k \text{and } B \subseteq \mathbf{R}^{n-k} \text{ convex bounded open sets}\right\}.$$

Given $\Omega \in \mathcal{A}$, let us denote with $\partial_p \Omega$ the part of the boundary of Ω such that $\partial_p \Omega = E((\partial A) \times B)$.

For every $\Omega \in \mathcal{A}$ and every $u_0 \in W^{1,\infty}_{loc}(\mathbf{R}^n)$, let us consider the integral functional

$$G^0(u_0, \Omega, \cdot) : u \in W \mapsto \int_\Omega g(\nabla u)\, dx,$$

with $W = \left\{ u \in W^{1,\infty}_{loc}(\mathbf{R}^n) : u = u_0 \text{ on } \partial_p\Omega \right\}$ and its lower semicontinuous envelope in tl $L^1(\Omega)$-topology defined by

$$\overline{G}^0(u_0, \Omega, u) = \inf\{\liminf_h \textstyle\int_\Omega g(\nabla u_h)\, dx : u_h \in W^{1,\infty}_{loc}(\mathbf{R}^n), \\ u_h \to u \text{ in } L^1(\Omega),\ u_h = u_0 \text{ on } \partial_p\Omega\}, \tag{71}$$

for $u \in L^1(\Omega)$.

We will obtain a representation result for $\overline{G}^0(u_0, \Omega, \cdot)$ in the $L^1(\Omega)$-topology for $u \in BV(\Omega)$, and $u_0 \in T(g, \Omega)$, in the case in which $\Omega \in \mathcal{A}$ and $(\text{dom} g)^{\text{o}} = \emptyset$.

For every $\Omega \in \mathcal{A}$, there exist $A \subseteq \mathbf{R}^k$ and $B \subseteq \mathbf{R}^{n-k}$ convex bounded open sets such that $E(A \times B)$, for some affine transformation E verifying (10) and (70).

Let g_E a Borel function as in (12); it verifies assumption (21), (60) with g_E in place of g.

For any $u_0 \in T(g, \Omega)$, let us denote with $(u_0)^E_T$ the function as in (48) with u_0 in place of u.

Remark 4.3. If $u_0 \in T(g, \Omega)$, then $(u_0)^E_T \in T(g_E, A \times B)$.
By definition of $T(g, \Omega)$, there exist $x_0 \in \Omega$ and a compact set $K \subseteq \text{ri}(\text{dom} g)$ such that $T[x_0]u_0 - (x_0)$ is positively 1-homogeneous and $\nabla u_0(x) \in K$ for a.e. $x \in \mathbf{R}^n$.

If we set $x_0 = E(y_0)$ and $K_E = M_E^T(K - z_0)$, we have that $T[y_0](u_0)^E_T - (u_0)^E_T(y_0) = u_0(M_E y + x_0) - u_0(x_0) - (M_E^T z_0)y$ is positively 1-homogeneous, K_E is a compact and $K_E \subseteq$ ri(dom g_E, $\nabla(u_0)^E_T(y) \in K_E$ for a.e. $y \in \mathbf{R}^n$. □

Let us consider the functional

$$\overline{G}^0_E\left((u_0)^E_T, A \times B, u\right) = \inf\left\{\liminf_h \textstyle\int_{A\times B} g_E(\nabla u_h)\, dy : u_h \in W^{1,\infty}_{loc}(\mathbf{R}^n), \\ u_h \to u \text{ in } L^1(A \times B),\ u_h = (u_0)^E_T \text{ on } (\partial A) \times B\right\},$$

for $u \in L^1(A \times B)$.

Let us observe that

$$\overline{G}^0_E\left((u_0)^E_T, E^{-1}(\Omega), u^E_T\right) = \overline{G}^0(u_0, \Omega, u), \qquad \text{for every } \Omega \in \mathcal{A} \text{ and } u \in BV(\Omega). \tag{72}$$

In the following we consider unit outward vectors normal to different sets. Thus we will denote with $\mathbf{n}_{A\times B}$ and $\mathbf{n}_\Omega$ the unit outward vector normal to $(\partial A) \times B$ and to $\partial_p\Omega$.

Theorem 4.4. *Let g be a Borel function as in (4) verifying (21), (59), $\overline{G}^0$ be given by (71) and $T(g, \cdot)$ by (23), then*

$$\overline{G}^0(u_0, \Omega, u) = \int_\Omega g^{**}(\nabla u)\, dx + \int_\Omega (g^{**})^\infty \left(\frac{dD^s u}{d|D^s u|}\right) d|D^s u| + \int_{\partial_p\Omega} (g^{**})^\infty ((u_0 - u)\mathbf{n}_\Omega)\, dH^{n-1},$$

for every $\Omega \in \mathcal{A}$, $u_0 \in T(g,\Omega)$, $u \in BV(\Omega)$ and where $\mathbf{n}_\Omega$ denotes the unit outward vecto normal to $\partial_p\Omega$.

Proof. By Theorem 4.2, with u_T^E and $\overline{G}_E^0$ in place of u and $\overline{G}^0$, and (72)

$$\begin{aligned}\overline{G}^0(u_0,\Omega,u) &= \overline{G}_E^0\left((u_0)_T^E, A\times B, u_T^E\right) = \\ &= \int_{A\times B} g_E^{**}\left(\nabla u_T^E(y)\right) dy + \int_{\overline{A}\times B} \left(g_E^{**}\right)^\infty \left(\frac{dD^s\overline{u}_T^E}{d\left|D^s\overline{u}_T^E\right|}(y)\right) d\left|D^s\overline{u}_T^E\right|.\end{aligned} \tag{73}$$

Then, by (13) and (50), taking $x = E(y)$, we have

$$\begin{aligned}\int_{A\times B} g_E^{**}\left(\nabla_y u_T^E(y)\right) dy &= \int_{E^{-1}(\Omega)} g_E^{**}\left(M_E^T \nabla_x u(E(y)) - M_E^T z_0\right) dy = \\ &= \int_{E^{-1}(\Omega)} g^{**}\left(M_E\left(M_E^T \nabla_x u(E(y)) - M_E^T z_0 + M_E^T z_0\right)\right) dy = \\ &= \int_\Omega g^{**}(\nabla_x u(x)) dx.\end{aligned} \tag{74}$$

By (14) and (55)

$$\begin{aligned}\int_{\overline{A}\times B} \left(g_E^{**}\right)^\infty \left(\frac{dD^s\overline{u}_T^E}{d\left|D^s\overline{u}_T^E\right|}(y)\right) d\left|D^s\overline{u}_T^E\right| &= \int_{\overline{A}\times B} (g^{**})^\infty \left(M_E \frac{dD^s\overline{u}_T^E}{d\left|D^s\overline{u}_T^E\right|}(y)\right) d\left|D^s\overline{u}_T^E\right| \\ &= \int_{\overline{A}\times B} (g^{**})^\infty \left(\frac{dD^s\overline{u}}{d\left|D^s\overline{u}\right|}(E(y))\right) d\left|D^s\overline{u}_T^E\right|.\end{aligned} \tag{75}$$

By (75), (56), with $\psi(\cdot) = \chi_{\overline{A}\times B}(\cdot)(g^{**})^\infty\left(\frac{dD^s\overline{u}}{d|D^s\overline{u}|}(E(\cdot))\right)$ we obtain

$$\int_{\overline{A}\times B} (g^{**})^\infty \left(\frac{dD^s\overline{u}}{d\left|D^s\overline{u}\right|}(E(y))\right) d\left|D^s\overline{u}_T^E\right| = \int_{\Omega\cup\partial_p\Omega} (g^{**})^\infty \left(\frac{dD^s\overline{u}}{d\left|D^s\overline{u}\right|}(x)\right) d\left|D^s\overline{u}\right|. \tag{76}$$

By Remark 2.7, we have

$$\begin{aligned}&\int_{\Omega\cup\partial_p\Omega} (g^{**})^\infty\left(\tfrac{dD^s\overline{u}}{d|D^s\overline{u}|}(x)\right) d\left|D^s\overline{u}\right| = \\ &= \int_\Omega (g^{**})^\infty\left(\tfrac{dDu}{d|D^s u|}(x)\right) d\left|D^s u\right| + \int_{\partial_p\Omega} (g^{**})^\infty\left((u_0-u)\,\mathbf{n}_\Omega\right) dH^{n-1}.\end{aligned} \tag{77}$$

So by (73), (74), (76) and (77) we obtain the thesis. □

5 Applications to Dirichlet Minimum Problem

In this section we study the Dirichlet minimum problem for integrals of the type $\int_\Omega g(\nabla u)\,dx$ by assuming that g is a Borel function as in (4) verifying the coerciveness condition (78).

Theorem 5.1. *Let g be a Borel function as in (4) verifying (21), (59), and*

$$|z| \le g(z) \qquad \textit{for every } z \in \mathbf{R}^n. \tag{78}$$

If $\Omega \in \mathcal{A}$, $T(g,\Omega)$ is given by (23), $\beta \in L^\infty(\Omega)$, $\lambda > ||\beta||_{L^\infty(\Omega)}$ and $u_0 \in T(g,\Omega)$, then:

$$\begin{aligned}&\inf\left\{\int_\Omega g(\nabla u)\,dx + \int_\Omega \beta u dx + \lambda \int_\Omega |u|\,dx : u \in W^{1,\infty}(\Omega) \text{ such that } u = u_0 \text{ su } \partial_p\Omega\right\} = \\ &= \min\left\{\int_\Omega g^{**}(\nabla u)\,dx + \int_\Omega (g^{**})^\infty\left(\frac{dD^s u}{d|D^s u|}\right) d\,|D^s u| + \right. \\ &\left. + \int_{\partial_p\Omega}(g^{**})^\infty((u_0 - u)\,\mathbf{n}_\Omega)\,dH^{n-1} + \int_\Omega \beta u dx + \lambda\int_\Omega |u|\,dx : u \in BV(\Omega)\right\}.\end{aligned} \tag{79}$$

the minimizing sequences of the left hand-side of (79) are compact in $L^1(\Omega)$ and the converging subsequences converge to solutions of the right hand-side of (79).

Proof. By Poincaré inequality, (78) and the compactness in $BV(\Omega)$ in the $L^1(\Omega)$-topology of the subsets of $BV(\Omega)$ bounded in the $BV(\Omega)$-norm, we get that the functional

$$u \in W \to \int_\Omega g(\nabla u)\,dx + \int_\Omega \beta u dx + \lambda \int_\Omega |u|\,dx,$$

where $W = \left\{u \in W^{1,\infty}(\Omega) : u = u_0 \text{ on } \partial_p\Omega\right\}$, is coercive on $BV(\Omega)$ with the topology of $L^1(\Omega)$. By virtue of this, well known results in relaxation theory (cf. §1.3 of [3]) and Theorem 4.4, we have that its relaxed functional in the $L^1(\Omega)-$topology is given by

$$u \in L^1(\Omega) \to \begin{cases} \int_\Omega g^{**}(\nabla u)\,dx + \int_\Omega (g^{**})^\infty\left(\frac{dD^s u}{d|D^s u|}\right) d\,|D^s u| + \\ \qquad + \int_{\partial_p\Omega}(g^{**})^\infty((u_0 - u)\,\mathbf{n})\,dH^{n-1} + \int_\Omega \beta u dx + \lambda \int_\Omega |u|\,dx \text{ if } u \in BV(\Omega), \\ +\infty \text{ if } u \notin BV(\Omega). \end{cases}$$

Then, again by standard arguments in relaxation theory, we have (79). □

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