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A VISCOELASTIC FLUID FLOW THROUGH MIXING GRIDS

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Abstract. *We study the asymptotic behaviour of a viscoelastic fluid in a porous medium* Ω_ε, $(\varepsilon > 0)$ *obtained by removing from an open set* Ω *some small obstacles* $(T_\nu^\varepsilon)_{1\leq\nu\leq n(\varepsilon)}$ *of size* a_ε *periodically distributed on a hyperplane H which intersects* Ω. *We establish that the fluid behaves differently depending on whether the size* a_ε *is greater than or smaller than a critical size* c_ε. *If* $a_\varepsilon = c_\varepsilon$, *a convolution term appears in the limit problem. This corresponds to a long memory effect. If* a_ε *is smaller than* c_ε, *the fluid behaves as if there where no obstacles. If* a_ε *is greater than* c_ε *or is of the order of the period, the fluid adheres on the hyperplane H which plays a thin solid plate role and the fluid behaves separately on each side of this plate.*

1 Introduction

Viscoelastic materials have a gift for a "continuous memory" in the sense that the stresses, at any moment t, depend on the history of all deformations previously subjected by the material. There are many substances in a industrial feeding which are viscoelastic example: gelatinous liquid. In biomechanics, we study viscoelastic fluids (saliva, ...) and viscoelastic solids (skin, biological textures, ...).

In this study, we consider linearized vibrations of a viscous fluid. The model obtained is a viscoelastic medium. More precisely the fluid adheres to solid particles which are similar to small holes $(T_\nu^\varepsilon)_{\nu=1}^{n(\varepsilon)}$ (ε being a small parameter tending to 0 and $n(\varepsilon)$ designates the number of holes). These obstacles are periodically distributed on a hyperplane H which intersects the domain occupied by the fluid. This provides a mathematical model for fluid flows through mixing grids. The aim of theses particles is to deflect and to set the fluid in rotational motion in order to mix and homogenize the profile of the velocity field or to make values of quantities carried by the fluid, such as its temperature or concentration of a polluting, uniform cross a section.

The problem of mixing grids has been studied by several authors. In [10], G. Nguetseng considered the Dirichlet problem for the Laplace equation where the size of the holes is of the order of the period of their distribution. In this case, the solution tends to a function taking null values on the plane section H when the period tends to 0. For the same problem, D. Cioranescu and F. Murat in [7] obtained a "strange term" when the size a_ε of the holes is of the order of the critical size c_ε given by:

$$c_\varepsilon = C_0\varepsilon^{n-1/n-2} \quad \text{for } n \geq 3, \quad c_\varepsilon = e^{-1/C_0\varepsilon} \quad \text{for } n = 2 \quad \text{with } C_0 > 0. \tag{1}$$

G. Allaire (see [1]) obtained at the limit, in the case of Stokes and Navier-Stokes equations, Brinkman's laws. In another point of view, these problems were studied by [11] and [3] for Laplace and Stokes operator respectively. The holes considered are entierely included in the surface H.

The case of Stokes equations in a domain which contains a periodically perforated sieve is studied by C. Conca in [8]. We also refer the reader to [13] and [14].

In our problem, we first study the asymptotic behaviour (problem (P_ε)) of the fluid when the size a_ε of the holes is of the order of c_ε given by (1). A long memory effect appears (problem (P_{C_0})). This models the presence of obstacles in the fluid.
Next, we shall be interested to what happens when the size a_ε is less or greater than c_ε. We denote by (P_0) and (P_∞) the limit problems obtained respectively.
Finally we consider the case where the size of the holes is of the order of the period. We obtain the same model (P_∞). (See Figure 1 for the different problems studied).

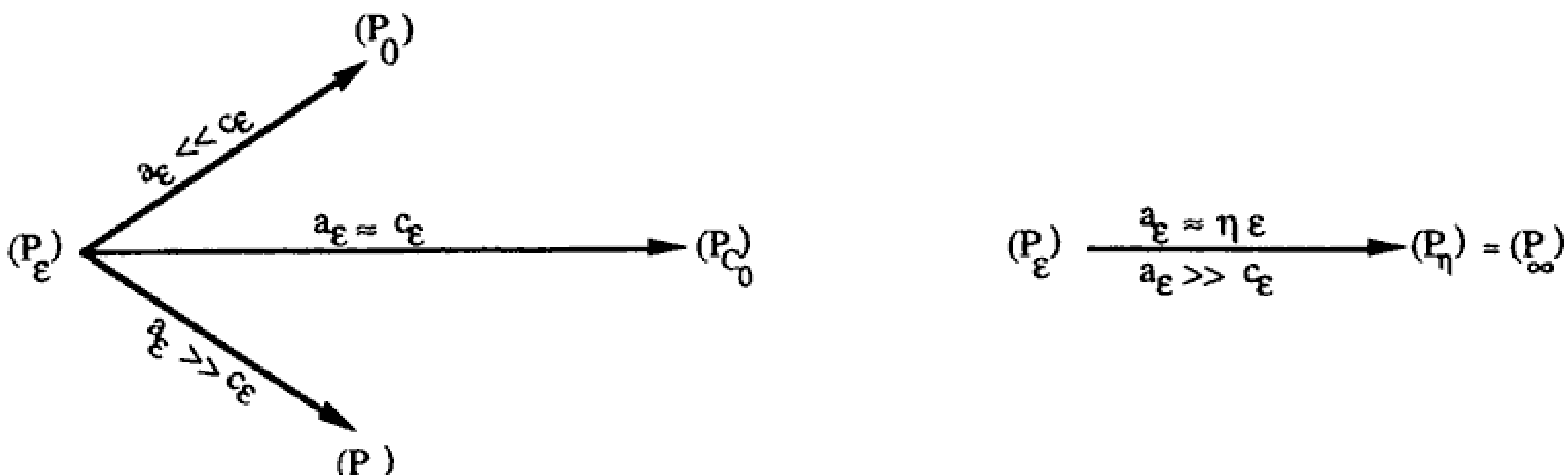

Figure 1

2 Formulation of the problem

2.1 Geometrical considerations

Let Ω be a bounded and connected open set in $\mathbb{R}^n (n \geq 2)$, whose boundary is C^1 by parts. We assume that Ω has a non-empty intersection with the hyperplane

$$H = \{x \in \mathbb{R}^n / x_n = 0\}. \tag{2}$$

We define the open set H_ε to be a slice of Ω of thickness 2ε near H (see Figure 2) by

$$H_\varepsilon = \{x \in \Omega / |x_n| < \varepsilon\}. \tag{3}$$

The set H_ε is covered with a regular mesh of size 2ε. Each cell is a cube P_ν^ε identical to $(-\varepsilon, +\varepsilon)^n$ up to a translation. At the center of each P_ν^ε included in H_ε there is a hole T_ν^ε, each of which is similar to the same closed set T rescaled to the size a_ε. We assume that:

$$\exists \alpha,\ 0 < \alpha < 1 \quad \text{such that} \quad B_\alpha \subset T \subset B_1 \tag{4}$$

where B_α and B_1 are the open balls centered at the origin with radius respectively α and 1,

$$\lim_{\varepsilon \to 0} \frac{a_\varepsilon}{\varepsilon} = 0. \tag{5}$$

Elementary geometrical considerations give the number of holes:

$$n(\varepsilon) = \frac{|\Omega \cap H|}{(2\varepsilon)^{n-1}} [1 + o(1)] \tag{6}$$

where $|\Omega \cap H|$ is the measure of $\Omega \cap H$ in $\mathbb{R}^{n-1}$ and $o(1)$ is a function of ε such that $\lim_{\varepsilon \to 0} o(1) = 0$.

We define the set Ω_ε (see Figure 3) by:

$$\Omega_\varepsilon = \Omega \backslash \cup_{\nu=1}^{n(\varepsilon)} T_\nu^\varepsilon.$$

Because we considered only the cells entirely included in Ω, we are sure that no obstacle intersects the boundary $\partial\Omega$.

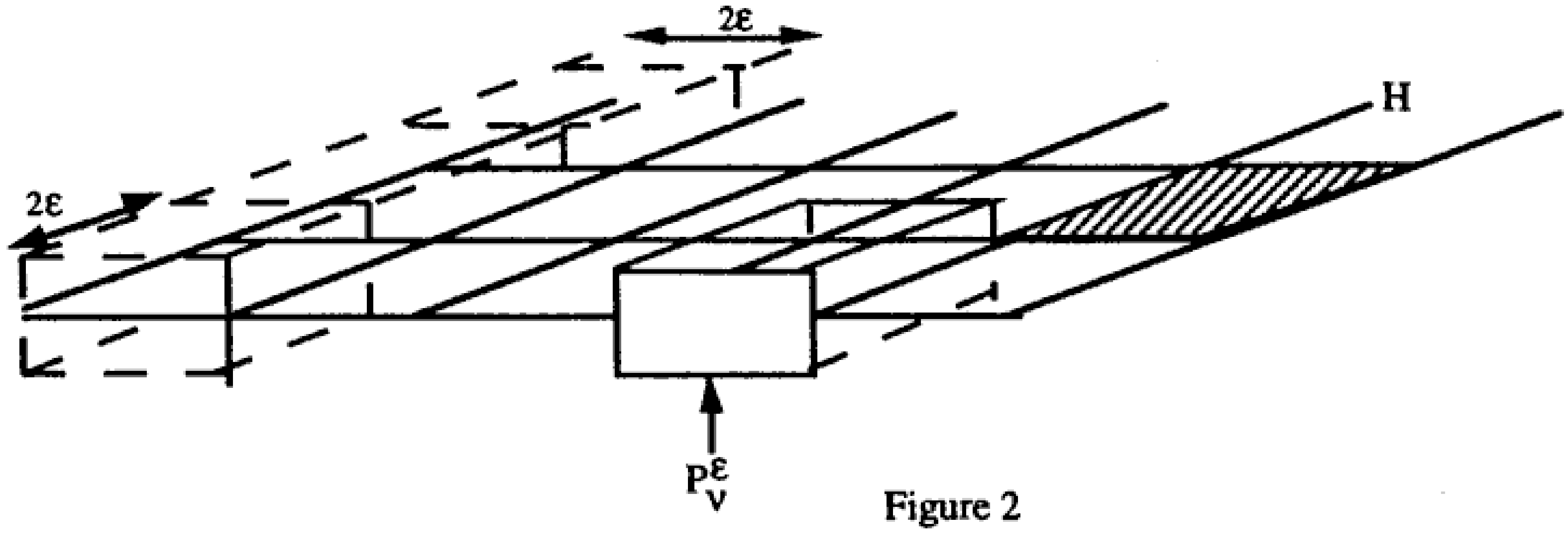

Figure 2

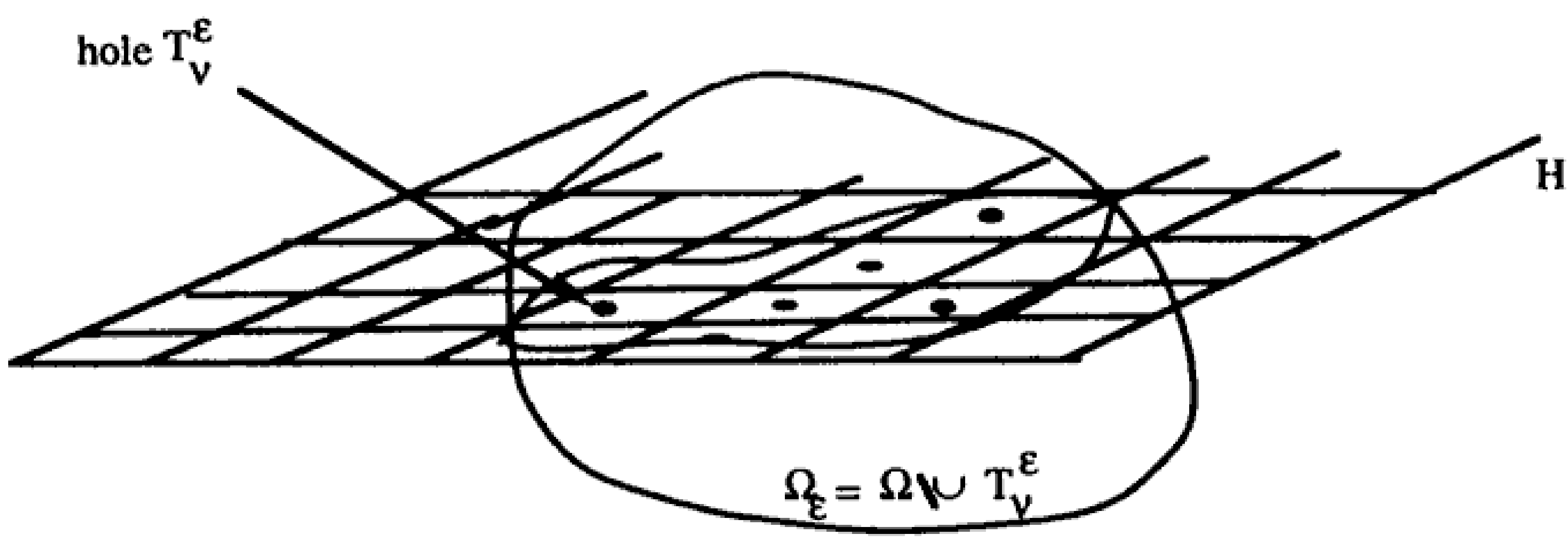

Figure 3

2.2 Statement of the problem

We consider a reference state characterized by a stress tensor field with components $\sigma_{ij}^0 = -p_0\delta_{ij}$, associated with a constant pressure p_0. (δ_{ij} being the Kronecker symbol). Now, under the hypothesis of small perturbations and denoting $U_\varepsilon(x,t)$ the displacement field, the stress perturbation tensor is given by

$$\sigma_{ij}(U_\varepsilon) = c_0^2\rho_0\delta_{ij}\delta_{kh}\varepsilon_{kh}(U_\varepsilon) + (\eta\delta_{ij}\delta_{kh} + \mu(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}))\varepsilon_{kh}\left(\frac{\partial U_\varepsilon}{\partial t}\right), \tag{7}$$

where ρ_0 is the volumic mass of the fluid in the unperturbed state, c_0 is the velocity of the sound, $\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$, η and μ are the viscosity coefficients and satisfy [12]:

$$n\eta + 2\mu\alpha^* > 0 \quad \text{with} \quad 0 < \alpha^* < 1 \quad \mu > 0. \tag{8}$$

The perturbation displacement field U_ε satisfies:

$$\left\{\begin{array}{lll} \rho_o \frac{\partial U_{\varepsilon,i}}{\partial t^2} = \partial_j \sigma_{ij}(U_\varepsilon) + F_i & \text{in} & \Omega_\varepsilon \\ U_\varepsilon(x,t) = 0 & \text{on} & \partial\Omega_\varepsilon \\ U_\varepsilon(x,0) = 0, \quad U'_\varepsilon(x,0) = 0 & \text{in} & \Omega_\varepsilon, \end{array}\right. \qquad (P_\varepsilon)$$

where the body force F is such that $F \in L^2(0,+\infty,(L^2(\Omega))^n)$.

For the existence and uniqueness of a solution of (P_ε), we have the following theorem (see [9]):

Theorem 1 *There exists a unique solution U_ε to Problem (P_ε) satisfying:*

$$U_\varepsilon \in L^\infty(0,T,[H_0^1(\Omega_\varepsilon)]^n) \cap L^2(0,T,[H_0^1(\Omega_\varepsilon)]^n)$$

$$U'_\varepsilon \in L^\infty(0,T,[L^2(\Omega_\varepsilon)]^n) \cap L^2(0,T,[H_0^1(\Omega_\varepsilon)]^n), \quad (\rho_o U'_\varepsilon) \in L^2(0,T,[H^{-1}(\Omega_\varepsilon)]^n).$$

Moreover $||U_\varepsilon||_{L^\infty(0,T,[H_0^1(\Omega_\varepsilon)]^n)}$, $||U'_\varepsilon||_{L^\infty(0,T,[L^2(\Omega_\varepsilon)]^n)}$, $||U'_\varepsilon||_{L^2(0,T,[H_0^1(\Omega_\varepsilon)]^n)}$ *and* $||(\rho_o U'_\varepsilon)'||_{L^2(0,T,[H^{-1}(\Omega_\varepsilon)]^n)}$ *are bounded independently of* ε.

Now we are going to see what happens when $\varepsilon \to 0$ following the considerations described by Figure 1.

To make the study easy, we use in (P_ε) the Laplace transform to obtain a stationary problem of elasticity type. If we denote by L the Laplace transform, $u_\varepsilon = L(U_\varepsilon)$ and $f = L(F)$, we obtain

$$\left\{\begin{array}{l} u_\varepsilon \in [H_0^1(\Omega_\varepsilon)]^n \\ \lambda^2 \rho_0 u_{\varepsilon i} - \partial_j(a_{ijlh}\varepsilon_{lh}(u_\varepsilon)) = f_i \quad \text{in} \quad \Omega_\varepsilon \quad i = 1,\dots,n \end{array}\right. \tag{9}$$

with

$$\left\{\begin{array}{l} a_{ijlh} = c_0^2 \rho_0 \delta_{ij}\delta_{lh} + \lambda(\delta_{ij}\delta_{lh} + \mu(\delta_{il}\delta_{jh} + \delta_{ih}\delta_{jl})), \\ \text{for } \lambda \in \mathbb{C} \quad Re\lambda > 0 \text{ sufficiently large.} \end{array}\right. \tag{10}$$

In order to prove the convergence of the homogenization process, we use the energy method developped by L. Tartar [15]. First, let us consider the weak formulation of (9):

$$\left\{\begin{array}{l} \text{Find } u_\varepsilon \in [H_0^1(\Omega_\varepsilon)]^n \text{ such that :} \\ \int_{\Omega_\varepsilon} \lambda^2 \rho_o u_\varepsilon \overline{w} dx + \int_{\Omega_\varepsilon} a_{ijlh}\varepsilon_{lh}(u_\varepsilon)\varepsilon_{ij}(\overline{w}) dx = \int_{\Omega_\varepsilon} f\overline{w} dx \quad \forall w \in [H_0^1(\Omega_\varepsilon)]^n \end{array}\right. \tag{11}$$

where $\overline{w}$ denotes the complex conjugate of w and $H_0^1(\Omega_\varepsilon)$ is considered as a $\mathbb{C}$ vector space.

2.3 A priori estimate

Taking $w = u_\varepsilon$ in (11), we easily get $||\tilde{u}_\varepsilon||_{H_0^1\Omega)}$ bounded by a constant independent of ε, where $\tilde{\ }$ denotes the extension operator by 0 in $\Omega \backslash \Omega_\varepsilon$. Then there exists a subsequence still denoted $(\tilde{u}_\varepsilon)_{\varepsilon>0}$ and $u \in [H_0^1(\Omega)]^n$ such that

$$\tilde{u}_\varepsilon \rightharpoonup u \quad \text{in} \quad [H_0^1(\Omega)]^n. \tag{12}$$

Using a Lemma due to Allaire [1], we have also the following estimate:

$$|u_\varepsilon|_{L^2(H_\varepsilon)} \leq c\sigma_\varepsilon |\nabla u_\varepsilon|_{L^2(H_\varepsilon)} \tag{13}$$

where c is a constant independent of ε and σ_ε is given by:

$$\sigma_\varepsilon = \begin{cases} \left(\frac{\varepsilon^n}{a_\varepsilon^{n-2}}\right)^{1/2} & \text{for} \quad n \geq 3 \\ \varepsilon \left| log \frac{a_\varepsilon}{\varepsilon} \right|^{1/2} & \text{for} \quad n = 2. \end{cases} \tag{14}$$

3 Test functions

In this section, we construct suitable test functions $(\omega_k^\varepsilon)_{1 \leq k \leq n}$ which allows us to pass to the limit in problem (11). The cases $n = 2$ and $n \geq 3$ will be treated separately.

3.1 The two dimensional case

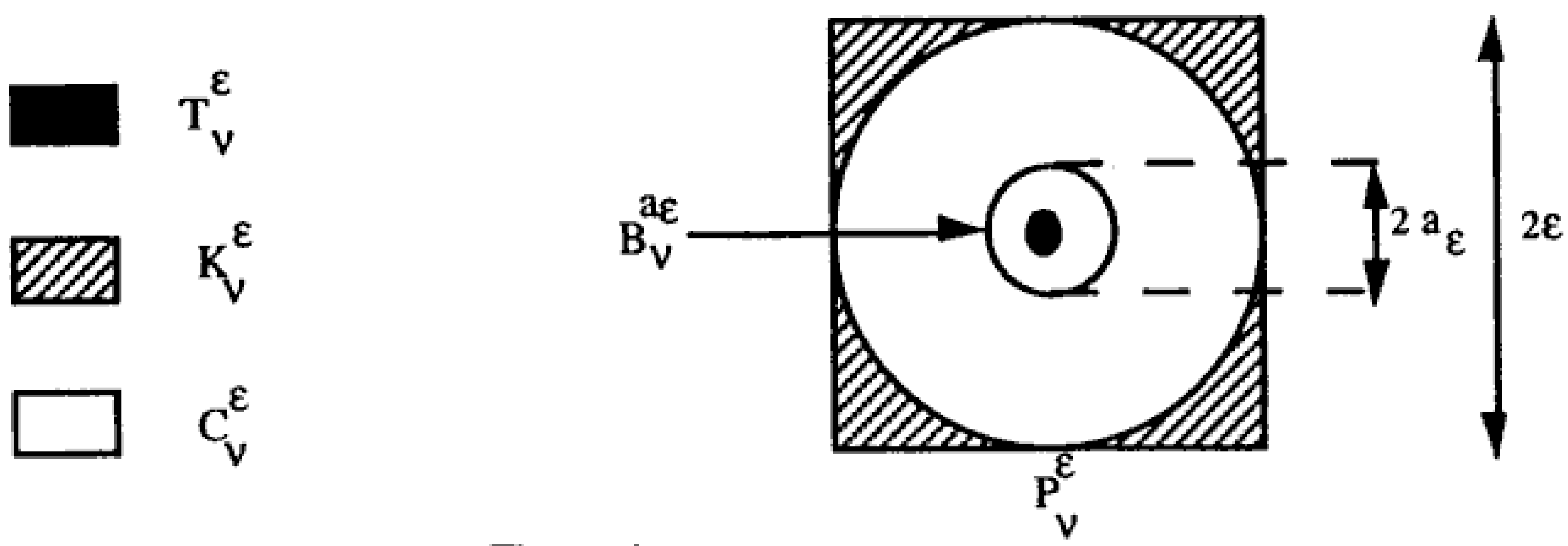

Figure 4

We decompose each cube P_ν^ε such as indicated by Figure 4. The set $B_\nu^{a_\varepsilon}$ (resp. B_ν^ε) is the ball of radius a_ε (resp. ε) centered in P_ν^ε, $C_\nu^\varepsilon = B_\nu^\varepsilon \backslash T_\nu^\varepsilon$ and $K_\nu^\varepsilon = P_\nu^\varepsilon \backslash \overline{B}_\nu^\varepsilon$.

We define functions $(\omega_k^\varepsilon)_{1 \leq k \leq 2}$ on the open set Ω by:

$$\begin{cases} \text{For} & \text{each cube entirely} & \text{included in } H_\varepsilon : & & & \\ & \omega_k^\varepsilon = e_k & \text{in } K_\nu^\varepsilon & \text{and} & -\partial_j(a_{ijlh}\varepsilon_{lh}(\omega_k^\varepsilon)) = 0 & \text{in} \quad C_\nu^\varepsilon \\ & \omega_k^\varepsilon = 0 & \text{in } T_\nu^\varepsilon & \text{and} & w_k^\varepsilon \in [H^1(P_\nu^\varepsilon)]^2. & \\ -\text{and} & \text{by}: & & & & \\ & \omega_k^\varepsilon = e_k & \text{elsewhere} & \text{in} & \Omega \backslash \cup_{\nu=1}^{n(\varepsilon)} P_\nu^\varepsilon. & \end{cases} \tag{15}$$

To estimate these functions, we compare them with functions $(\omega_{0k}^\varepsilon)_{1 \leq k \leq 2}$ defined by replacing T_ν^ε by $B_\nu^{a_\varepsilon}$ in (15). It is possible to compute explicitly ω_{0k}^ε in $C_\nu^\varepsilon \backslash B_\nu^{a_\varepsilon}$. Denote by r the radial coordinate and e_r the unit vector in each $C_\nu^\varepsilon \backslash B_\nu^{a_\varepsilon}$. Proceeding as in [4], we get: $\omega_{0k}^\varepsilon = x_k r f(r) e_r + g(r) e_k$ for $r \in [a_\varepsilon, \varepsilon]$ where we seek for f and g expansions of the form: $f(r) = \sum_{k=-\infty}^{+\infty} \alpha_k r^k$ and $g(r) = \sum_{k=-\infty}^{+\infty} \beta_k r^k - A log r$. We establish that

$$\begin{cases} f(r) &= \frac{1}{(1+2m)\sigma_\varepsilon^2} \left[-\frac{\varepsilon^2}{r^2} + \frac{\varepsilon^2 a_\varepsilon^2}{r^4} + 1 \right] [1 + o(1)], \quad m = \frac{\lambda\mu}{(c_0^2\rho_0 + \lambda\mu + \eta\mu)} \\ g(r) &= \frac{1}{\sigma_\varepsilon^2} \left(\varepsilon^2 log r - \tilde{m}\frac{\varepsilon^2 a_\varepsilon^2}{2r^2} - 2m + 3\tilde{m}\frac{r^2}{2} - \varepsilon^2 log \varepsilon \right) [1 + o(1)] + 1, \quad \tilde{m} = \frac{1}{1+2m}. \end{cases} \tag{16}$$

Arguing as in [4], we deduce:

Lemma 2 *The functions* $\omega_{0k}^{\varepsilon}$ *satisfy:*

$$|\nabla \omega_{0k}^{\varepsilon}|_{L^2(\Omega)}^2 \leq c\frac{\varepsilon}{\sigma_\varepsilon^2}, \qquad |\omega_{0k}^{\varepsilon} - e_k|_{L^2(\Omega)} \leq c\varepsilon^{1/2}\left(\frac{\varepsilon}{\sigma_\varepsilon}\right)^2$$

and

$$(a_{ijlh}\varepsilon_{lh}(\omega_{0k}^{\varepsilon})(e_r^{\nu}\cdot e_j))\delta_\nu^{a_\varepsilon} = 2\lambda\mu\frac{1+m}{1+2m}\frac{\varepsilon^2}{a_\varepsilon\sigma_\varepsilon^2}[1+o(1)]e_k\delta_\nu^{a_\varepsilon}$$

where $\delta_\nu^{a_\varepsilon}$ *denotes the measure defined by:* $<\delta_\nu^{a_\varepsilon}, \varphi> = \int_{\partial B_\nu^{a_\varepsilon}} \varphi(s)ds \quad \forall\varphi \in \mathcal{D}(\Omega)$.

Now let us introduce the difference $\omega_k^{\varepsilon'}$ between ω_k^{ε} and $\omega_{0k}^{\varepsilon}$

$$\omega_k^{\varepsilon'} = \omega_k^{\varepsilon} - \omega_{0k}^{\varepsilon}. \tag{17}$$

Then we have

Lemma 3

$$\omega_k^{\varepsilon'} \in [H_0^1(\Omega)]^2, \qquad |\nabla\omega_k^{\varepsilon'}|_{L^2(\Omega)} \leq c\varepsilon^{1/2}\frac{\varepsilon}{\sigma_\varepsilon^2} \quad \textit{and} \quad |\omega_k^{\varepsilon'}|_{L^2(\Omega)} \leq c\varepsilon^{1/2}\frac{\varepsilon}{\sigma_\varepsilon^2} \quad k=1,2$$

where c *does not depend on* ε.

Proof. We take in account the number of holes (see (6)), we procede as in [5] (Lemma 6.1) and the result follows. □

So we deduce from Lemma 2, Lemma 3 and (17):

Theorem 4 *The functions* $(\omega_k^{\varepsilon})_{1\leq k\leq 2}$ *defined by (15) satisfy*

$$|\omega_k^{\varepsilon} - e_k|_{L^2(\Omega)} \leq c\varepsilon^{1/2}\frac{\varepsilon}{\sigma_\varepsilon^2}, \qquad |\nabla\omega_k^{\varepsilon}|_{L^2(\Omega)}^2 \leq c\frac{\varepsilon}{\sigma_\varepsilon^2}$$

where the constant c *does not depend on* ε.

3.2 The case $n \geq 3$

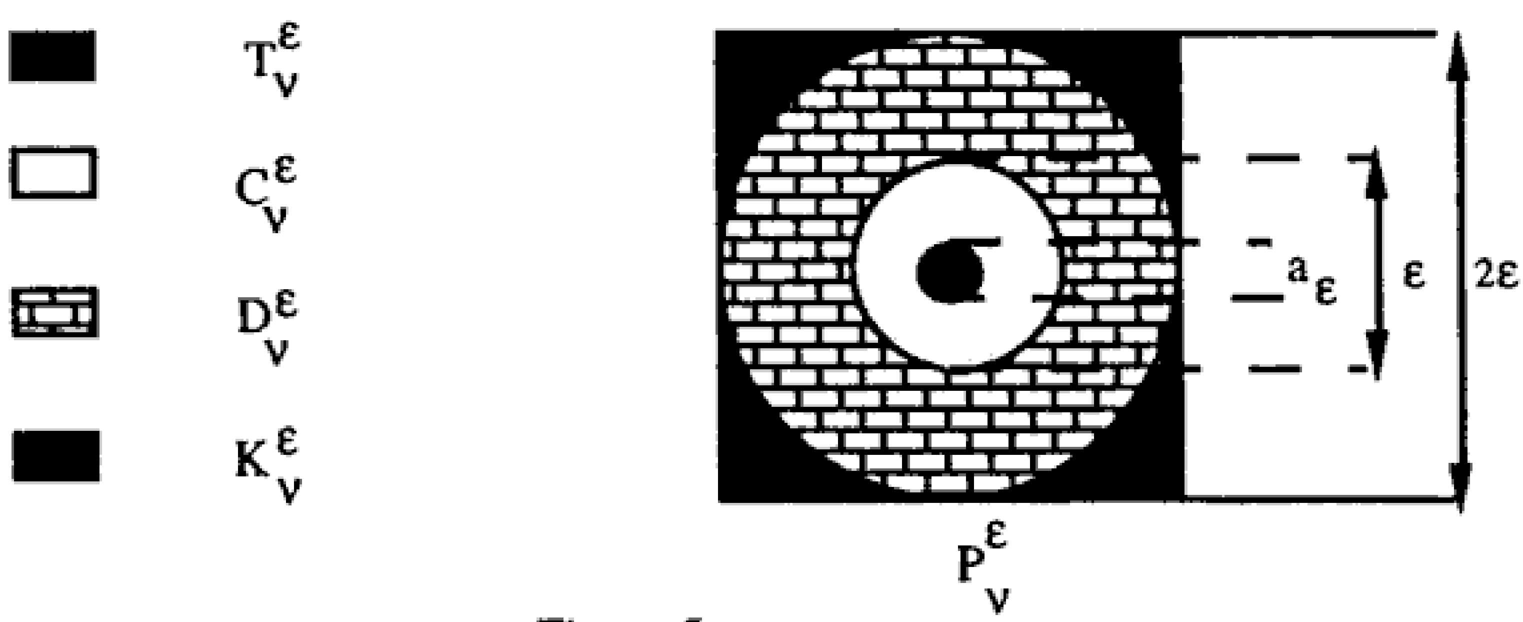

Figure 5

We decompose each cube P_ν^ε as indicated by Figure 5, where

C_ν^ε is the open ball of radius $\varepsilon/2$ centered in P_ν^ε and perforated by T_ν^ε,

B_ν^ε is the ball of radius ε and centered in P_ν^ε,
D_ν^ε is equal to B_ν^ε perforated by $\overline{C}_\nu^\varepsilon \cup \overline{T}_\nu^\varepsilon$,
$K_\nu^\varepsilon = P_\nu^\varepsilon \backslash (T_\nu^\varepsilon \cup \overline{C}_\nu^\varepsilon \cup \overline{D}_\nu^\varepsilon)$.

Then we define the functions $(\omega_k^\varepsilon)_{1 \le k \le n}$ as follows:

$$\begin{cases} -\text{For each cube entirely included in } \quad H_\varepsilon, \\ \quad \omega_k^\varepsilon = e_k \quad \text{in } K_\nu^\varepsilon, \quad -\partial_j(a_{ijlh}\varepsilon_{lh}(\omega_k^\varepsilon)) = \quad \text{in } D_\nu^\varepsilon \\ \quad \omega_k^\varepsilon = \omega_k\left(\frac{x}{a_\varepsilon}\right) \quad \text{in } C_\nu^\varepsilon, \quad \omega_k^\varepsilon = 0 \quad \text{in } T_\nu^\varepsilon \\ \quad \omega_k^\varepsilon \in [H^1(P_\nu^\varepsilon)]^n, \\ -\text{and by :} \\ \quad \omega_k^\varepsilon = e_k \quad \text{elsewhere in} \quad \Omega \backslash \cup_{\nu=1}^{n(\varepsilon)} P_\nu^\varepsilon \end{cases} \tag{18}$$

where ω_k is the unique solution (see [5]) of:

$$\begin{cases} -\partial_j(a_{ijlh}\varepsilon_{lh}(\omega_k)) = \text{ in } \mathbb{R}^n \backslash T, \quad \sum_{ij} |\varepsilon_{ij}(\omega_w)|^2_{L^2(\mathbb{R}^n \backslash T)} < +\infty, \\ \qquad \omega_k = 0 \quad \text{on } \partial T, \qquad \omega_k = e_k \quad \text{at infinity.} \end{cases} \tag{19}$$

Taking into account the number of holes and proceeding as in the case of distributions of holes in the whole volume (see [5]), we get:

Theorem 5 *The functions $(\omega_k^\varepsilon)_{1 \le k \le n}$ defined in (18) satisfy:*

$$|\nabla \omega_k^\varepsilon|^2_{L^2(\Omega)} \le c\frac{\varepsilon}{\sigma_\varepsilon^2},$$

$$|\omega_k^\varepsilon - e_k|^2_{L^2(\Omega)} \le c\varepsilon \quad \begin{cases} \left(\frac{\varepsilon}{\sigma_\varepsilon}\right)^4 = \left(\frac{a_\varepsilon}{\varepsilon}\right)^2 & if \quad n = 3 \\ \left(\frac{a_\varepsilon}{\varepsilon}\right)^n |log \frac{a_\varepsilon}{\varepsilon}| & if \quad n = 4 \\ \left(\frac{a_\varepsilon}{\varepsilon}\right)^n & if \quad n \ge 5 \end{cases}$$

where the constant c does not depend on ε.

In the following section we describe the case where a_ε is of the order of c_ε.

4 Case $a_\varepsilon \simeq c_\varepsilon$: The size a_ε is of the order of the critical size c_ε

Assume that:

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon log a_\varepsilon} = -C_0 \quad \text{if } n = 2, \quad \lim_{\varepsilon \to 0} \frac{a_\varepsilon}{\varepsilon^{n-1/n-2}} = C_0 \quad \text{if } n \ge 3, \tag{20}$$

then we have

Theorem 6 *Let u_ε be the unique solution of (9). Then*

$$\tilde{u}_\varepsilon \rightharpoonup u \quad in \ [H_0^1(\Omega)]^n$$

where u is the unique solution of the following problem:

$$\begin{cases} \text{Find } u \in [H_0^1(\Omega)]^n \\ \lambda^2\rho_0 u_i - \partial_j(a_{ijlh}\varepsilon_{lh}(u)) + \mu_i \cdot u = f_i \quad in \quad \Omega \quad i = 1,\dots,n, \end{cases} \tag{21}$$

and for $i = 1, \dots, n$, μ_i is given by:

$$\mu_i = \begin{cases} 2\pi C_0 \frac{\lambda\mu(c_0^2\rho_0+\lambda(\eta+2\mu))}{c_0^2\rho_0+\lambda(\eta+3\mu)} e_i\delta_H & for\ n = 2, \\ \frac{C_0^{n-2}}{2^n}\left(\frac{2nm^2+nm+m+n-1}{nm(m+1)}\right) F_i(\lambda)\delta_H & for\ n \geq 3, \end{cases}$$

$F_k(\lambda) = \int_{\partial T} a_{ijlh}\varepsilon_{lh}(\omega_k)n_j e_i d\sigma$, $\mathbf{n} = (n_j)$ is the outward unit normal to ∂T, δ_H denotes the measure defined as the unit mass concentrated on H i.e., $< \delta_H, \varphi >_{\mathcal{D}',\mathcal{D}(\mathbb{R}^n)} = \int_H \varphi(s)ds$ for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

Proof. Let $\varphi \in \mathcal{D}(\Omega)$. Take $\varphi\overline{\omega}_k^\varepsilon|_{\Omega_\varepsilon}$ as a test function in (11) and integrate by parts after extending u_ε by 0 in $\Omega\backslash\Omega_\varepsilon$, we get

$$\int_\Omega \lambda^2\rho_o\tilde{u}_\varepsilon\varphi\omega_k^\varepsilon dx + \int_\Omega a_{ijlh}\varepsilon_{lh}(\tilde{u}_\varepsilon)\frac{1}{2}[\partial_i\varphi(\omega_k^\varepsilon)_j + \partial_j\varphi(\omega_k^\varepsilon)_i]dx +$$

$$+ < -\partial_h(a_{ijlh}\varepsilon_{ij}(\omega_k^\varepsilon)), \varphi(\tilde{u}_\varepsilon)_l > - \int_\Omega a_{ijlh}\varepsilon_{lh}(\omega_k^\varepsilon)(\partial_h\varphi(\tilde{u}_\varepsilon)_l)dx = \int_\Omega f\varphi\omega_k^\varepsilon dx.$$

Since the size a_ε satisfies (20), Theorems 4 and 5 lead to

$$\omega_k^\varepsilon \rightharpoonup e_k \qquad \text{in } [H^1(\Omega)]^n, \tag{22}$$

$$\omega_k^\varepsilon \to e_k \qquad \text{in } [L^2(\Omega)]^n. \tag{23}$$

So we can easily pass to the limit in all terms of the above equality, except for the term

$$< -\partial_h(a_{ijlh}\varepsilon_{ij}(\omega_k^\varepsilon)), \varphi(\tilde{u}_\varepsilon)_l >$$

To find this limit, we distinguish two cases.

Case $n = 2$. We remark that we have:

$$-\partial_j(a_{ijlh}\varepsilon_{lh}(\omega_k^\varepsilon)) = \mu_{k,i}^\varepsilon - \gamma_{k,i}^\varepsilon \qquad \text{in } H^{-1}(\Omega)$$

where

$$\begin{cases} \mu_k^\varepsilon = \sum_{\nu=1}^{n(\varepsilon)} a_{ijlh}\varepsilon_{lh}(\omega_{0k}^\varepsilon)(e_r^\nu \cdot e_j)e_i\delta_\nu^\varepsilon + \sum_{\nu=1}^{n(\varepsilon)} a_{ijlh}\varepsilon_{lh}(\omega_k^{\varepsilon'})(e_r^\nu \cdot e_j)e_i\delta_\nu^\varepsilon \\ \gamma_k^\varepsilon = \sum_{\nu=1}^{n(\varepsilon)} a_{ijlh}\varepsilon_{lh}(\omega_k^\varepsilon)(e_r^\nu \cdot e_j)e_i\delta_{T_\nu^\varepsilon}, \end{cases}$$

δ_ν^ε is the unit mass concentrated on the sphere $\partial B_\nu^\varepsilon$ and $\partial_{T_\nu^\varepsilon}$ is the unit mass concentrated on $\partial T_\nu^\varepsilon$. Note that we have: $\gamma_k^\varepsilon = 0$ in $[H^{-1}(\Omega_\varepsilon)]^2$ i.e. $< \gamma_k^\varepsilon, v >= 0$, $\forall v \in [H_0^1(\Omega)]^2$ with $v = 0$ on $\cup_{\nu=1}^{n(\varepsilon)} T_\nu^\varepsilon$, so $< \gamma_k^\varepsilon, \varphi(\tilde{u}_\varepsilon) >= 0$. Then it suffices to prove that there exists $\mu_k \in [H^{-1}(\Omega)]^2$ such that: $\mu_k^\varepsilon \to \mu_k$ in $[H^{-1}(\Omega)]^2$.

First, we have by a simple computation:

$$\sum_{\nu=1}^{n(\varepsilon)} a_{ijlh}\varepsilon_{lh}(\omega_{0k}^{\varepsilon})(e_r^{\nu}\cdot e_j)e_j\delta_{\nu}^{\varepsilon} = \left[\frac{\lambda\mu}{C_o}\left(1-\frac{2m+3}{(1+2m)^2}\right)\left(\sum_{\nu=1}^{n(\varepsilon)}\delta_{\nu}^{\varepsilon}\right)e_k + \right.$$

$$\left. +\frac{4\lambda\mu}{(1+2m)C_o}\left(1+\frac{1}{1+2m}\right)\left(\sum_{\nu=1}^{n(\varepsilon)}(e_r^{\nu}\cdot e_k)e_r^{\nu}\delta_{\nu}^{\varepsilon}\right)\right][1+o(1)].$$

Letting $\varepsilon \to 0$, we get:

$$\sum_{\nu=1}^{n(\varepsilon)} a_{ijlh}\varepsilon_{lh}(\omega_{0k}^{\varepsilon})(e_r^{\nu}\cdot e_j)e_i\delta_{\nu}^{\varepsilon} \to \mu_k \quad \text{in } [H^{-1}(\Omega)]^2 \tag{24}$$

since (see [1]), we have for $n \geq 2$:

$$\sum_{\nu=1}^{n(\varepsilon)} \delta_{\nu}^{\varepsilon} \to \frac{S_n}{2^{n-1}}\delta_H \quad \text{in } H^{-1}(\Omega) \tag{25}$$

$$\sum_{\nu=1}^{n(\varepsilon)} (e_r^{\nu}\cdot e_k)e_r^{\nu}\delta_{\nu}^{\varepsilon} \to \frac{S_n}{n2^{n-1}}e_k\delta_H \quad \text{in } [H^{-1}(\Omega)]^n \tag{26}$$

where S_n denotes the area of the unit sphere in $\mathbb{R}^n$.
Next, we prove that:

$$\sum_{\nu=1}^{n(\varepsilon)} a_{ijlh}\varepsilon_{lh}(\omega_k^{\varepsilon'})(e_r^{\nu}\cdot e_j)e_i\delta_{\nu}^{\varepsilon} \to 0 \quad \text{in } [H^{-1}(\Omega)]^2. \tag{27}$$

Let $v_\varepsilon \in [H_0^1(\Omega)]^2$ such that $v_\varepsilon \rightharpoonup v$ in $[H_0^1(\Omega)]^2$ and set

$$\Delta_\varepsilon = < \sum_{\nu=1}^{n(\varepsilon)} a_{ijlh}\varepsilon_{lh}(\omega_k^{\varepsilon'})(e_r^{\nu}\cdot e_j)e_i\delta_{\nu}^{\varepsilon}, v_\varepsilon >_{H^{-1}(\Omega),H_0^1(\Omega)} .$$

We remark that

$$-\partial_j(a_{ijlh}\varepsilon_{lh}(\omega_k^{\varepsilon'})) = \sum_{\nu=1}^{n(\varepsilon)} a_{ijlh}\varepsilon_{lh}(\omega_{0k}^{\varepsilon})(e_r^{\nu}\cdot e_j)e_i\delta_{\nu}^{a_\varepsilon} + \sum_{\nu=1}^{n(\varepsilon)} a_{ijlh}\varepsilon_{lh}(\omega_k^{\varepsilon'})(e_r^{\nu}\cdot e_j)e_i\delta_{\nu}^{\varepsilon} -$$

$$-\sum_{\nu=1}^{n(\varepsilon)} a_{ijlh}\varepsilon_{lh}(\omega_k^{\varepsilon'})(e_r^{\nu}\cdot e_j)e_i\delta_{T_{\nu}^{\varepsilon}}.$$

Let R_ε be a mapping satisfying:

Proposition 7 *(see [5]) There exists an operator R_ε satisfying:*
(i) $R_\varepsilon \in L([H_0^1(\Omega)]^n;[H_0^1(\Omega_\varepsilon)]^n)$, (ii) $u \in [H_0^1(\Omega_\varepsilon)]^n \Rightarrow R_\varepsilon\tilde{u} = u$ in Ω_ε.
(ii) There exists a constant $c > 0$ which does not depend on ε or u, such that:

$$||R_\varepsilon u||_{[H_0^1(\Omega_\varepsilon)]^n} \leq c[|\nabla u|^2_{L^2(\Omega)} + \frac{1}{\sigma_\varepsilon^2}|u|^2_{L^2(H_\varepsilon)}].$$

We consider the mapping R_ε in the case where the holes are the balls $B_\nu^{a_\varepsilon}$ so:

$$\begin{aligned}\Delta_\varepsilon &= <\sum_{\nu=1}^{n(\varepsilon)} a_{ijlh}\varepsilon_{lh}(\omega_k^{\varepsilon'})(e_r^\nu\cdot e_j)e_i\delta_\nu^\varepsilon, \widetilde{R_\varepsilon v_\varepsilon}>_{H^{-1},H_0^1(\Omega)}\\ &= <-\partial_j(a_{ijlh}\varepsilon_{lh}(\omega_k^{\varepsilon'})), (\widetilde{R_\varepsilon v_\varepsilon})_i>_{H^{-1},H_0^1(\Omega)} = \int_\Omega a_{ijlh}\varepsilon_{lh}(\omega_k^{\varepsilon'})\varepsilon_{ij}(\widetilde{R_\varepsilon v_\varepsilon})\end{aligned}$$

then

$$\begin{aligned}|\Delta_\varepsilon| &\leq c|\nabla\omega_k^{\varepsilon'}|_{L^2(\Omega)}|\nabla(R_\varepsilon v_\varepsilon)|_{L^2(\Omega_\varepsilon)}\\ &\leq c\varepsilon^{1/2}\frac{\varepsilon}{\sigma_\varepsilon^2}\left[|\nabla v_\varepsilon|^2_{L^2(\Omega)} + \frac{1}{\sigma_\varepsilon^2}|v_\varepsilon|^2_{L^2(H_\varepsilon)}\right].\end{aligned}$$

Taking into account the fact that v_ε vanishes on $\partial\Omega$, one can establish that:

$$|v_\varepsilon|^2_{L^2(H_\varepsilon)} \leq c\varepsilon|\nabla v_\varepsilon|^2_{L^2(\Omega)}.$$

Then

$$\begin{aligned}|\Delta_\varepsilon| &\leq c\varepsilon^{1/2}\frac{\varepsilon}{\sigma_\varepsilon^2}\left[1+\frac{\varepsilon}{\sigma_\varepsilon^2}\right]|\nabla v_\varepsilon|^2_{L^2(\Omega)}\\ &\leq c\varepsilon^{1/2}\frac{\varepsilon}{\sigma_\varepsilon^2}\left[1+\frac{\varepsilon}{\sigma_\varepsilon^2}\right] \quad \text{since } v_\varepsilon \text{ is bounded in } [H^1(\Omega)]^2\\ &\leq c\varepsilon^{1/2} \quad \text{since } \frac{\varepsilon}{\sigma_\varepsilon^2}\to C_0.\end{aligned}$$

So $\Delta_\varepsilon \to 0$ when $\varepsilon \to 0$ and (27) holds.

Case $n \geq 3$. We also remark that:

$$\begin{cases} -\partial_j(a_{ijlh}\varepsilon_{lh}(\omega_k^\varepsilon)) = \mu_{k,i}^\varepsilon - \gamma_{k,i}^\varepsilon\\ \text{with} \quad \gamma_k^\varepsilon = 0 \quad \text{in} \quad [H^{-1}(\Omega_\varepsilon)]^n\\ \text{and} \quad \mu_k^\varepsilon = \sum_{\nu=1}^{n(\varepsilon)}(a_{ijlh}\varepsilon_{lh}(\omega_k^\varepsilon)(e_r^\nu\cdot e_j)e_i)\delta_\nu^{\varepsilon/2} - \partial_j(a_{ijlh}\varepsilon_{lh}(\omega_k^\varepsilon)\chi_\varepsilon)e_i,\end{cases}$$

where $\delta_\nu^{\varepsilon/2}$ is the unit mass concentrated on the sphere $\partial C_\nu^\varepsilon \cap \partial D_\nu^\varepsilon$ and χ_ε is the characteristic function of $\cup_{\nu=1}^{n(\varepsilon)} D_\nu^\varepsilon$. Before passing to the limit, we need the following lemma proved in [5]:

Lemma 8 *(i) The solution ω_k of (19) satisfies at infinity*

$$\begin{cases} \omega_k = e_k - \frac{\gamma}{\lambda\mu r^{n-2}}\left[\frac{2m+1}{n-2}f_k + (F_k\cdot e_r)e_r\right][1+o(1)]\\ \nabla\omega_k = O\left(\frac{1}{r^{n-1}}\right), \qquad \gamma = \frac{1}{2S_n(m+1)}\\ a_{ijlh}\varepsilon_{lh}(\omega_k)n_je_i = \frac{\gamma}{r^{n-2}}\left[2mF_k + \frac{nm+m+n-1}{m}(F_k\cdot e_r)e_r\right] + O\left(\frac{1}{r^n}\right)\end{cases} \tag{28}$$

where $O(h)$ denotes a function such that $|O(h)| \leq c.h$ with c a constant independent of h.
(ii) $\lambda \in \{Re\lambda > \beta\} \subset \mathbb{C} \to F_k(\lambda)$ is holomorphic and $|F_k(\lambda)| \leq cP(|\lambda|)$ where $P(|\lambda|)$ is a polynomial.

It follows from definition of ω_k^ε in D_ν^ε and the behaviour of ω_k at infinity that:

$$|\nabla\omega_k^\varepsilon|^2_{L^2(\cup_{\nu=1}^{n(\varepsilon)}D_\nu^\varepsilon)} \leq c\varepsilon\left(\frac{\varepsilon}{\sigma_\varepsilon^2}\right)^2 \tag{29}$$

then

$$-\partial_j(a_{ijlh}\varepsilon_{lh}(\omega_k^\varepsilon)\chi_\varepsilon)e_i \to 0 \qquad \text{strongly in } [H^{-1}(\Omega)]^n.$$

We have also by (18) and (28):

$$a_{ijlh}\varepsilon_{lh}(\omega_k^\varepsilon)(e_r^\nu \cdot e_j)e_i = 2^{n-1}\gamma\frac{\varepsilon}{\sigma_\varepsilon^2}\left[2mF_k + \frac{nm+m+n-1}{m}(F_k \cdot e_\nu^r)e_\nu^r\right] + O\left(\frac{a_\varepsilon}{\sigma_\varepsilon^2}\right). \quad (30)$$

Then using (25)-(26), we deduce the result.
The uniqueness of the solution of (21) holds since the matrix $[\mu_k]_{k=1}^n$ is positive (see [5]). □

Now, we give the evolution problem (P_{C_0}) corresponding to (21). First let us introduce the following Hilbert space $W(\Omega)$ defined by

$$W(\Omega) = \{v \in L^2(0,T,[H_0^1(\Omega)]^n), v' \in L^2(0,T,[H_0^1(\Omega)]^n), (\rho_o v')' \in L^2(0,T,[H^{-1}(\Omega)]^n)\}$$

and equipped with the norm:

$$||v|| = \left(||v||^2_{L^2(0,T,[H_0^1(\Omega)]^n)} + ||v'||^2_{L^2(0,T,[H_0^1(\Omega)]^n)} + ||(\rho_o v')'||^2_{L^2(0,T,[H^{-1}(\Omega)]^n)}\right)^{1/2}.$$

Theorem 9 *Let U_ε be the unique solution of (P_ε) in $W(\Omega_\varepsilon)$ and u be the unique solution of (21). Then*

i) there exists $U \in \mathcal{D}'(]0,+\infty[,\ [H_0^1(\Omega)]^n)$ such that: $u(\lambda) = L(U) \quad Re\lambda > 0$ sufficiently large,

ii) $\tilde{U}_\varepsilon \rightharpoonup U \quad$ in $W(\Omega)$.

iii) The distribution U is the unique solution of the problem (P_{C_0}):

$$\begin{cases} i) & \rho_o \frac{\partial^2 U_i}{\partial t^2} = \partial_j \sigma_{ij}(U) - (G * U)_i + F_i \quad in \quad \Omega \quad i = 1,\dots,n \\ ii) & U(x,0) = 0, \quad U'(x,0) = 0 \quad in \quad \Omega \\ iii) & U(x,t) = 0 \quad on \quad \partial\Omega, \end{cases} \qquad (P_{C_0})$$

where $G(t) = L^{-1}(M(\lambda))$ for $Re\lambda > 0$, sufficiently large. The matrix $M(\lambda)$ is defined by its columns $[\mu_k(\lambda)]_{k=1}^n$.

Proof. It is easy to see that ii) is a consequence of estimates given by Theorem 1 For proof of i) and iii), we refer the reader for the case of distribution of holes in the whole volume Ω (see [5]). □

Remark 10 *Problem (P_{C_0}) describes a viscoelastic medium with a long memory. The convolution term $(G * U)$ and the mass δ_H model the resistance of holes and their distribution on the hyperplane H.*
We can give G explicity in the case of spherical holes since F_k is well known (see [4]).
We also remark that G is independent of the form of holes in the two dimensional case contrary to the case $n \geq 3$ where G depends on F_k. The quantity F_k appears as a force exerted by the fluid on T.

Now, we treat the case where the size a_ε is much smaller than the critical one.

5 Case $a_\varepsilon << c_\varepsilon$: The size a_ε is less than the critical c_ε

We suppose that:

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon\log a_\varepsilon}=0 \quad \text{if } n=2, \quad \lim_{\varepsilon\to 0}\frac{a_\varepsilon}{\varepsilon^{n-1/n-2}}=0 \quad \text{if } n\geq 3, \tag{31}$$

then we have

Theorem 11 *Let u_ε be the unique solution of (9). Then*

$$\tilde{u}_\varepsilon \to u \quad in\ [H_0^1(\Omega)]^n \tag{32}$$

where u is the unique solution of the problem:

$$\begin{cases} u\in [H_0^1(\Omega)]^n \\ \lambda^2\rho_0 u_i - \partial_j(a_{ijlh}\varepsilon_{lh}(u)) = f_i \quad in\ \Omega \quad i=1,\ldots,n. \end{cases} \tag{33}$$

Proof. Let $\varphi\in\mathcal{D}(\Omega)$ and ω_k^ε be the test functions built in Section 3. From assumption (31) and Theorem 4, 5, we deduce:

$$\omega_k^\varepsilon \to e_k \quad \text{in } [H_0^1(\Omega)]^n. \tag{34}$$

If we take $\varphi\overline{\omega}_k^\varepsilon|_{\Omega_\varepsilon}$ as a test function in (11), we have:

$$\int_\Omega \lambda^2\rho_o\tilde{u}_\varepsilon\varphi\omega_k^\varepsilon dx + \int_\Omega a_{ijlh}\varepsilon_{lh}(\tilde{u}_\varepsilon)\varepsilon_{ij}(\varphi\omega_k^\varepsilon)dx = \int_\Omega f\varphi\omega_k^\varepsilon dx. \tag{35}$$

Since (34) holds, we get (33) by letting $\varepsilon\to 0$ in (35).

Note that we have:

$$\int_\Omega \lambda^2\rho_o|\tilde{u}_\varepsilon|^2dx + \int_\Omega a_{ijlh}\varepsilon_{lh}(\tilde{u}_\varepsilon)\varepsilon_{ij}(\overline{\tilde{u}_\varepsilon})dx = \int_\Omega f\overline{\tilde{u}_\varepsilon}dx \to \int_\Omega f\overline{u}dx$$

$$= \int_\Omega \lambda^2\rho_o|u|^2dx + \int_\Omega a_{ijlh}\varepsilon_{lh}(u)\varepsilon_{ij}(\overline{u})dx$$

from which we deduce

$$\int_\Omega a_{ijlh}\varepsilon_{lh}(\tilde{u}_\varepsilon)\varepsilon_{ij}(\overline{\tilde{u}_\varepsilon})dx \to \int_\Omega a_{ijlh}\varepsilon_{lh}(u)\varepsilon_{ij}(\overline{u})dx$$

since by (12) and rellich's theorem we have $\tilde{u}_\varepsilon\to u$ in $[L^2(\Omega)]^n$. This leads to

$$\int_\Omega a_{ijlh}\varepsilon_{lh}(\tilde{u}_\varepsilon - u)\varepsilon_{ij}(\overline{\tilde{u}_\varepsilon - u})dx \to 0.$$

By coerciveness of coefficients a_{ijlh} and Korn's inequality, we deduce (32). □

The corresponding evolution problem to (33) is given by:

Theorem 12 *Let U_ε be the unique solution of (P_ε) in $W(\Omega_\varepsilon)$ and u be the unique solution of (33). Then*

i) there exists $U \in \mathcal{D}'(]0,+\infty[, [H_0^1(\Omega)]^n)$ such that: $u(\lambda) = L(U) \quad Re\lambda > 0$ sufficiently large,

ii) $\tilde{U}_\varepsilon \rightharpoonup U$ in $W(\Omega)$.

iii) The distribution U is the unique solution of the problem (P_0):

$$\begin{cases} i) & \rho_o \frac{\partial^2 U_i}{\partial t^2} = \partial_j \sigma_{ij}(U) + F_i & in \quad \Omega i = 1,\dots,n \\ ii) & U(x,0) = 0, \quad U'(x,0) = 0 & in \quad \Omega \\ iii) & U(x,t) = 0 & on \quad \partial\Omega. \end{cases} \qquad (P_0)$$

Remark 13 *We note that, when the holes are very small, the fluid behaves as there are no obstacles. This phenomenon was also observed in [1], [6] and [7] for a volume distribution of the holes.*

6 Case $a_\varepsilon >> c_\varepsilon$: The size a_ε is greater than the critical size c_ε

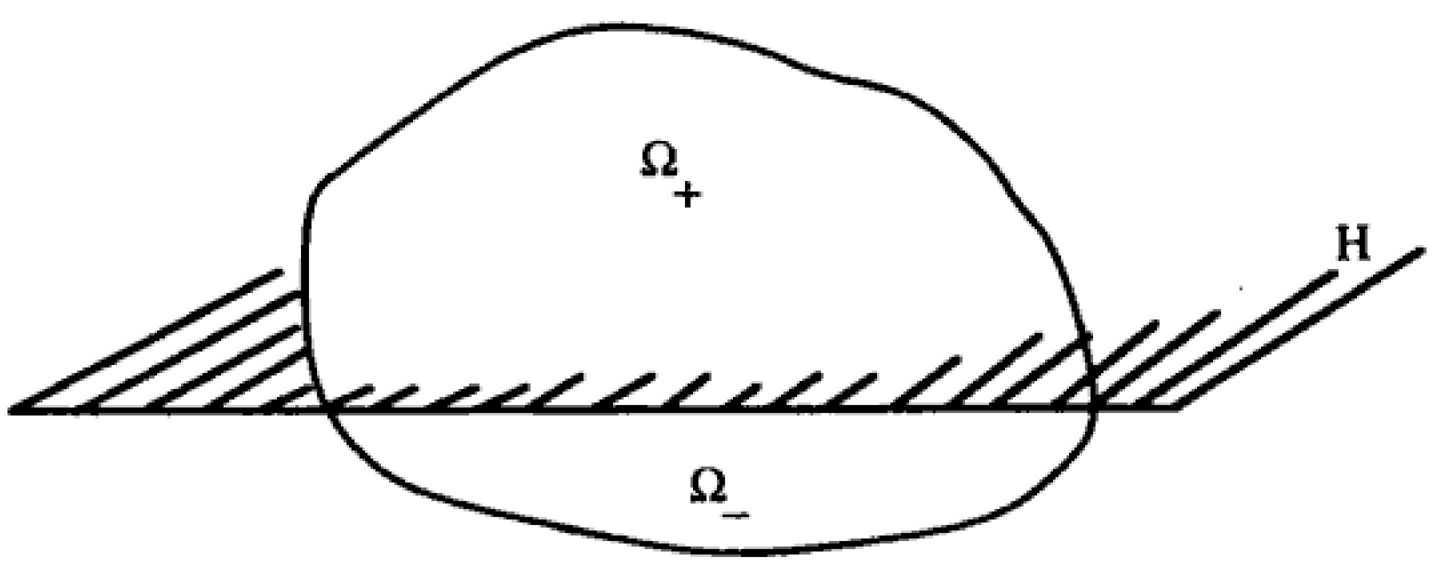

Figure 6

We suppose that:

$$\lim_{\varepsilon \to 0} \varepsilon \log a_\varepsilon = 0 \quad \text{if} \quad n = 2, \quad \lim_{\varepsilon \to 0} \frac{\varepsilon^{n-1/n-2}}{a_\varepsilon} = 0 \quad \text{if} \quad n \geq 3. \qquad (36)$$

We denote by Ω_+ and Ω_- (see Figure 6) the sets $\Omega_+ = \Omega \cap [x_n > 0]$, $\Omega_- = \Omega \cap [x_n < 0]$ where $[x_n > 0]$ (resp. $[x_n < 0]) = \{x = (x', x_n) \in \mathbb{R}^n / x_n > 0 \text{ (resp. } x_n < 0)\}$. We set $\partial_+\Omega = \partial\Omega \cap [x_n > 0]$, $\partial_-\Omega = \partial\Omega \cap [x_n < 0]$. The function u^+ (resp. u^-) denotes the restriction of a function u to Ω_+ (resp. Ω_-).

Theorem 14 *Let u_ε be the unique solution of (9) and u such that*

$$\begin{cases} \lambda^2 \rho_0 u_i^+ - \partial_j (a_{ijlh} \varepsilon_{lh}(u^+)) = f_i & in\, \Omega_+ & i = 1,\dots,n, \\ \lambda^2 \rho_0 u_i^- - \partial_j (a_{ijlh} \varepsilon_{lh}(u^-)) = f_i & in\, \Omega_- & i = 1,\dots,n, \\ \quad u^+ = 0 \quad on\, \partial_+\Omega, & u^- = 0 & on\, \partial_-\Omega \\ \quad u^+ = 0 \quad on\, H, & u^- = 0 & on\, H. \end{cases} \qquad (37)$$

Then we have

$$\tilde{u}_\varepsilon \rightharpoonup u \quad in\, [H_0^1(\Omega)]^n.$$

Proof. We prove the theorem for $n \geq 3$. The case $n = 2$ can be treated similarly. From Theorem 4 and Theorem 5, functions (ω_k^ε) built in Section 3 are such that

$$|\nabla\omega_k^\varepsilon|^2_{L^2(\Omega)} \leq c\frac{\varepsilon}{\sigma_\varepsilon^2} \quad \text{with } \frac{\varepsilon}{\sigma_\varepsilon^2} \to +\infty.$$

Proceeding as in Section 4, we have for $\varphi \in \mathcal{D}(\Omega)$

$$\begin{aligned} &\int_\Omega \lambda^2\rho_0\tilde{u}_\varepsilon\varphi\omega_k^\varepsilon dx + \int_\Omega a_{ijlh}\varepsilon_{ij}(\tilde{u}_\varepsilon)\tfrac{1}{2}[\partial_i\varphi(\omega_k^\varepsilon)_j + \partial_j\varphi(\omega_k^\varepsilon)_i]dx + \\ &\quad + < -\partial_h(a_{ijlh}\varepsilon_{ij}(\omega_k^\varepsilon)), \varphi(\tilde{u}_\varepsilon)_l > - \int_\Omega a_{ijlh}\varepsilon_{lh}(\omega_k^\varepsilon)\partial_h\varphi(\tilde{u}_\varepsilon)_l dx = \int_\Omega f\varphi\omega_k^\varepsilon dx \end{aligned} \tag{38}$$

with

$$\begin{cases} -\partial_h(a_{ijlh}\varepsilon_{ij}(\omega_k^\varepsilon)) = \mu_{k,i}^\varepsilon - \gamma_{k,i}^\varepsilon \\ \mu_k^\varepsilon = \sum_{\nu=1}^{n(\varepsilon)}(a_{ijlh}\varepsilon_{lh}(\omega_k^\varepsilon)(e_r^\nu \cdot e_j)e_i)\delta_\nu^{\varepsilon/2} - \partial_j(a_{ijlh}\varepsilon_{lh}(\omega_k^\varepsilon)\chi_\varepsilon)e_i. \end{cases}$$

First from (30), we have

$$\begin{aligned} &\tfrac{\sigma_\varepsilon^2}{\varepsilon}\sum_{\nu=1}^{n(\varepsilon)}((a_{ijlh}\varepsilon_{lh}(\omega_k^\varepsilon))(e_r^\nu \cdot e_j)e_j)\delta_\nu^{\varepsilon/2} = 2^{n-1}\gamma\Big[2mF_k\sum_{\nu=1}^{n(\varepsilon)}\delta_\nu^{\varepsilon/2} + \\ &\quad + \tfrac{nm+m+n-1}{m}\sum_{\nu=1}^{n(\varepsilon)}(F_k \cdot e_\nu^r)e_\nu^r\delta_\nu^{\varepsilon/2}\Big] + O\left(\tfrac{a_\varepsilon}{\varepsilon}\right)\sum_{\nu=1}^{n(\varepsilon)}\delta_\nu^{\varepsilon/2}. \end{aligned} \tag{39}$$

Using (25), (26) and letting ε goes to 0 in (39), we deduce

$$\frac{\sigma_\varepsilon^2}{\varepsilon}\sum_{\nu=1}^{n(\varepsilon)}((a_{ijlh}\varepsilon_{lh}(\omega_k^\varepsilon))(e_r^\nu \cdot e_j)e_i)\delta_\nu^{\varepsilon/2} \to \mu_k^0 \quad \text{in } [H^{-1}(\Omega)]^n \tag{40}$$

where

$$\mu_k^0 = \frac{1}{C_0^{n-2}}\mu_k \quad \text{and} \quad (M_0 = [\mu_k^0]_{k=1}^n = \frac{1}{C_0^{n-2}}M). \tag{41}$$

Moreover, we derive from (29)) that

$$\frac{\sigma_\varepsilon^2}{\varepsilon}(-\partial_j(a_{ijlh}\varepsilon_{lh}(\omega_k^\varepsilon)\chi_\varepsilon)e_i) \to 0 \quad \text{in } [H^{-1}(\Omega)]^n$$

Then

$$\frac{\sigma_\varepsilon^2}{\varepsilon}\mu_k^\varepsilon \longrightarrow \mu_k^0 \quad \text{in} \quad [H^{-1}(\Omega)]^n. \tag{42}$$

We have also ty Theorem 5

$$\frac{\sigma_\varepsilon^2}{\varepsilon}\nabla\omega_k^\varepsilon \to 0 \text{ in } [L^2(\Omega)]^n, \qquad \frac{\sigma_\varepsilon^2}{\varepsilon}\omega_k^\varepsilon \to 0 \text{ in } L^2(\Omega). \tag{43}$$

Then letting ε goes to 0 in (38), after multiplying (38) by $\frac{\sigma^2}{\varepsilon}$, we get by (42)-(43):

$$< M_0 u, \varphi >= 0 \quad \forall\varphi \in \mathcal{D}(\Omega)$$

from which we deduce by density since $u \in [H_0^1(\Omega)]^n$

$$< M_o u, \overline{u} >= 0$$

i.e.

$$\int_H |u|^2 ds = 0 \quad \text{for } n=2 \quad \text{and} \quad \int_H ([F_k]_{k=1}^n) u.\bar{u} ds = 0 \quad \text{for } n \geq 3.$$

So $u = 0$ on H $\forall n \geq 2$ since $[F_k]_{k=1}^n$ is symetrical and invertible matrix (see [1]).

Now to get the equations satisfied by u^+ and u^- in (37), we take in (11), for ε sufficiently small, φ_0^+ (resp. φ_0^-) as a test function, where φ_0^+ (resp. φ_0^-) is the extention by 0 to Ω of a function φ^+ (resp. φ^-) $\in \mathcal{D}(\Omega_+)$ (resp. $\in \mathcal{D}(\Omega_-)$) and we deduce the result. It is clear that u is the unique solution of (37). □

Then one can prove as we did for Theorem 9, the following result:

Theorem 15 *Let U_ε be the unique solution of (P_ε) in $W(\Omega_\varepsilon)$ and u the unique solution of (37). Then*
i) there exists $U^+ \in \mathcal{D}'(]0,+\infty[,[H_0^1(\Omega_+)]^n)$ (resp. $U^- \in \mathcal{D}'(]0,+\infty[,[H_0^1(\Omega_-)]^n)$) such that:

$$u^+(\lambda) = L(U^+), \qquad u^-(\lambda) = L(U^-) \qquad \text{for } Re\lambda > 0 \text{ sufficiently large},$$

ii) $\tilde{U}_\varepsilon \rightharpoonup U$ in $W(\Omega)$.
iii) The distribution U is the unique solution of the problem (P_∞):

$$\begin{cases} i) & \rho_o \frac{\partial^2 U_i^+}{\partial t^2} = \partial_j \sigma_{ij}(U^+) + F_i & \text{in } \Omega_+ \; i = 1,\ldots,n \\ ii) & \rho_o \frac{\partial^2 U_i^-}{\partial t^2} = \partial_j \sigma_{ij}(U^-) + F_i & \text{in } \Omega_- \; i = 1,\ldots,n \\ iii) & U(x,0) = 0, \; U'(x,0) = 0 & \text{in } \Omega \\ iv) & U(x,t) = 0 & \text{on } \partial\Omega \cup H. \end{cases} \qquad (P_\infty)$$

Remark 16 *We note that the fluid behaves in two domain (Ω_+ and Ω_-) separated by the solid obstacle H. This models the effect of big holes.*

The following section gives us some remarks about problems (21), (37) and (33).

7 Relationships between problems $(P_\infty), (P_0)$ and (P_{C_0})

We denote by U_{C_0} the unique solution of (P_{C_0}) and $u_{C_0} = L(U_{C_0})$.
Our purpose in this section to see that one can obtain problems (33) and (37) as a limit of problem (21) when $C_0 \to 0$ and $C_0 \to +\infty$ respectively.

First, let us take u_{C_0} as a test function in the weak formulation associated to problem (21). We obtain

$$\int_\Omega \lambda^2 \rho_o |u_{C_0}|^2 dx + \int_\Omega a_{ijlh} \varepsilon_{lh}(u_{C_0}) \varepsilon_{ij}(\bar{u}_{C_0}) dx + h(C_0) < M_0 u_{C_0}, \bar{u}_{C_0} > = \int_\Omega f \bar{u}_{C_0} dx$$

with

$$h(C_0) = \begin{cases} C_0 & \text{for } n = 2 \\ C_0^{n-2} & \text{for } n \geq 3. \end{cases}$$

So by coerciveness of the coefficients a_{ijlh}, positivity of the matrix M_0 and Korn's inequality, we have

$$|u_{C_0}|_{H_0^1(\Omega)} \leq c$$

where c is a constant independent of C_0.
By Rellich's theorem and the continuity of the trace function from $[H_0^1(\Omega)]^n$ to $[L^2(H)]^n$, we deduce the above estimate:

$$\begin{aligned} u_{C_0} &\rightharpoonup u \quad \text{in } [H_0^1(\Omega)]^n \\ u_{C_0} &\rightharpoonup u \quad \text{in } [L^2(H)]^n \end{aligned} \tag{44}$$

when $C_0 \to 0$ or $C_0 \to +\infty$.

Now we have by (21) for all $v \in [H_0^1(\Omega)]^n$

$$\int_\Omega \lambda^2 \rho_o u_{C_0} v dx + \int_\Omega a_{ijlh} \varepsilon_{lh}(u_{C_0}) \varepsilon_{ij}(v) dx + h(C_0) < M_0 u_{C_0}, v >= \int_\Omega f v dx. \tag{45}$$

i) Using (44), we Let C_0 go to 0 in (45). The term $h(C_0) < M_0 u_{C_0}, v >$ takes null value at the limit, so u is solution of (33).
ii) If we devide each term of (45) by $h(C_0)$ and let C_0 go to $+\infty$, we get: $< M_0 u, v >= 0$ $\forall v \in [H_0^1(\Omega)]^n$ and then $u = 0$ on H.
Now taking $v \in \mathcal{D}(\Omega_+)$ (resp. $\mathcal{D}(\Omega_-)$) in (45) and letting $C_0 \to +\infty$, we obtain (37).

We summarize the above results in the following theorem:

Theorem 17 *Let u_{C_0} be the unique solution of (21). We have*

$$u_{C_0} \rightharpoonup u \quad in\ [H_0^1(\Omega)]^n$$

when $C_0 \to 0$ (resp. $C_0 \to +\infty$) where u is the unique solution of (33) (resp. (37)).

8 Case $a_\varepsilon \simeq \eta\varepsilon$: The size a_ε is of the order of the size of the period

In this section, we assume that

$$\lim_{\varepsilon \to 0} \frac{a_\varepsilon}{\varepsilon} = \eta \qquad \text{with } 0 < \eta < 1. \tag{46}$$

First, we need the following lemma proved in [10].

Lemma 18 *There exists a constant $c > 0$ independent of ε such that:*

$$|u|^2_{L^2(H)} \leq c\varepsilon |\nabla u|^2_{L^2(\Omega)} \quad \forall u \in H_0^1(\Omega), \quad u = 0 \ on\ \partial T_\nu^\varepsilon, \quad \nu = 1, n(\varepsilon).$$

Theorem 19 *Let u_ε be the unique solution of (9) and u be the unique solution of (37). Then*

$$\tilde{u}_\varepsilon \rightharpoonup u \quad in\ [H_0^1(\Omega)]^n.$$

Proof. In section 2, we have established that $||\tilde{u}_\varepsilon||_{H_0^1(\Omega)}$ is bounded independently of ε, then by Lemma 18, we get $|\tilde{u}_\varepsilon|^2_{L^2(H)} \leq c\varepsilon$ and $\tilde{u}_\varepsilon|_H \to 0$ in $[L^2(H)]^n$. But by the continuity of the trace mapping $u \longmapsto u|_H$ from $[H_0^1(\Omega)]^n$ to $[L^2(H)]^n$ we deduce from (12) that: $\tilde{u}_\varepsilon|_H \rightharpoonup u|_H$ in $[L^2(H)]^n$. So $u = 0$ on H.
Arguing as the end of the proof of Theorem 14, we achieve this proof. □

The evolution problem corresponding to (37) is (P_∞) given by Theorem 15.

Remark 20 *We note that we obtain the same limit problem as the case of holes with size greater than the critical size c_ε. When the holes are of the order of the period, the fluid adheres on the hyperpane H which plays a thin solid plate role and the fluid behaves separately on each side of this plate.*
This differs from the case of the distribution of holes in the whole volume where the limit problems are different depending on whether the size of holes is greater than the critical one or of the order of the period. In the case of the Stokes problem, Allaire [2] proved a relationship between the two homogenized problems.

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