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A REPRESENTATION FORMULA FOR WEAKLY COMPACT STARSHAPED SETS

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Abstract. *Let S be a nonconvex weakly compact and weakly connected subset of a real locally convex topological linear space L and D a relatively weakly open subset of S containing the set* $\mathrm{lnc}_w S$ *of local nonconvexity points of S with respect to the weak topology. It is proved that* $\ker S=\bigcap\{\mathrm{cl\,conv}S_z : z \in D\cap \mathrm{reg}S\}$, *where* reg$S$ *denotes the set of regular points of S and* $S_z = \{s \in S : z$ *is visible from s via S*$\}$. *This substantially strengthens a recent result of Stavrakas in which the intersection above was taken over the whole set* reg$S$. *The intersection formula is shown to hold also for a nonconvex connected weakly compact subset S of L with D being a relatively weakly open subset of S containing the set* lnc$S$ *of local nonconvexity points of S.*

1 Introduction

Let S be a nonempty closed subset of a real locally convex topological linear space L. A point x is *visible* from y via S if and only if the closed line segment $[x,y]$ lies in S. For z in S, we define $S_z = \{s \in S : z \text{ is visible from } s \text{ via } S\}$. We say that a nonempty subset T of S has the *finite visibility property* if and only if every finite subset of T is visible via S from a common point. S is *starshaped* if and only if there is some point p in S such that p sees via S each point of S, and the set of all such points p is denoted by ker$S$ and called the *kernel* of S. A point s in S is called a *point of local convexity* (lc point) of S if and only if there is some neighbourhood N of s in L such that $S\cap N$ is convex. If S fails to be locally convex at q, then q is called a *point of local nonconvexity* (lnc point) of S. For a closed planar subset H of L, a point s in $S\cap H$ is called an *H-lc* (*H-lnc*) *point* of S if and only if $S\cap H$ is locally convex (locally nonconvex) at s. Following [9,Def.6.4], a point s in S is called a *regular point* (reg point) of S if and only if there exists a closed halfspace in L which has s in its bounding closed hyperplane and contains all points visible from s via S. The sets of reg, lc and lnc points of S will be denoted by reg$S$, lc$S$ and lnc$S$, respectively. Besides, $\mathrm{lc}_w S$ and $\mathrm{lnc}_w S$ will denote sets of local convexity and local nonconvexity points of S with respect to the weak topology. Of course, $\mathrm{lc}_w S \subseteq \mathrm{lc}S$ and $\mathrm{lnc}S \subseteq \mathrm{lnc}_w S$. For distinct points x and y, $R(x,y)$ will denote the closed halfline emanating from x via y.

A central theorem of combinatorial geometry due to Krasnosel'skii [9,Th.6.17] states in the simplest form that a compact subset S of R^d is starshaped if and only if every $d+1$ boundary points of S are visible from a common point via S. In the proof one applies Helly's theorem to a formula representing ker$S$ as an intersection of convex hulls of some visibility sets associated with boundary points of S. Such formulae are of independent interest in an infinite-dimensional setting. In [7] Stavrakas proved that for a compact connected nonconvex subset S of a real Banach space $\ker S = \bigcap\{\mathrm{cl\,conv}S_z : z \in D\}$, where D is a relatively open subset of S containing lnc$S$. Recently, he has proved that for S weakly compact in L, ker$S$= $\bigcap\{\mathrm{cl\,conv}S_z : z \in \mathrm{reg}S\}$ [8,Th.1]. Because of the use of convex components and the convergence in Hausdorff metric in [7,Lemma 1] the proof of the former of these results cannot be combined with that of the latter one in order to obtain their common generalization in L. However, in the meantime the author developed effective techniques generating various similar intersection formulae [2] - [5]. In the present paper, we apply them to derive a substantial sharpening of [8,Th.1] stated in the abstract. Also we characterize starshapedness in terms of finite visibility, sharpening [8,Cor.1]. The reader is referred to [6] and [9] for details concerning machinery used in the paper.

2 Lemmas

We present below results helpful in the proof of the main intersection formula. The first of them combines [2,Lemmas 2.3 and 2.5] with [8,Th.1].

Lemma 2.1 *Let S be a weakly compact subset of a real locally convex topological linear space L and p,s distinct points in L, $s \in S$. If s is a boundary point of $S \cap [p,s]$ relative to $[p,s]$, then in every weakly open neighbourhood of s there is a regular point z of S such that $p \notin \mathrm{cl\,conv}S_z$.*

Proof. Select an arbitrary weakly open convex neighbourhood V of s. Since S is weakly closed, there exists a point $y \in (p,s) \cap V \sim S$ and a weakly open convex neighbourhood $U \subseteq V$ of y disjoint from S. Without loss of generality, assume that y is the origin of L and that $U = \bigcap_{i=1}^{n} f_i^{-1}((-1,1))$, where $f_i \in L'$ for $1 \leq i \leq n$. We define a mapping $F : L \to R^n$ by $F(x) = [f_1(x), ..., f_n(x)]^T$. Let R^n be endowed with the usual Euclidean norm. F is weakly continuous and open. We have $\mathbf{0}= F(y) \in F(U)$. Moreover, $F(U) \cap F(S) = \varnothing$, since otherwise we would have $F(u) = F(t)$ for some $u \in U$ and $t \in S$, whence $F(t) = [f_1(t), ..., f_n(t)]^T = F(u) \in \prod_{i=1}^{n}(-1,1)$ implying $t \in U \cap S = \varnothing$, a contradiction. Thus in particular $F(y) \notin F(S)$ and $F(S)$ is compact. Let $N_{F(s)} \subseteq F(V)$ be a closed nondegenerate ball at $F(s)$ and N a closed nondegenerate ball at $\mathbf{0}$ $=F(y)$ such that $N \subseteq (N_{F(s)} - F(s)) \cap F(U)$. Translate N towards $F(s)$ along the line segment $[F(y), F(s)]$ until $F(S)$ is intersected and let u be a point in bdry$N' \cap F(S)$, where N' is the translate of N. Easily, $u \in F(V)$. Let H_u be the unique hyperplane in R^n supporting N' at u. Since N is smooth, u is a regular point of $F(S)$. Denote by H_u^+ the closed halfspace determined by H_u, not containing $F(y)$ $=\mathbf{0}$. Select a linear functional h_u on R^n such that $H_u = \{r \in R^n : h_u(r) = 1\}$. Define the continuous linear functional on L by $G = h_u \circ F$. Let $z \in S \cap V$ be a point such that $F(z) = u$. Of course, $z \in$ bdry$S$ and $G(z) = 1$. We claim that $S_z \subseteq \{x \in L : G(x) \geq 1\}$. Take a point $q \in S, q \neq z$, such that $[q,z] \subseteq S$. Suppose that $[q,z] \nsubseteq \{x \in L : G(x) \geq 1\}$, i.e. $h_u \circ F(q) = G(q) < 1$, whence $F(q) \in \sim H_u^+$. Since F is linear and $[q,z] \subseteq S$, we have $[F(q), F(z)] = [F(q), u] \subseteq F(S)$. Since N' is smooth and $u \in$ bdryN', it must be $F(q) = u$, whence $G(q) = h_u(u) = 1$, a contradiction. Consequently, $z \in$ reg$S$. Furthermore, $\mathrm{cl\,conv}S_z \subseteq \{x \in L : G(x) \geq 1\}$ and p, y lie in the complement of this halfspace because $y \in (p,s), G(y) = 0$. Finally, $p \notin \mathrm{cl\,conv}S_z$, as required. □

In Lemma 2.1 we incidentally established the following fact (cf. [2,Lemma 2.2]).

Corollary 2.2 *If S is a weakly compact subset of a real locally convex topological linear space L, then* reg$S$ *is a set everywhere dense in* bdry$S$ *in the weak topology.*

Lemma 2.1 yields also the following result.

Lemma 2.3 *Let S be a nonconvex weakly compact and weakly connected subset of a real locally convex topological linear space L. Let $x \in \bigcap_{z\in D\cap \mathrm{reg}S} \mathrm{cl}\,\mathrm{conv}S_z$, where D is a relatively weakly open subset of S containing* $\mathrm{lnc}_w S$, *and let $[a,b] \subseteq S$. If points x,a,b are noncollinear, then there exists a convex neighbourhood U of the origin such that $S\cap \mathrm{conv}\{x,a,b\} \cap ([a,b]+U)$ consists exclusively of* $\mathrm{conv}\{x,a,b\}$ *- lc points of S. Hence,if, in addition, $[u,a] \subseteq S$ for some point $u \in [x,a)$, then* $\mathrm{conv}\{u,a,b\} \subseteq S$. *This holds true also for a nonconvex connected weakly compact subset S of L with D being a relatively weakly open subset of S containing* lnc$S$.

Proof. We establish the assertion in the first of specified settings. By Tietze's theorem [9,Th.4.4], $\mathrm{lnc}_w S$ is nonempty. Suppose, to reach a contradiction, that no such U exists. Since $[a,b]$ and the set of $\mathrm{conv}\{x,a,b\}$ -lnc points of S are compact sets, there must be in $[a,b]$ a $\mathrm{conv}\{x,a,b\}$- lnc point y. Of course, $y \in \mathrm{lnc}S \subseteq \mathrm{lnc}_w S \subseteq D$, so that there exists a convex weakly open neighbourhood V of the origin such that $y+V \subseteq D$ and $x \notin y+ \mathrm{cl}V$. y is a $\mathrm{conv}\{x,a,b\}$ -lnc point of S, so that there exist points $r,s \in S\cap \mathrm{conv}\{x,a,b\} \cap (y+V)$ such that $[r,s] \nsubseteq S$ and, additionally, $R(x,r) \cap [a,b] \subseteq y+V$ and $R(x,s) \cap [a,b] \subseteq y+V$. Select a point $t \in [r,s] \sim S$ and let w be a point of S lying on $R(x,t) \sim [x,t]$ as close as possible to t. Then w is a relative boundary point of $S \cap [x,w]$, so that Lemma 2.1 implies that there exists in $y+V$ a regular point z of S for which $x \notin \mathrm{cl}\,\mathrm{conv}S_z$, a contradiction. Now let $[u,a] \cup [a,b] \subseteq S$ for some $u \in [x,a)$. Consider the set $\mathcal{P}$ of these points $v \in [u,a]$ for which $\mathrm{conv}\{v,a,b\} \subseteq S$. The above considerations together with a planar result called Valentine's lemma [10],[2,Lemma 2.7] imply that $\mathcal{P}$ is relatively open in $[u,a]$. But S is weakly compact, i.e. also closed, in L, whence $\mathcal{P}$ is also relatively closed in $[u,a]$. Consequently, $\mathcal{P} \equiv [u,a]$, i.e. $\mathrm{conv}\{u,a,b\} \subseteq S$, as desired. The argument holds word for word also for the second case. □

3 Main results

The following theorem is the main result of the paper.

Theorem 3.1 *If S is a nonconvex weakly compact and weakly connected subset of a real locally convex topological linear space L and D a relatively weakly open subset of S containing* $\mathrm{lnc}_w S$, *then*

$$\mathrm{ker}S = \bigcap_{z\in D\cap \mathrm{reg}S} \mathrm{cl}\,\mathrm{conv}S_z$$

This holds true also for a nonconvex connected weakly compact subset S of L with D being a relatively weakly open subset of S containing lnc$S$.

Proof. We consider the first assertion. It is enough to prove the inclusion $\bigcap_{z\in D\cap \mathrm{reg}S} \mathrm{cl}\,\mathrm{conv}S_z \subseteq$ ker$S$. Select any point $p \in \bigcap_{z\in D\cap \mathrm{reg}S} \mathrm{cl}\,\mathrm{conv}S_z$ and any point $s \in D\cap \mathrm{reg}S$, $p \neq s$, in order to prove first that $[p,s] \subseteq S$. It must be $[s,t] \subseteq S$ for some point $t \in [p,s)$, since otherwise Lemma 2.1 would imply the existence of a regular point z in D with $p \notin \mathrm{cl\,conv} S_z$ which is contradictory. If $t = p$, then we are done, thus let $[s,t]$ be the longest proper subsegment of $[p,s]$ lying in S. Then t must be an lc_w point of S, since otherwise Lemma 2.1 would lead to an extension of $[s,t]$ in S beyond t, contradicting the choice of t. Hence, let U_t be a weakly open convex neighbourhood of t such that $S \cap U_t$ is convex. Without loss of generality, suppose that t is the origin of L. Since S is weakly compact, it is also weakly bounded, i.e. $S \subseteq \alpha U_t$ for some $\alpha > 0$. Let $M = \min\{\frac{1}{2}, \frac{1}{\alpha+1}\}$. We claim that $[\circ, Mp] \subseteq S$. Of course, $u = Mp \in U_t$ because $p \in \mathrm{cl\,conv} S_s \subseteq \mathrm{cl\,conv} S \subseteq (\alpha+1)U_t \subseteq \frac{1}{M}U_t$. Let $U \subseteq U_t$ be an arbitrarily small weakly open convex neighbourhood of u. Then $V = \frac{1}{M}U$ is an open convex neighbourhood of p. By assumption, $p \in \mathrm{cl\,conv} S_s$, i.e. there are points $x_{1,U}, \ldots, x_{n(U),U} \in S_s$ such that $V \cap \mathrm{conv}\{x_{1,U}, \ldots, x_{n(U),U}\} \neq \varnothing$. Select a point y in this intersection. Applying Lemma 2.3, we obtain $[x_{i,U}, t] \subseteq \mathrm{conv}\{x_{i,U}, t, s\} \subseteq S$ for all $1 \leq i \leq n(U)$. Furthermore, $My \in U$ and, since $[\circ, Mx_{i,U}] \subseteq S \cap M\alpha U_t \subseteq S \cap U_t$ for all $1 \leq i \leq n(U)$, the choice of U_t implies $[\circ, My] \subseteq \mathrm{conv}(\bigcup_{i=1}^{n(U)} [\circ, Mx_{i,U}]) \subseteq S \cap U_t$. Since U was selected arbitrarily, we infer that each point of $[\circ, Mp]$ lies in the weak closure of S,i.e. in S itself. Summarizing, we produced a subsegment $[Mp, s]$ of $[p,s]$ longer than $[t,s]$, which is contradictory. Hence, it must be $[p,s] \subseteq S$, as required.

The rest is easy. Select in S an arbitrary point a different from p. By [1,Lemma 2.3], $[w,a] \subseteq S$ for some lnc_w point w of S. By Corollary 2.2, w is in the weak closure of $\mathrm{reg} S$, so that by virtue of the above argument $[p,w] \subseteq S$. Hence, $[p,w] \cup [w,a] \subseteq S$ and Lemma 2.3 implies, in the only nontrivial case when p, w, a are noncollinear, that $[p,a] \subseteq S$. Consequently, $p \in \ker S$, finishing the argument in the first case.

It remains to check where this argument should be modified to ensure the validity of the second assertion. Tietze's theorem in the original topology yields $\mathrm{lnc} S \neq \varnothing$. t must be an lc point of S and U_t an open convex neighbourhood of S such that $S \cap U_t$ is convex. By Mackey's theorem [6, \$21.11.(7)], S is bounded, so that, we can choose $\alpha > 0$ such that $S \subseteq \alpha U_t$. U is an arbitrarily small open convex neighbourhood of u contained in U_t. We deduce that each point of $[\circ, Mp]$ lies in the closure of S, i.e. in S too since this set is closed. Finally, [1,Lemma 2.3] in the original topology yields $[w,a] \subseteq S$ for some lnc point w of S. By Corollary 2.2, w is in the weak closure of $\mathrm{reg} D$ and S is weakly closed, so that $[p,w] \subseteq S$. An application of the second case of Lemma 2.3 also yields required $p \in \ker S$. □

Proposition 3.2 *Let S be a nonconvex connected weakly compact subset of a real locally convex topological linear space and D a relatively weakly open subset of S containing* $\mathrm{lnc} S$. *Then S is starshaped if and only if every finite subset of $D \cap \mathrm{reg} S$ is visible via S from a common point.*

Proof. The sufficiency is the only nontrivial task. By Tietze's theorem in the original topology, $\mathrm{lnc} S \neq \varnothing$. Consider the family $\mathfrak{F} = \{S_z : z \in D \cap \mathrm{reg} S\}$ consisting of weakly closed subsets of the weakly compact set S which have the finite intersection property. Hence, there is a point $p \in \bigcap_{z \in D \cap \mathrm{reg} S} S_z$. Since, by Corollary 2.2, $\mathrm{reg} S$ is a set weakly everywhere dense in $\mathrm{bdry} S$ and S is weakly closed, we have $p \in \bigcap_{z \in D \cap \mathrm{bdry} S} S_z$. Select any point t in S to show that $[p,t] \subseteq S$. By [1,Lemma 2.3], $[t,q] \subseteq S$ for some $q \in \mathrm{lnc} S$. For nontriviality, let points p, q, t be noncollinear. Consider the plane $F = \mathrm{aff}\{p,q,t\}$ equiped with the Euclidean norm and the component C of $S \cap F$ containing p. If there are no local nonconvexity points of C relative to F, then, by Tietze's theorem, C is convex, whence $[p,t] \subseteq S$, as desired. If C is nonconvex, then $p \in C \cap \bigcap_{z \in D \cap \mathrm{bdry} S} S_z \subseteq \bigcap_{z \in D \cap \mathrm{rel\,bdry} C} C_z = \ker C$, where the last equality follows from [2, Th.3.1]. This again yields $[p,t] \subseteq S$ and finally desired $p \in \ker S$. $\square$

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