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CUBIC EXTENSIONS OF FLAG-TRANSITIVE PLANES, II. ODD ORDER

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Abstract. *The finite translation planes with spreads in $PG(5,q)$ which are odd cubic extensions of flag-transitive planes and admit solvable groups are completely determined.*

1 Introduction

In a previous article, we considered even order cubic extensions of flag-transitive planes. In this article, we continue this study with consideration of odd order planes.

In particular, we consider the analysis of translation planes of order q^3 that admit collineatior groups G which leave invariant a subplane π_o of order q, act flag transitively on π_o and act transitively on the set of components not in π_o.

In two previous articles (see [14] and [13]), the general study of translation planes which are extensions of flag-transitive planes is undertaken.

An 'extension of a flag-transitive plane' is an affine plane π containing an affine subplane π_o and a collineation group which leaves π_o invariant, acts flag-transitively on π_o and acts transitively on the parallel classes of π not in π_o.

The reader is referred the Part I, Even Order for the complete statements of the main results of [14] and [13]). We shall give a short version of the results here for convenience.

Theorem 1 *(Hiramine, Jha, Johnson ([14] and [13]). Let π be a non-Desarguesian translation plane of order q^n where $q > 4$ that is an extension of a flag-transitive plane and let G denote the associated collineation group.*

Then π is Hall or the derived Walker plane of order 25 in either of the following two situations:

(i) $n = 2$ or

(ii) $n \neq 3$ and G is solvable.

When $n = 3$, there are problems in the general classification of extensions of flag-transitive planes. In particular, there is a tremendous variety of of translation planes called generalized Desarguesian planes of order q^3 that admit $GL(2,q)$. There are many mutually nonisomorphic planes of this type and where the kernel of the plane may be chosen in a variety of ways.

For such planes, the associated vector space is a standard $GF(q)GL(2,q)$ module. What this means is that a group isomorphic to $SL(2,q)$ is generated by elation groups and that $GL(2,q)$ leaves invariant each subplane of order q incident with the zero vector in the associated $GF(q)$-regulus net defined by the elation axes of $SL(2,q)$. In addition, there are always infinite orbits of lengths $q+1$ and q^3-q so that we obtain cubic extensions of a Desarguesian flag-transitive plane admitting non-solvable collineation groups when $q > 3$.

We have seen in a previous article (see Hiramine, Jha, Johnson [12]) that the Lüneburg-Tits plane of order 2^{18} is a cubic extension of a Lüneburg-Tits subplane of order 2^6.

Furthermore, the authors prove the following theorem.

Theorem 2 *(Hiramine, Jha, Johnson [12]Let π be a cubic extension translation plane of even order q^3 with subplane π_o of order $q > 4$.*

(1) Then π_o is Desarguesian or Lüneburg-Tits and the full collineation group G contains a group isomorphic to $SL(2,q)$ or $S_z(\sqrt{q})$ respectively where the involutions are elations.

(2) If $q = 2^r$ and r is odd then π_o is Desarguesian and G is isomorphic to $SL(2,q)$ and generated by elations.

The various articles of the authors on extensions of flag-transitive planes form a theory which is, in some sense, a continuation of the ideas of the second author [15] who studied autotopism groups in translation planes of order q^n with an orbit of length $q^n - q$. We are replacing the assumption that the group is an autotopism group with the assumption that the group leaves a subplane invariant and acts transitively on the flags of the subplane.

Furthermore, Jha posed the following problems P and Q:

Problem (P): Classify all spreads within $PG(2n-1,q)$ admitting an automorphism group G such that G fixes globally a set Δ of $q+1$ components and acts transitively on the remaining components.

Actually, this problem originally had the further assumption that $n \geq 3$ as it was considered too difficult to complete when $n = 2$ due to the many known examples.

Problem (Q): Classify all translation planes of order p^n that admit collineation groups with a slope orbit of length $p^n - p$ where p is a prime.

What we are considering in this article includes the study of problems (P) and (Q) when there is an invariant subplane of order q in the first problem and of order p in the second problem and asking when the collineation group is solvable and transitive on the flags of the subplane.

Hence, we can make some progress towards the problems (P) and (Q) of Jha by adding some hypotheses regarding the action on subplanes.

We analyze the collineation groups of cubic extensions and are able to generally formulate a classification.

We have seen previously that the generalized Desarguesian planes and the Lüneburg-Tits plane of order 2^{18} appear here (see Hiramine, Jha, Johnson [12]).

In this article, we consider only planes of odd order. Since there are various problems encountered for odd order plane which are not present when the plane has even order, we usually only consider spreads in $PG(5,q)$. In this case when $q \equiv 1 \mod 4$, we show that the group must be nonsolvable and involve $SL(2,q)$. When $q \equiv -1 \mod 4$, although the group is not completely determined, there is a possible class of solvable cubic extensions the form of which is completely determined.

Using the implied nonsolvability of the various groups, we then can basically complete the classification of results on solvable extensions of flag-transitive planes of order q^n at least when $q \equiv 1 \mod 4$ or when q is even.

For convenience, we repeat some definitions.

Definition 3 *If an affine plane π of order q^n admits a collineation group G which has infinite point orbits of lengths $q+1$ and (q^n-q), we shall call π a '$(q+1,q^n-q)$-transitive plane' and G a '$(q+1,q^n-q)$-transitive group'.*

If π is a translation plane whose kernel contains $GF(q)$ and the group G is in the linear translation complement, we shall call π a 'linear $(q+1,q^n-q)$-transitive plane' and G a 'linear $(q+1,q^n-q)$-transitive group'.

If G leaves a subplane π_o of order q invariant within the net of length $q+1$ and there is a collineation group transitive on the sets of affine and infinite points of π_o and the infinite points of $\pi-\pi_o$ then π_o is a flag-transitive affine plane and we shall call π an 'extension of π_o'.

If the group of an extension is solvable, we shall call the plane a 'solvable extension'.

2 Cubic Extensions when the spread is in $PG(5,q), q\equiv 1$ mod 4

The problem of whether there exist solvable cubic extensions is completely unresolved. We begin the study of cubic extensions with spread in $PG(5,q)$. We shall require the following result to Foulser.

Theorem 4 *Foulser [7].*

Let π be a translation plane of order q^3 that admits a planar p-group G fixing a subplane π_o of order q pointwise. Then G is elementary Abelian of order dividing the order of the kernel of π_o.

Proof. See Foulser [7] (3.4) part (5) to observe that G is an additive subgroup of the kernel K_o of π_o. □

Assume, for the remainder of this section, that π is a cubic extension translation plane with spread in $PG(5,q)$ of odd order of a translation plane π_o of order q and G is a collineation group in the translation complement which is a $(q+1,q^3-q)$-group.

Lemma 5 *If the group $G|\pi_o$ is solvable on π_o then one of the following occurs:*

(1) the subplane is desarguesian of order 9 or Hall of order 9,

(2) the group induced on the infinite points of π_o is isomorphic to A_4 or S_4 and $q=5,7,11$ or 23,

(3) the group is a subgroup of $\Gamma L(1,q^2)$ or

(4) $q=3$ and the group induced contains $SL(2,3)$.

Proof. Apply Foulser [4], Theorem 1, p. 459) and Foulser and Kallaher [8] (1.2). □

Lemma 6 *Let S be a Sylow 2-subgroup of G_L where L is a component of π_o. Then S is faithful on π_o.*

Proof. If S is not faithful on π_o, there is an involution fixing π_o pointwise which implies that there is a subplane of order $q^{3/2}$ which contains a subplane of order q which cannot occur. Hence, S is faithful on π_o. □

Lemma 7 *If the group $G|\pi_o$ is solvable on π_o then either π_o is Desarguesian or the case that the group is a subgroup of $\Gamma L(1,q^2)$ does not occur.*

Proof. Assume that the plane is not Desarguesian and the group induced on π_o is in $\Gamma L(1,q^2)$ and hence has order on π_o of order divisible by $(q^2-1)r$ where $q=p^r$. Assume that the order of a p group fixing π_o pointwise is p^a. By 2.1, if $p^a > \sqrt{q}$ then the plane π_o is Desarguesian. Hence, $p^a \leq \sqrt{q}$ and a p-group acting faithfully on π_o has order at least $q/p^a > \sqrt{q}$. Let the order of a faithful p-group be $p^{r/2+t}$ where $t>0$. However, the group is isomorphic to a subgroup of $\Gamma L(1,q^2)$ and the Sylow p-subgroup has order $(2r)_p$ (the p-part of $2r$). Since p is odd, let $r=p^b f$ where $(p,f)=1$ so $p^{r/2+t} \geq p^b$. However, by induction, is follows that $r < p^{r/2+t}$. □

Corollary 8 *Under the assumptions of the previous lemma, π_o is a K-subspace where the spread is in $PG(5,K)$ and K isomorphic to $GF(q)$.*

Proof. The previous argument shows that there is an element $GL(6,q)$ that fixes π_o pointwise. □

Note we may assume that G contains the kernel homology group of order $q-1$.

Lemma 9 *Let π_o be Desarguesian and the group $G|\pi_o$ is in $\Gamma L(1,q^2)$. Let ℓ be a component of π_o and let S be a Sylow 2-subgroup of G_ℓ of order divisible by $(q-1)_2^2$ and let S_1 denote the subgroup of $GL(6,q)$ in S.*

Then the full subgroup $S_{\top}$ of S_1 which fixes a component of π_o pointwise has order exactly 2.

Thus, $(q-1)_2$ divides $2r_2$.

Proof. We see that the group acting on ℓ has order divisible by $q(q-1)^2$. We note that $GL(2,q)$ commutes with the kernel homology group of order $q-1$ which also faithfully induces the kernel homology group of π_o. We see that we may assume that S_1 has a subgroup $S_{\top}$ which fixes $\pi_o \cap \ell$ pointwise. The subgroup of $\Gamma L(1,q^2)$ which fixes a line of π_o pointwise has order 2 and induces an involutory homology on π_o. Hence, if $|S_{\top}|>2$ or equivalently if $|S_1|>2(q-1)_2$, we have a contradiction. Note that the order of a Sylow 2-group of subgroup of $\Gamma L(1,q^2)$ that fixes a component of π_o has order $(q-1)_2 2r_2$. Hence, it follows that $(q-1)_2$ must divide $2r_2$. □

Lemma 10 *If $q \equiv 1 \bmod 4$, the situation described in the previous lemma cannot occur.*

Proof. Let $r=2^a t$ where t is odd. First assume a is not zero. Then $(q-1)=(p^t-1)(p^t+1)(p^{2t}+1)(p^{2^2t}+1)(p^{2^3t}+1)\dots(p^{2^{a-1}t}+1)$. Let $2^d=(p^{2t}-1)_2$ so that $(q-1_2=2^{d+a-1}$ where $d\geq 3$. Note if $a=0$ then $(q-1)_2=(p-1)_2$. Now if $a=0$ then $(q-1)_2=(p-1)_2>2$.

Thus, in either case, $(q-1)_2 > 2r_2$. □

Lemma 11 *Assume that the group induced on the infinite points of π_o is isomorphic to A_4 or S_4 and $q=5,7,11$ or 23. Then $q \neq 5$ or 23.*

Proof. Since there is a planar p-group acting trivially on π_o, it follows that π_o is a kernel subspace so there is a kernel group of π of order $q-1$ leaving π_o invariant. Similarly, as in the previous lemmas, we have a linear 2-group of order $(q-1)_2^2$ acting on π_o and fixing a component so it again follows that $(q-1)_2$ divides $2r = 2$ which eliminates $q = 5$. So, we have a group acting trivially on the infinite points of π_o of order divisible by $q(q^2-1)$ $(q+1)/24$ which implies that if $(q+1)_2/8 > 1$, there is an involution fixing π_o pointwise. Hence, $q \neq 23$. □

Theorem 12 *Let π be a cubic extension translation plane of odd order order q^3 and kernel containing $K \simeq GF(q)$ which contains a subplane π_o of order $q > 3$.*

If G is a collineation group which leaves π_o invariant and acts as a $(q+1, q^3-q)$-transitive group G and $q \equiv 1 \bmod 4$ then G is nonsolvable.

For any odd order if G is non-solvable then

(a) π_o is Desarguesian and $G|\pi_o$ contains $SL(2,q)$ or

(b) $q = 9$ and $G|\pi_o\ell_\infty \simeq A_5$.

Proof. Note that if G is non-solvable then it must induce a non-solvable group on π_o since the 2-groups are faithful on π_o. Now assume, in general, that G restricted to π_o is nonsolvable. Then the plane is Desarguesian, Hall or order 9 or Hering of order 3^3 by the results of Buekenhout et al [1].

First assume that π_o is Hall or order 9. The full collineation group of π_o which fixes the zero vector has order $2^8 \cdot 3 \cdot 5$. The group G has order divisible by $9(9^2-1) = 2^4 \cdot 3^2 \cdot 5$. In order that π_o be Hall and the group is transitive on the infinite points of π_o, it follows that there must be a group of order $2 \cdot 3 \cdot 5$ induced. Since the group induced on π_o restricted to the infinite points is a subgroup of S_5, it follows the group is nonsolvable.

If the subplane is Hering of order 27, then the subplane of order $q = 3^3$ admits a collineation group isomorphic to $SL(2,13)$ which is normalized by G. Furthermore, any odd order subgroup which centralizes $SL(2,13)$ must fix the infinite points π_o since each Sylow 13-group fixes exactly two infinite points of π_o. Moreover, the outer automorphism group is trivial so there is a 3-group of order divisible by 3^3 of which there is a subgroup of order 9 that centralizes this copy of $SL(2,13)$.

Hence, it follows that there is a group of order 9 which fixes π_o pointwise. However, this implies by Foulser, Theorem (3.1), that π_o is Desarguesian.

If π_o is Desarguesian and assume that the group G induced on π_o is non-solvable then by Foulser [4] (12.1), either $SL(2,q)$ is induced on π_o or the group induced contains the preimage of A_5 (acting on the infinite points) and $q = 11, 19, 29$ or 59.

In each case, there is a p-group fixing the subplane π_o pointwise. Furthermore, the group induces A_5 on the line at infinity of the subplane π_o. Hence, there is a kernel homology group acting on the subplane of order divisible by 2,6,14,58 respectively as $q = 11, 19, 29$ or 59.

Hence, there is a 2-group of order $(q-1)_2^2$ that fixes a component and a subgroup of order $(q-1)_2$ which fixes a component of π_o pointwise. Hence, the subgroup fixing the infinite points of π_o has order $q(q^2-1)(q-1)/60$ and since the 2-part is strictly larger than $(q-1)_2$ in each instance, there must be an involution fixing π_o pointwise which cannot occur.

We have seen the G cannot be solvable if $q \equiv 1 \bmod 4$. This completes the proof. □

3 Linear groups for spreads in $PG(5,q), q \equiv -1 \bmod 4$

Now assume that we have $q \equiv -1$ mod 4, the spread is linear and the group G is linear (within $GL(6,q)$).

Assume that the group is solvable. We have noted that there is a planar p-group of order q which fixes π_o pointwise. Furthermore, we may assume that π_o is Desarguesian and the group induced upon π_o is a subgroup of $\Gamma L(1,q^2)$. Since the group is solvable, there is a subgroup $G_{p'}$ of order order $|G|/|G|_p$. Since $G_{p'}$ fixes π_o, there is a Maschke complement C_1 of dimension 4 over the kernel K isomorphic to $GF(q)$, as we are assuming that the group G is linear.

Since we now also have the kernel homology group of order $q-1$ acting on the plane and on π_o, it follows that the group $G_{p'}$ has order divisible by $(q-1)^2(q+1)$. We note that the group fixing a component ℓ of π_o has order divisible by $q(q-1)^2$. Furthermore, as the group induced on π_o is a subgroup of $\Gamma L(1,q)^2$, it follows that there is a subgroup of order $q(q-1)/2$ which fixes π_o pointwise. In addition, $G_{[\pi_o]}$ has order dividing $q(q^2-1)_{2'}$. If there is an element of odd order u dividing $q+1$ then, since the group is linear, we have a linear planar group of order $p^\alpha u^\beta$ for α and β positive integers. However, this says that there is a normal u-group by Jha [15] which implies that the p-group fixes more than q element on a line of π_o which cannot occur.

Hence, the full groups fixing π_o pointwise has order exactly $q(q-1)/2$.

The preceding section says that we are finished or there exist involutory homologies with axes in the net containing π_o. We consider $\Gamma L(1,q^2)$ acting on the cosets of $GF(q)$ as components. In this representation, th einvolutory homologies are of the form

$\sigma_a : x \longmapsto w^q a$ where $a^{q+1} = 1$. Letting $a = b^{1-q}$, then the line $GF(q)b$ is fixed pointwise by σ_a. Also, the coaxis of σ_a is $GF(q)b\omega^{(q^2-1)/4}$ where ω is a primitive element of $GF(q^2)^*$. The group generated by the homologies has two orbits of lenghts $(q+1)/2$ on the line at infinity of π_o. Moreover, each homology must commute with $G_{[\pi_o]}$ so we have a normal group of order divisible by $2(q+1)$ which commute with $G_{[\pi_o]}$ and has two orbits of lenght $(q+1)/2$ on the line at infinity of π_o.

We claim that there is a unique Sylow p-subgroup of order q in $G_{[\pi_o]}$. Proof: The group acts faithfully on any component ℓ and furthermore, acts faithfully on the set or 1-dimensional K-subspaces on ℓ. Hence, there is a faithful group induces on the projective geometry $PG(2,q)$ induced from the 3-dimensional vector space ℓ. Thus, it remains to show that there is a unique Sylow p-subgroup.

Acting on $PG(2,q)$, we have a collineation group which fixes a point say ∞. It follows easily that any fixed-point-free group of order q which fixes ∞ and a line ℓ_∞ has the form:

$\langle (x,y) \longmapsto (x+a, xf(a)+y+g(a)); a \in GF(q) \rangle$ where f and g are appropriate functions and $f(a) = 0$ if and only if $a = 0$ if and only if the corresponding element is the identity.

If there is another Sylow p-subgroup then the second group cannot fix ℓ_∞ since otherwise there would be a generated p-group with additional fixed points. This p-element would determine a collineation of π which fixes more than q point on ℓ.

Hence, either we have a unique Sylow p-subgroup or there is an element in $G_{[\pi_o]}$ which moves ℓ_∞. Note that Sylow p-subgroup S as above is regular on the lines incident with ∞ other than ℓ_∞. Hence, it follows that the group generated by the two Sylow p-subgroups is doubly transitive and hence the group $G_{[\pi_o]}$ must have order divisible by $q(q+1)$ which is a contradiction.

Thus, we must have a normal Sylow p-group S in $G_{[\pi_o]}$.

Hence, the group $G_{p'}$ permutes transitively the q^2-1 orbits of length q of S.

Let C_1 be any $G_{p'}$ complement of π_o. Then the intersection of C_1 with any component is at least 1-dimensional. Assume that some component is contained in C_1. Then q^2-1 components are contained in C_1 which cannot occur. Hence, the components intersect C_1 in 1- and 2-dimensional subspaces. Let a and b denote the number of components which intersect C_1 in 1- and 2-dimensional subspaces respectively.

Then, $a+b=q^3+1$ and $q^4-1=a(q-1)+b(q^2-1)$

so that we have $q^4+q^3=aq+bq^2=(q^3+1-b)q+bq^2$.

This last equation is valid if and only if $q^3-q=b(q^2-q)$ if and only if $b=q+1$.

Hence, the components in the net N_{π_o} intersect any $G_{p'}$-Maschke complement is a $G_{p'}$-Maschke complement.

We shall say that such Maschke complements are 'strongly embedded'.

Assume that $q+q\neq 2^a$. Let u be an odd divisor of $q+1$. Then there is an Abelian subgroup A of order qu^β for some non-negative integer β. Suppose that there is a unique strongly embedded A-Maschke complement. Then the group S of order q must leave this complement invariant which cannot occur.

Hence, there are at least qA-Maschke complements C_i, $i=1,2,\ldots,q$. We see that $C_i\cap C_j$ for $i\neq j$ is at least two dimensional and since $(q^3-1,q+1)=2$, it follows that $C_i\cap C_j$ must be 2-dimensional. Since the orbits of $G_{p'}$ are divisible by q^2-1 on the components not on π_o, it follows that two A-Maschke complements which are images of a strongly embedded complement can intersect only on components of π_o. It follows easily that $C_i\cap C_j$ must be a subplane of the net N_{π_o} containing π_o. Hence, there are at least $1+q$ subplanes of order q incident with the zero vector in N_{π_o}. We assert that a group H in $G_{[\pi_o]}$ of order $(q-1)/2$ must fix exactly three 2-dimensional $GF(q)$-subspaces. Acting on a component taken as $PG(2,q)$, we see that the group H fixes a point (∞) of $PG(2,q)$. The associated normal p-group fixes (∞) and a line ℓ_∞. Note that no non-trivial element of H can commute with a non-trivial element of S_p. Hence, H is a subgroup of the group of an affine Desarguesian plane and thus is a subgroup of W_0T where T is the associated translation group. Since G is linear, it follows that H acting on $PG(2,q)$ is in $GL(2,q)T$. Hence, $H/H\cap T\simeq H$ is isomorphic to a subgroup of $GL(2,q)$ and it follows that $H\cap Z(GL(2,q))=\langle 1\rangle$. Since $(q-1)/2$ is odd, it follows that H is cyclic. Thus, we may assume that H fixes three points of $PG(2,q)$ of a triangle. It follows that H fixes three subplanes of order q of the net N_Δ not all in the direct sum of any two of them. Hence, we may diagonalize H.

For another viewpoint, we redress the argument in matrix form.

We let π_o be represent by $\{(x_1,0,0,y_1,0,0);x_1,y_1\in GF(q)\}$ where the translation plane is represented with respect to $x=0$, $y=0$ in for the form

$\{(x_1,x_2,x_3,y_1,y_2,y_3);x_i,y_i\in GF(q), \text{for } i=1,2,3\}$.

Hence, we may represent elements of the p-group fixing π_o pointwise in the form

$$diag\begin{bmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ d & e & 1 \end{bmatrix}.$$

We first assert that for the group of order q the set of elements e in the $(3,1)$-entries is $GF(q)$. Suppose there are two elements with the same $(3-1)$-entries e say $diag\begin{bmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ d & e & 1 \end{bmatrix}$ and $diag\begin{bmatrix} 1 & 0 & 0 \\ c^* & 1 & 0 \\ d^* & e & 1 \end{bmatrix}$.

Then, we obtain an element of the form

$$diag\begin{bmatrix} 1 & 0 & 0 \\ c^* & 1 & 0 \\ d^* & e=e^* & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ d & e & 1 \end{bmatrix} = diag\begin{bmatrix} 1 & 0 & 0 \\ c-c^* & 1 & 0 \\ d-d^* & 0 & 1 \end{bmatrix}$$

as $diag\begin{bmatrix} 1 & 0 & 0 \\ c^* & 1 & 0 \\ d^* & e & 1 \end{bmatrix}^{-1} = diag\begin{bmatrix} 1 & 0 & 0 \\ -c^* & 1 & 0 \\ -d^*+c^*e^* & -e^*=-e & 1 \end{bmatrix}$.

This element fixes a point on $x=0$ say (x_1,x_2,x_3) if and only if

$$(c-c^*)x_2+(d-d^*)x_3=0.$$

If $(c-c^*)(d-d^*)\neq 0$ then there exist additional fixed point on $x=0$ which cannot occur. If $c-c^*=0$ but $d-d^*\neq 0$ then $(x_1,x_2,0)$ is fixed pointwise. Hence, $c-c^*=0$ if and only if $d-d^*=0$.

The above agrument is basically symmetric so that it follows that the set of $(2,1)$-entries is $GF(q)$. It thus follows that $c=f(e)$ for all $e\in GF(q)$ where f is a $1-1$ function on $GF(q)$ which must be additive since the p-group is elementary Abelian. Furthermore, it similarly follows that $d=g(e)$ where g is a function on $GF(q)$.

Since the group is commutative, it is direct to verify that we obtain the following conditions on the functions:

$$g(a)+ag(b)+g(b)=g(b)+bf(a)+g(a)=g(a+b)$$

and so

$$bf(a)=af(b)$$

for all $a,b\in GF(q)$.

The second equation implies that $f(a)=af$ for some non-zero element f of $GF(q)$. Hence,

$$g(a)+g(a)+abf=g(a+b).$$

Represent g as follows:

$$g(a)=\sum_{i=0}^{q-1} g_i a^i \text{ for } g_i\in GF(q).$$

Let $a=b$ in the above equation so that

$$2g(a)=a^2f=g(2a).$$

Hence,

$$\begin{aligned} & 2g_o + 2g_1 a + (2g_2 + f)a^2 + \Sigma_{i=3}^{q-1}(2g_i)a^i \\ = \; & g_2 + 2g_1 a + 4g_2 a^2 + \Sigma_{i=3}^{q-1}(g_i 2^i)a^i \\ \text{for all } a \; \in \; & GF(q). \end{aligned}$$

Hence, we obtain

$$g_o = 0, f = 2g_2, g_i = 0 \text{ for all } i \geq 3.$$

Hence,

$$\begin{aligned} g(a) &= g_1 a + fa^2/2 \\ \text{for all } a &\in GF(q). \end{aligned}$$

Now since H fixes three subplanes, we may choose a basis without alternating the form for S_p so that element of H have the form $diag\begin{bmatrix} 1 & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & m(b) \end{bmatrix}$ for the order of b dividing $(q-1)/2$ and m a function on $GF(q)$.

Since S_p is normal then

$$\begin{aligned} & diag\begin{bmatrix} 1 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & m(b) \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ af & 1 & 0 \\ g(a) & a & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0m(b) & \end{bmatrix}^{-1} \\ = \; & diag\begin{bmatrix} 1 & 0 & 0 \\ b^{-1}af & 1 & 0 \\ m(b)^{-1}g(a) & bm(b0^{-1}a & 1 \end{bmatrix}. \end{aligned}$$

Hence, this implies that

$$bm(b)^{-1} = b^{-1} \text{ and hence } m(b) = b^2.$$

This also provides

$$b^{-2}g(a) = g(b^{-1}a) \text{ for all } a \text{ for all } b \text{ of order dividing } (q-1)/2.$$

Thus, we obtain

$$c^2(g_1 a + fa^2/2) = g_1 ca + fc^2 a^2/2$$

so that $c^2 g_1 = g_1 c$ for all c of order dividing $(q-1)/2$. Hence, either $q = 3$ or $g_1 = 0$.

Thus, for $q > 3$,

We may represent the group $G_{[\pi_o]}$ as follows:

$$\left\langle Diag\begin{bmatrix} 1 & 0 & 0 \\ fa & 1 & 0 \\ fa^2/2 & a & 1 \end{bmatrix}; a \in GF(q) \right\rangle.$$

$$\left\langle Diag\begin{bmatrix} 1 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b^2 \end{bmatrix}; \; order\ of\ b\ divides\ (q-1)/2 \right\rangle.$$

However, two of the 2-dimensional $GF(q)$-subspaces are subplanes of N_{π_o} and hence are fixed by the group generated by the homologies. It follows that the third 2-space is then fixed. However, this means that the subplane contains the centers and axes of all of the involutory homologies. That is, the subplane in N_{π_o}. Let Then, since we now have a direct product of three subplanes lying in the net it is possible to use the Krull-Schmidt to show all are isomorphic are $\mathcal{E}$-modules where $\mathcal{E}$ is the enveloping algebra of the net so that by Liebler [19], it follows that the net N_{π_o} is a regulus net. The centralizer within $GL(6,q)$ of $G_{[\pi_o]}$ is isomorphic to $GL(2,q)$ and has the form $\left\langle \begin{bmatrix} \alpha I_3 & \beta I_3 \\ \delta I_3 & \gamma I_3 \end{bmatrix}; \alpha\gamma - \delta\beta \neq 0 \right\rangle$. The group generated by the homologies commutes with $G_{[\pi_o]}$ and thus, we see that this group is faithful on π_o.

Theorem 13 *Theorem. If a translation plane π of odd order q^3 with spread in $PG(5,q)$ admits a solvable collineation group G in the linear translation complement which fixes a subplane π_o of order q and acts transitively on the components of π_o and transitively on the components of $\pi - \pi_o$ then*

(1) $q \equiv -1 \bmod 4$,

(2) the net N_Δ defined by the components of π_o is a regulus net (corresponds to a regulus n $PG(5,q)$),

(3) G is the direct product of two groups F and N such that F fixes π_o pointwise and has order $q(q-1)/2$, and N has order $2(q^2-1)$.

Furthermore, if $q > 3$ then bases may be chosen so that the group F has the following form:

$$\left\langle diag \begin{bmatrix} 1 & 0 & 0 \\ fa & 1 & 0 \\ fa^2/2 & a & 1 \end{bmatrix} \right\rangle \left\langle diag \begin{bmatrix} 1 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b^2 \end{bmatrix} \right\rangle$$

where the order of b divides $(q-1)/2$ and for all a in $GF(q)$.

The group N acting on π_o is faithful and has the following form:

$$\left\langle \sigma = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, g = \begin{bmatrix} \delta I & \gamma\theta I \\ \gamma I & \delta I \end{bmatrix} \right\rangle$$

where g has order q^2-1 and δ, γ in $GF(q)$ and θ is a non-square in $GF(q)$.

(4) Let $M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 1 & 0 & 0 \end{bmatrix}$ and take the component $y = xM$.

The image of $y = xM$ under FN is

$$y = x\left(\begin{bmatrix} 1 & 0 & 0 \\ -b^{-1}fa & b^{-1} & 0 \\ b^{-2}fa^2/2 & -b^{-1}a & b^{-2} \end{bmatrix} M \begin{bmatrix} 1 & 0 & 0 \\ fa & b & 0 \\ fa^2/2 & ba & b^2 \end{bmatrix}\right)$$

$= xM_{a,b}$ by $G_{[\pi_o]}$ then

$y = xM_{a,b}$ onto $y = x(\delta + \gamma(\pm M_{a,b})^{-1}(\gamma\theta + (\pm M_{a,b}))$.

(5) Hence, the spread is

$$\begin{aligned} x &= 0, y = x\beta I, y = x(\delta + \gamma(\pm M_{a,b})^{-1}(\gamma\theta + (\pm M_{a,b})) \\ for\ all\ \beta,\delta,\gamma &\in GF(q). \end{aligned}$$

(6) The spread is a union of a set of $q(q-1)/2$ Desarguesian spreads sharing the regulus net of degree $q+1$.

Proof. By 3-transitivity choose an involutory homology ρ with axis $y = x$ such that the group generated by the homologies is $\langle \sigma, \rho \rangle$.

ρ has the form $\begin{bmatrix} \alpha & 1+\alpha \\ 1-\alpha & -\alpha \end{bmatrix}$. $\rho\sigma$ is $\begin{bmatrix} \alpha & -1-\alpha \\ 1-\alpha & \alpha \end{bmatrix}$.

The group generated by the homologies has order divisible by $2(q+1)$ and contains $-I$ in the kernel group of order $q-1$ and has two orbits of length $(q+1)/2$ on the infinite points of π_o. The order of $\rho\sigma$ is $(q+1)$. The centralizer of this element is

$$\left\{ \begin{bmatrix} \delta & \gamma g \\ \gamma\delta & \end{bmatrix} ; \delta, \gamma \in GF(q) - \{0\} \right\}$$

so this is the field of order $q^2 - 1$ where $(\alpha - 1)/(\alpha + 1) = \theta^{-1}$.

Let $M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 1 & 0 & 0 \end{bmatrix}$ and take the component $y = xM$.

In order to have a spread we need that the group

$\left\langle \sigma = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, g = \begin{bmatrix} \delta I & \gamma g I \\ \gamma I & \delta I \end{bmatrix} \right\rangle \times$

$\left\langle diag \begin{bmatrix} 1 & 0 & 0 \\ fa & 1 & 0 \\ fa^2/2 & a & 1 \end{bmatrix} \right\rangle \left\langle diag \begin{bmatrix} 1 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b^2 \end{bmatrix} \right\rangle$ maps $y = xM$ onto

$y = x \left(\begin{bmatrix} 1 & 0 & 0 \\ -b^{-1}fa & b^{-1} & 0 \\ b^{-2}fa^2/2 & -b^{-2}a & b^{-2} \end{bmatrix} M \begin{bmatrix} 1 & 0 & 0 \\ fa & b & 0 \\ fa^2/2 & ba & b^2 \end{bmatrix} \right)$

$= xM_{a,b}$ by $G_{[\pi_o]}$ then

$y = xM_{a,b}$ onto $y = x(\delta + \gamma(\pm M_{a,b})^{-1}(\gamma g + (\pm M_{a,b}))$.

Hence, it suffices to have a spread that

$(\delta + \gamma(\pm M_{a,b})^{-1})(\gamma g + \delta(\pm M_{a,b}))$,

$(\delta + \gamma(\pm M_{a,b})^{-1})(\gamma g + \delta(\pm M_{a,b})) + \rho I$,

$(\delta + \gamma(\pm M_{a,b})^{-1})(\gamma g + \delta(\pm M_{a,b})) - M$ and

$M + \tau I$ are all nonsingular.

Note that if $M + \tau I$ is nonsingular for all $\tau \in GF(q)$ then

$\begin{bmatrix} 1 & 0 & 0 \\ -b^{-1}fa & b^{-1} & 0 \\ b^{-2}fa^2/2 & -b^{-2}a & b^{-2} \end{bmatrix} (M + \tau I) \begin{bmatrix} 1 & 0 & 0 \\ fa & b & 0 \\ fa^2/2 & ba & b^2 \end{bmatrix} =$

$M_{a,b} + \tau I$ is nonsingular so that $\beta + \delta(\pm M_{a,b})$ is nonsingular.

Furthermore, in this latter case, $(\delta + \gamma(\pm M_{a,b})^{-1})$ $(\gamma g + \delta(\pm M_{a,b})) + \rho I$ is nonsingular unless $\delta\rho = -\gamma g$ and $\gamma\rho = -\delta$ which is valid if and only if $\delta^2 = \gamma^2 g$, a contradiction.

Hence, a spread is obtained if and only if

$M + \tau I$ and $(\gamma g + \delta(\pm M_{a,b})) - (\delta + \gamma(\pm M_{a,b}))M$ are nonsingular or identically zero for all $\tau, \delta\gamma \in GF(q)$.

If these two sets are nonsingular or identically zero, a spread is obtained with components $x = 0, y = x\gamma I, y = x(\delta + \gamma(\pm(M_{a,b})^{-1}(\gamma g + \delta(\pm M_{a,b}))$.

If $M + \tau I$ for all $\tau \in GF(q)$ is non-singular then M has no eigenvalues and since M satisfies its characteristic polynomial of degree 3, it follows that the characteristic polynomial is irreducible. Thus, whenever $M+\tau I$ is non-singular for all $\tau \in GF(q)$, then the module generated by M over $GF(q)$ is a field of order q^3-1.

Furthermore, it then also follows that $\langle M_{a,b}, GF(q)\rangle$ is a field of order q^3-1. Note that the group $(x,y) \mapsto (x,y)$ is a collineation of each such spread.

Hence, we have a set of Desarguesian partial spreads
$S_{a,b} = \{y = x(\delta + \gamma(\pm M_{a,b})^{-1}(\gamma g + \delta(\pm M_{a,b}));$
$\delta, \gamma \in GF(q) - \{(0,0)\}$ each of degree $2(q+1)$. □

Remark 14 *When $q=3$, for example, $q(q-1)/2=3$ and there is a planar element of order 3 that acts on one (all) of the above Desarguesian spreads. Hence, the Desarguesian plane of order 3^3 provides an example of a translation plane that admits the group indicated.*

However, we have used GAP to show that there are no possible examples when $q=7$.

4 Cubic extensions of order q^3 with arbitrary kernels, $q \equiv -1 \bmod 4$

In this section, we assume that the associated translation plane of order q^3 does not admit affine involutory homologies. In this case, our arguments may be generalized and do not require an assumption regarding the kernel.

We begin with a fundamental lemma:

Lemma 15 *Assume that q is not 3. If $G^* = G|\pi_o$ is a solvable transitive subgroup of $\Gamma L(2,q)$ then any 2-group of G^* which fixes a component and which is linear is in the kernel of π_o. Furthermore, the involution is the kernel involution of the superplane.*

Proof. First assume that q^2-1 has a p-primitive divisor u and let g be an element of order u. Then we assert that g is linear. Assume that $u|r$. Then u divides $(p^{u-1}-1, p^{2r}-1) = (p^{(u-1,2r)}-1)$ which implies that $(u-1,2r)=2r$ since u is p-primitive so that if $u|r$ we have a contradiction.

Now we assert that $< gZ >$ is normal in $GL(2,q)/Z$ where Z is the center since G^*/Z is a dihedral group of order dividing $2(q+1)$ or A_4, or S_4. In the latter case $u=3$. SInce the 2-group of G is faithful on π_o, it follows that $(q^2-1)_2$ is at least 8 so that the group must be S_4. So, $(q+1)/(r,q+1) \leq 24$ so that the only possibilities are when $q = 3, 3^2, 3^3, 3^4, 5, 5^2, 7, 11,$ 13, $17, 19, 23$. However, the only survivors of $u=3$ and the order of the Sylow 2-subgroup having order 8 are $q=5$ and $q=11$.

Hence, we may assume that $G^* \cap GL(2,q)$ modulo Z is a subgroup of a dihedral group of order $2(q+1)$ and there is a characteristic subgroup of order u. Hence, there is a normal subgroup $< g > Z$ in G^*. Since u does not divide the order of Z then $< g >$ is normal and characteristic in G^* since there cannot be two distinct u-subgroups in a solvable subgroup. Moreover, G^* acts irreducibly on π_o and $< g >$ is a cyclic normal subgroup¿ There is a kernel involution of the plane which we may assume is G^*. Thus, there is a 2-group fixing a component ℓ of order 4. THe linear 2-group in $\Gamma L(1,q^2)$ has order $(q^2-1)_2 2$ so the 2-groups stabilizing components have orders 4. Moreover, there is an unique involution which fixes a component of π_o pointwise in $\Gamma L(1,q^2)$. The 2-group of the kernel of π_o has order 2. Hence, there must be an involution which fixes π_o pointwise which does not occur. □

Theorem 16 *Let π be a translation plane of order q^3 which admits a collineation group G that fixes a subplane π_o of order q and is a $(q+1, q^3-q)$-transitive group. Assume that there does not exist involutory homologies with axes in π_o.*

If $q \equiv -1 \bmod 4$ and q is not 3 then G is nonsolvable.

Furthermore, the subplane is Desarguesian and $G|\pi_o$ contains $SL(2,q)$.

Proof. If the group is nonsolvable acting on π_o then the previous argument shows that the Hering plane or order 27 does not occur.

We note that $(q-1)_2 = 2$. Assume that $G|\pi_o$ is solvable. If the plane is not Desarguesian then either $q = 9$ or the group induced on the subplane is a subgroup of $\Gamma L(1,q^2)$. If there exists a planar p-group S of order q where $q = p^r$ then by Foulser [7], the subplane π_o is Desarguesian. Assume that p is not 3. If π_o is non-Desarguesian then the p-group fixing π_o pointwise has order less than or equal $\sqrt{q}$. Let $p^{r/t}$ denote the order of the largest possible proper subfield of $GF(q)$. Then there is a faithful p-group of order at least $p^{r-r/t}$ which is larger than r_p. Hence, there is an elation group of π_o on each axis which generates a non-solvable group except in the case when $p = 3$ and $SL(2,3)$ is generated. In this case, $p^{r-r/t} > 3r_3$ so that $SL(2,3^z > 3)$ is generated. Thus, in all cases, π_o is Desarguesian.

Now assume that there is not a p-primitive divisor so that $q = p$ and $p+1 = 2^a$ for some integer a. It follows that the group induced in $PGL(2,q)$ is a subgroup of a dihedral group of order $2(p+1)$ or $2(p-1)$ or A_4 or S_4. We see that $(p+1)/2$ must divide the order of the group so in the second possibility we can only have $2^a = 2$ or 4. Hence, $p = q = 3$. In the latter two case, $(p+1)/2$ must divide 8. Hence, $2^a = 2, 4, 8$ or 16. However, $2^a - 1 = 3$ or 7. If $q = 3$ then $(3^2-1)_2 = 8$ and there must be an element of order 3 induced. An element of order 3 on a Desarguesian plane must be an elation so that the group generated is $SL(2,3)$ which is possible. However, we have excluded the case when $q = 3$. Assume that $q = 7$. Then the order of a Sylow 2-subgroup of G is at least 16. We shall come back to this order.

Hence, the induced group is a subgroup of a dihedral group of order $2(p+1)$. Assume that the intersection with the kernel with the full group is trivial. Hence, there is a dihedral group of order $2(p+1)$ in G^*. Hence, there is an elementary Abelian group of order 4 acting linearly on a Desarguesian affine plane π_o and there are no Baer involutions in G^* (Kallaher and Foulser [8]). Hence, there must also be a kernel intolution in the generated group.

Thus, we must have a kernel involution acting in G^* which leads to a contradiction exactly as previously since there are no involutory homologies.

Now assume that $q = 5$ or 11 and the induced group in $PGL(2,q)$ is S_4. First assume that (q^2-1) divides the order of G^*. When $q = 5$ then $q^2 - 1 = 24$ and this is possible. When $q = 11$ then $(q^2-1) = 120 = 5 \cdot 24$ so perhaps this is possible also.

Since $q = 5$ is not congruent to -1 mod 4, we are left to consider $q = 7$ and $q = 23$. The only problem arises when where is no kernel involution induced on π_o for if a kernel involution is induced then it must be kernel of the plane and this implies by the transitivity requirement that there is an extra 2-element which, in turn, implies that there is a planar 2-element whose fixed points contains those of π_o which cannot occur.

If $q = 7$ and there is no kernel involution induced then 16 must divide 24 as the only solvable such group would necessarily be S_4 or A_4.

If $q = 11$ and there is no kernel involution induced then it is possible that S_4 is induced as a subgroup of π_o. However, there is an elementary Abelian subgroup of order 4 which contains no Baer involutions so that there must be a generated kernel involution on π_o. □

5 Final Conclusions

Our results tend to imply that it might be possible to classify the generalized Desarguesian planes directly by the fact that they are cubic extensions of flag-transitive planes. However, it may be possible that even order planes of order q^3 admit $S_z(\sqrt{q})$. For example, consider the Lüneburg-Tits planes of order h^6 admitting $S_z(h^3)$. If $h = 8$, we obtain a cubic extension. We leave open the following problem:

Completely determine the cubic extensions of order q^3 admitting a collineation group isomorphic to $S_z(q)$.

We also now can essentially complete the general theory of solvable extensions when there are no involutory homologies.

COmbining our general classification result mentioned in the introduction together with our results on cubic extensions and the results on even order [12], we obtain:

Corollary 17 *Let π be a solvable extension of order $q^n, q > 4$ of a proper flag-transitive plane of order q.*

If the spread for π is in $PG(2n-1, q)$ and $q \equiv 1 \bmod 4$ then π is the Hall plane or order q^2 or the derived likeable Walker plane of order 25.

Corollary 18 *Let π be a solvable extension of order $q^n, q > 4$ of a proper flag-transitive plane of order q.*

If $q \equiv -1 \bmod$ and π contains no involutory homologies than π is the Hall plane or order q^2.

We note that when $q \equiv -1$, we also have a possibly class of translation planes where we have completely determined the equations for the components. Furthermore, while there is at least one example in this class, and GAP shows that no examples are possible when $q = 7$, **the question of existence for other orders is completely open**.

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