CUBIC EXTENSIONS OF FLAG-TRANSITIVE PLANES, II. ODD ORDER

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Abstract. The finite translation planes with spreads in PG(5, q) which are odd cubic extensions of flag-transitive planes and admit solvable groups are completely determined.

1 Introduction

In a previous article, we considered even order cubic extensions of flag-transitive planes. In this article, we continue this study with consideration of odd order planes.

In particular, we consider the analysis of translation planes of order \( q^3 \) that admit collineation groups \( G \) which leave invariant a subplane \( \pi_o \) of order \( q \), act flag transitively on \( \pi_o \) and act transitively on the set of components not in \( \pi_o \).

In two previous articles (see [14] and [13]), the general study of translation planes which are extensions of flag-transitive planes is undertaken.

An 'extension of a flag-transitive plane' is an affine plane \( \pi \) containing an affine subplane \( \pi_o \) and a collineation group which leaves \( \pi_o \) invariant, acts flag-transitively on \( \pi_o \) and acts transitively on the parallel classes of \( \pi \) not in \( \pi_o \).

The reader is referred the Part I, Even Order for the complete statements of the main results of [14] and [13]). We shall give a short version of the results here for convenience.

Theorem 1 (Hiramine, Jha, Johnson ([14] and [13])). Let \( \pi \) be a non-Desarguesian translation plane of order \( q^n \) where \( q > 4 \) that is an extension of a flag-transitive plane and let \( G \) denote the associated collineation group.

Then \( \pi \) is Hall or the derived Walker plane of order 25 in either of the following two situations:

(i) \( n = 2 \) or
(ii) \( n \neq 3 \) and \( G \) is solvable.

When \( n = 3 \), there are problems in the general classification of extensions of flag-transitive planes. In particular, there is a tremendous variety of of translation planes called generalized Desarguesian planes of order \( q^3 \) that admit \( GL(2, q) \). There are many mutually nonisomorphic planes of this type and where the kernel of the plane may be chosen in a variety of ways.

For such planes, the associated vector space is a standard \( GF(q)GL(2, q) \) module. What this means is that a group isomorphic to \( SL(2, q) \) is generated by elation groups and that \( GL(2, q) \) leaves invariant each subplane of order \( q \) incident with the zero vector in the associated \( GF(q) \)-regulus net defined by the elation axes of \( SL(2, q) \). In addition, there are always infinite orbits of lengths \( q + 1 \) and \( q^3 - q \) so that we obtain cubic extensions of a Desarguesian flag-transitive plane admitting non-solvable collineation groups when \( q > 3 \).
We have seen in a previous article (see Hiramine, Jha, Johnson [12]) that the Lüneburg-Tits plane of order $2^{18}$ is a cubic extension of a Lüneburg-Tits subplane of order $2^6$.

Furthermore, the authors prove the following theorem.

**Theorem 2 (Hiramine, Jha, Johnson [12])** Let $\pi$ be a cubic extension translation plane of even order $q^2$ with subplane $\pi_o$ of order $q > 4$.

(1) Then $\pi_o$ is Desarguesian or Lüneburg-Tits and the full collineation group $G$ contains a group isomorphic to $SL(2, q)$ or $S_5(\sqrt{q})$ respectively where the involutions are elations.

(2) If $q = 2^r$ and $r$ is odd then $\pi_o$ is Desarguesian and $G$ is isomorphic to $SL(2, q)$ and generated by elations.

The various articles of the authors on extensions of flag-transitive planes form a theory which is, in some sense, a continuation of the ideas of the second author [15] who studied autotopism groups in translation planes of order $q^n$ with an orbit of length $q^n - q$. We are replacing the assumption that the group is an autotopism group with the assumption that the group leaves a subplane invariant and acts transitively on the flags of the subplane.

Furthermore, Jha posed the following problems $P$ and $Q$:

**Problem (P):** Classify all spreads within $PG(2n - 1, q)$ admitting an automorphism group $G$ such that $G$ fixes globally a set $\Delta$ of $q + 1$ components and acts transitively on the remaining components.

Actually, this problem originally had the further assumption that $n \geq 3$ as it was considered too difficult to complete when $n = 2$ due to the many known examples.

**Problem (Q):** Classify all translation planes of order $p^n$ that admit collineation groups with a slope orbit of length $p^n - p$ where $p$ is a prime.

What we are considering in this article includes the study of problems (P) and (Q) when there is an invariant subplane of order $q$ in the first problem and of order $p$ in the second problem and asking when the collineation group is solvable and transitive on the flags of the subplane.

Hence, we can make some progress towards the problems (P) and (Q) of Jha by adding some hypotheses regarding the action on subplanes.

We analyze the collineation groups of cubic extensions and are able to generally formulate a classification.

We have seen previously that the generalized Desarguesian planes and the Lüneburg-Tits plane of order $2^{18}$ appear here (see Hiramine, Jha, Johnson [12]).

In this article, we consider only planes of odd order. Since there are various problems encountered for odd order plane which are not present when the plane has even order, we usually only consider spreads in $PG(5, q)$. In this case when $q \equiv 1 \mod 4$, we show that the group must be nonsolvable and involve $SL(2, q)$. When $q \equiv -1 \mod 4$, although the group is not completely determined, there is a possible class of solvable cubic extensions the form of which is completely determined.

Using the implied nonsolvability of the various groups, we then can basically complete the classification of results on solvable extensions of flag-transitive planes of order $q^n$ at least when $q \equiv 1 \mod 4$ or when $q$ is even.

For convenience, we repeat some definitions.
Definition 3 If an affine plane \( \pi \) of order \( q^n \) admits a collineation group \( G \) which has infinite point orbits of lengths \( q+1 \) and \( (q^n-q) \), we shall call \( \pi \) a \(' (q+1, q^n-q) \)-transitive plane' and \( G \) a ‘\( (q+1, q^n-q) \)-transitive group’.

If \( \pi \) is a translation plane whose kernel contains \( GF(q) \) and the group \( G \) is in the linear translation complement, we shall call \( \pi \) a ‘linear \( (q+1, q^n-q) \)-transitive plane’ and \( G \) a ‘linear \( (q+1, q^n-q) \)-transitive group’.

If \( G \) leaves a subplane \( \pi_o \) of order \( q \) invariant within the net of length \( q+1 \) and there is a collineation group transitive on the sets of affine and infinite points of \( \pi_o \) and the infinite points of \( \pi - \pi_o \) then \( \pi_o \) is a flag-transitive affine plane and we shall call \( \pi \) an ‘extension of \( \pi_o \).

If the group of an extension is solvable, we shall call the plane a ‘solvable extension’.

2 Cubic Extensions when the spread is in \( PG(5, q), q \equiv 1 \mod 4 \)

The problem of whether there exist solvable cubic extensions is completely unresolved. We begin the study of cubic extensions with spread in \( PG(5, q) \). We shall require the following result to Foulser.

Theorem 4 Foulser [7].

Let \( \pi \) be a translation plane of order \( q^3 \) that admits a planar \( p \)-group \( G \) fixing a subplane \( \pi_o \) of order \( q \) pointwise. Then \( G \) is elementary Abelian of order dividing the order of the kernel of \( \pi_o \).

Proof. See Foulser [7] (3.4) part (5) to observe that \( G \) is an additive subgroup of the kernel \( K_o \) of \( \pi_o \).

Assume, for the remainder of this section, that \( \pi \) is a cubic extension translation plane with spread in \( PG(5, q) \) of odd order of a translation plane \( \pi_o \) of order \( q \) and \( G \) is a collineation group in the translation complement which is a \( (q+1, q^3-q) \)-group.

Lemma 5 If the group \( G|\pi_o \) is solvable on \( \pi_o \) then one of the following occurs:

1. the subplane is desarguesian of order 9 or Hall of order 9,
2. the group induced on the infinite points of \( \pi_o \) is isomorphic to \( A_4 \) or \( S_4 \) and \( q = 5, 7, 11 \) or 23,
3. the group is a subgroup of \( TL(1, q^2) \) or \( \sigma \)
4. \( q = 3 \) and the group induced contains \( SL(2, 3) \).

Proof. Apply Foulser [4]. Theorem 1, p. 459) and Foulser and Kallaher [8] (1.2).

Lemma 6 Let \( S \) be a Sylow 2-subgroup of \( G_L \) where \( L \) is a component of \( \pi_o \). Then \( S \) is faithful on \( \pi_o \).

Proof. If \( S \) is not faithful on \( \pi_o \), there is an involution fixing \( \pi_o \) pointwise which implies that there is a subplane of order \( q^{3/2} \) which contains a subplane of order \( q \) which cannot occur. Hence, \( S \) is faithful on \( \pi_o \).
Lemma 7 If the group $G|\pi_o$ is solvable on $\pi_o$ then either $\pi_o$ is Desarguesian or the case that the group is a subgroup of $\Gamma L(1,q^2)$ does not occur.

Proof. Assume that the plane is not Desarguesian and the group induced on $\pi_o$ is in $\Gamma L(1,q^2)$ and hence has order on $\pi_o$ of order divisible by $(q^2 - 1)r$ where $q = p'$. Assume that the order of a $p$ group fixing $\pi_o$ pointwise is $p^a$. By 2.1, if $p^a > \sqrt{q}$ then the plane $\pi_o$ is Desarguesian. Hence, $p^d \leq \sqrt{q}$ and a $p$-group acting faithfully on $\pi_o$ has order at least $q/p^d > \sqrt{q}$. Let the order of a faithful $p$-group be $p'^{2+2t}$ where $t > 0$. However, the group is isomorphic to a subgroup of $\Gamma L(1,q^2)$ and the Sylow $p$-subgroup has order $(2r)_p$ (the $p$-part of $2r$). Since $p$ is odd, let $r = p^b f$ where $(p,f) = 1$ so $p'^{2+2t} \geq p^b$. However, by induction, is follows that $r < p'^{2+2t}$. \[\Box\]

Corollary 8 Under the assumptions of the previous lemma, $\pi_o$ is a $K$-subspace where the spread is in $P G(5,K)$ and $K$ isomorphic to $G F(q)$.

Proof. The previous argument shows that there is an element $G L(6,q)$ that fixes $\pi_o$ pointwise. \[\Box\]

Note we may assume that $G$ contains the kernel homology group of order $q - 1$.

Lemma 9 Let $\pi_o$ be Desarguesian and the group $G|\pi_o$ is in $\Gamma L(1,q^2)$. Let $\ell$ be a component of $\pi_o$ and let $S$ be a Sylow 2-subgroup of $G|\ell$ of order divisible by $(q - 1)_2^2$ and let $S_1$ denote the subgroup of $G L(6,q)$ in $S$.

Then the full subgroup $S_1$ of $S$ which fixes a component of $\pi_o$ pointwise has order exactly 2.

Thus, $(q - 1)_2$ divides $2r_2$.

Proof. We see that the group acting on $\ell$ has order divisible by $q(q - 1)^2$. We note that $G L(2,q)$ commutes with the kernel homology group of order $q - 1$ which also faithfully induces the kernel homology group of $\pi_o$. We see that we may assume that $S_1$ has a subgroup $S_1$ which fixes $\pi_o \cap \ell$ pointwise. The subgroup of $\Gamma L(1,q^2)$ which fixes a line of $\pi_o$ pointwise has order 2 and induces an involutory homology on $\pi_o$. Hence, if $|S_1| > 2$ or equivalently if $S_1 > (q - 1)_2$, we have a contradiction. Note that the order of a Sylow 2-group of subgroup of $\Gamma L(1,q^2)$ that fixes a component of $\pi_o$ has order $(q - 1)_2 2r_2$. Hence, it follows that $(q - 1)_2$ must divide $2r_2$. \[\Box\]

Lemma 10 If $q \equiv 1 \mod 4$, the situation described in the previous lemma cannot occur.

Proof. Let $r = 2^d t$ where $t$ is odd. First assume $a$ is not zero. Then $(q - 1) = (p' - 1)(p'^2 + 1)(p'^t + 1)(p^{2t} + 1)(p^{3t} + 1) \ldots (p^{2^d + 1} - 1)$. Let $d = (p^2 - 1)_2$ so that $(q - 1)_2 = 2^d t - 1$ where $d > 3$. Note if $a = 0$ then $(q - 1)_2 = (p - 1)_2$. Now if $a = 0$ then $(q - 1)_2 = (p - 1)_2 > 2$.

Thus, in either case, $(q - 1)_2 > 2r_2$. \[\Box\]

Lemma 11 Assume that the group induced on the infinite points of $\pi_o$ is isomorphic to $A_4$ or $S_4$ and $q = 5, 7, 11$ or 23. Then $q 
\neq 5$ or 23.
Proof. Since there is a planar $p$-group acting trivially on $\pi_o$, it follows that $\pi_o$ is a kernel subspace so there is a kernel group of $\pi$ of order $q - 1$ leaving $\pi_o$ invariant. Similarly, as in the previous lemmas, we have a linear 2-group of order $(q - 1)2^2$ acting on $\pi_o$ and fixing a component so it again follows that $(q - 1)2$ divides $2r = 2$ which eliminates $q = 5$. So, we have a group acting trivially on the infinite points of $\pi_o$ of order divisible by $q(q^2 - 1)$ $(q + 1)/24$ which implies that if $(q + 1)/2 > 1$, there is an involution fixing $\pi_o$ pointwise. Hence, $q \neq 23$. □

Theorem 12 Let $\pi$ be a cubic extension translation plane of odd order order $q^3$ and kernel containing $K \cong GF(q)$ which contains a subplane $\pi_o$ of order $q > 3$.

If $G$ is a collineation group which leaves $\pi_o$ invariant and acts as a $(q + 1, q^3 - q)$-transitive group and $q \equiv 1 \mod 4$ then $G$ is nonsolvable.

For any odd order if $G$ is non-solvable then
(a) $\pi_o$ is Desarguesian and $G|\pi_o$ contains $SL(2,q)$ or
(b) $q = 9$ and $G|\pi_o\ell_\infty \cong A_5$.

Proof. Note that if $G$ is non-solvable then it must induce a non-solvable group on $\pi_o$ since the 2-groups are faithful on $\pi_o$. Now, assume, in general, that $G$ restricted to $\pi_o$ is nonsolvable. Then the plane is Desarguesian, Hall or order 9 or Hering of order $3^3$ by the results of Bukenhout et al [1].

First assume that $\pi_o$ is Hall or order 9. The full collineation group of $\pi_o$ which fixes the zero vector has order $2^8 \cdot 3 \cdot 5$. The group $G$ has order divisible by $9(9^2 - 1) = 2^4 \cdot 3^2 \cdot 5$. In order that $\pi_o$ be Hall and the group is transitive on the infinite points of $\pi_o$, it follows that there must be a group of order $2 \cdot 3 \cdot 5$ induced. Since the group induced on $\pi_o$, restricted to the infinite points is a subgroup of $S_5$, it follows the group is nonsolvable.

If the subplane is Hering of order 27, then the subplane of order $q = 3^3$ admits a collineation group isomorphic to $SL(2,13)$ which is normalized by $G$. Furthermore, any odd order subgroup which centralizes $SL(2,13)$ must fix the infinite points $\pi_o$ since each Sylow 13-group fixes exactly two infinite points of $\pi_o$. Moreover, the outer automorphism group is trivial so there is a 3-group of order divisible by $3^3$ of which there is a subgroup of order 9 that centralizes this copy of $SL(2,13)$.

Hence, it follows that there is a group of order 9 which fixes $\pi_o$ pointwise. However, this implies by Foulser, Theorem (3.1), that $\pi_o$ is Desarguesian.

If $\pi_o$ is Desarguesian and assume that the group $G$ induced on $\pi_o$ is non-solvable then by Foulser [4] (12.1), either $SL(2,q)$ is induced on $\pi_o$ or the group induced contains the preimage of $A_5$ (acting on the infinite points) and $q = 11, 19, 29$ or 59.

In each case, there is a $p$-group fixing the subplane $\pi_o$ pointwise. Furthermore, the group induces $A_5$ on the line at infinity of the subplane $\pi_o$. Hence, there is a kernel homology group acting on the subplane of order divisible by 2, 6, 14, 58 respectively as $q = 11, 19, 29$ or 59.

Hence, there is a 2-group of order $(q - 1)^2$ that fixes a component and a subgroup of order $(q - 1)^2$ which fixes a component of $\pi_o$ pointwise. Hence, the subgroup fixing the infinite points of $\pi_o$ has order $q(q^2 - 1)(q - 1)/60$ and since the 2-part is strictly larger than $(q - 1)^2$ in each instance, there must be an involution fixing $\pi_o$ pointwise which cannot occur.

We have seen the $G$ cannot be solvable if $q \equiv 1 \mod 4$. This completes the proof. □
3 Linear groups for spreads in $PG(5,q), q \equiv -1 \mod 4$

Now assume that we have $q \equiv -1 \mod 4$, the spread is linear and the group $G$ is linear (within $GL(6,q)$).

Assume that the group is solvable. We have noted that there is a planar $p$-group of order $q$ which fixes $\pi_o$ pointwise. Furthermore, we may assume that $\pi_o$ is Desarguesian and the group induced upon $\pi_o$ is a subgroup of $\Gamma L(1,q^2)$. Since the group is solvable, there is a subgroup $G_{p'}$ of order $[G]/[G]_{p'}$. Since $G_{p'}$ fixes $\pi_o$, there is a Maschke complement $C_1$ of dimension 4 over the kernel $K$ isomorphic to $GF(q)$, as we are assuming that the group $G$ is linear.

Since we now also have the kernel homology group of order $q - 1$ acting on the plane and on $\pi_o$, it follows that the group $G_{p'}$ has order divisible by $(q - 1)^2(q + 1)$. We note that the group fixing a component $\ell$ of $\pi_o$ has order divisible by $q(q - 1)^2$. Furthermore, as the group induced on $\pi_o$ is a subgroup of $\Gamma L(1,q^2)$, it follows that there is a subgroup of order $q(q - 1)/2$ which fixes $\pi_o$ pointwise. In addition, $G_{[\pi_o]}$ has order dividing $q(q^2 - 1)2$. If there is an element of odd order $u$ dividing $q + 1$ then, since the group is linear, we have a linear planar group of order $p^\alpha u^\beta$ for $\alpha$ and $\beta$ positive integers. However, this says that there is a normal $u$-group by Jha [15] which implies that the $p$-group fixes more than $q$ element on a line of $\pi_o$ which cannot occur.

Hence, the full groups fixing $\pi_o$ pointwise has order exactly $q(q - 1)/2$.

The preceding section says that we are finished or there exist involutory homologies with axes in the net containing $\pi_o$. We consider $\Gamma L(1,q^2)$ acting on the cosets of $GF(q)$ as components. In this representation, the involutory homologies are of the form

$$\sigma_a : x \mapsto w^a a$$

where $a^{q+1} = 1$. Letting $a = b^{1-q}$, then the line $GF(q)b$ is fixed pointwise by $\sigma_a$. Also, the coaxis of $\sigma_a$ is $GF(q)b^{(q^2-1)/4}$, where $a$ is a primitive element of $GF(q^2)^*$. The group generated by the homologies has two orbits of lengths $(q + 1)/2$ on the line at infinity of $\pi_o$. Moreover, each homology must commute with $G_{[\pi_o]}$ so we have a normal group of order divisible by $2(q + 1)$ which commute with $G_{[\pi_o]}$ and has two orbits of length $(q + 1)/2$ on the line at infinity of $\pi_o$.

We claim that there is a unique Sylow $p$-subgroup of order $q$ in $G_{[\pi_o]}$. Proof: The group acts faithfully on any component $\ell$ and furthermore, acts faithfully on the set or 1-dimensional $K$-subspaces on $\ell$. Hence, there is a faithful group induces on the projective geometry $PG(2,q)$ induced from the 3-dimensional vector space $\ell$. Thus, it remains to show that there is a unique Sylow $p$-subgroup.

Acting on $PG(2,q)$, we have a collineation group which fixes a point say $\infty$. It follows easily that any fixed-point-free group of order $q$ which fixes $\infty$ and a line $\ell_\infty$ has the form:

$$(x,y) \mapsto (x + a, xf(a) + y + g(a)) : a \in GF(q))$$

where $f$ and $g$ are appropriate functions and $f(a) = 0$ if and only if $a = 0$ if and only if the corresponding element is the identity.

If there is another Sylow $p$-subgroup then the second group cannot fix $\ell_\infty$ since otherwise there would be a generated $p$-group with additional fixed points. This $p$-element would determine a collineation of $\pi$ which fixes more than $q$ point on $\ell$.

Hence, either we have a unique Sylow $p$-subgroup or there is an element in $G_{[\pi_o]}$ which moves $\ell_\infty$. Note that Sylow $p$-subgroup $S$ as above is regular on the lines incident with $\infty$ other than $\ell_\infty$. Hence, it follows that the group generated by the two Sylow $p$-subgroups is
doubly transitive and hence the group \( G_{[\pi_o]} \) must have order divisible by \( q(q+1) \) which is a contradiction.

Thus, we must have a normal Sylow \( p \)-group \( S \) in \( G_{[\pi_o]} \).

Hence, the group \( G_{\pi} \) permutes transitively the \( q^2 - 1 \) orbits of length \( q \) of \( S \).

Let \( C_1 \) be any \( G_{\pi} \)-complement of \( \pi_{\pi} \). Then the intersection of \( C_1 \) with any component is at least 1-dimensional. Assume that some component is contained in \( C_1 \). Then \( q^2 - 1 \) components are contained in \( C_1 \) which cannot occur. Hence, the components intersect \( C_1 \) in 1- and 2-dimensional subspaces. Let \( a \) and \( b \) denote the number of components which intersect \( C_1 \) in 1- and 2-dimensional subspaces respectively.

Then, \( a + b = q^3 + 1 \) and \( q^4 - 1 = a(q - 1) + b(q^2 - 1) \)
so that we have \( q^4 + q^3 = aq + bq^2 = (q^3 + 1 - b)q + bq \).

This last equation is valid if and only if \( q^3 - q = b(q^2 - q) \) if and only if \( b = q + 1 \).

Hence, the components in the net \( N_{\pi_o} \) intersect any \( G_{\pi} \)-Maschke complement is a \( G_{\pi} \)-Maschke complement.

We shall say that such Maschke complements are 'strongly embedded'.

Assume that \( q + q \neq 2^i \). Let \( u \) be an odd divisor of \( q + 1 \). Then there is an Abelian subgroup \( A \) of order \( q^u \beta \) for some non-negative integer \( \beta \). Suppose that there is a unique strongly embedded \( A \)-Maschke complement. Then the group \( S \) of order \( q \) must leave this complement invariant which cannot occur.

Hence, there are at least \( qA \)-Maschke complements \( C_i \), \( i = 1, 2, \ldots, q \). We see that \( C_i \cap C_j \) for \( i \neq j \) is at least two dimensional and since \( (q^3 - 1, q + 1) = 2 \), it follows that \( C_i \cap C_j \) must be 2-dimensional. Since the orbits of \( G_{\pi} \) are divisible by \( q^2 - 1 \) on the components not on \( \pi_{\pi} \), it follows that two \( A \)-Maschke complements which are images of a strongly embedded complement can intersect only on components of \( \pi_{\pi} \). It follows easily that \( C_i \cap C_j \) must be a subplane of the net \( N_{\pi_o} \) containing \( \pi_{\pi} \). Hence, there are at least \( 1 + q \) subplanes of order \( q \) incident with the zero vector in \( N_{\pi_o} \). We assert that a group \( H \) in \( G_{[\pi_o]} \) of order \( (q - 1)/2 \) must fix exactly three 2-dimensional \( G_{\pi} \)-subspaces. Acting on a component taken as \( PG(2, q) \), we see that the group \( H \) fixes a point \( (\infty) \) of \( PG(2, q) \). The associated normal \( p \)-group fixes \( (\infty) \) and a line \( \ell_{\pi_o} \). Note that no non-trivial element of \( H \) can commute with a non-trivial element of \( S_p \). Hence, \( H \) is a subgroup of the group of an affine Desarguesian plane and thus is a subgroup of \( W_0T \) where \( T \) is the associated translation group. Since \( G \) is linear, it follows that \( H \) acting on \( PG(2, q) \) is in \( GL(2, q)T \). Hence, \( H/H \cap T \sim H \) is isomorphic to a subgroup of \( GL(2, q) \) and it follows that \( H \cap Z(GL(2, q)) = (1) \). Since \( (q - 1)/2 \) is odd, it follows that \( H \) is cyclic. Thus, we may assume that \( H \) fixes three points of \( PG(2, q) \) of a triangle. It follows that \( H \) fixes three subplanes of order \( q \) of the net \( N_{\pi_o} \) not all in the direct sum of any two of them. Hence, we may diagonalize \( H \).

For another viewpoint, we redress the argument in matrix form.

We let \( \pi_{\pi} \) be represent by \( \{(x_1, 0, 0, y_1, 0, 0); x_1, y_1 \in GF(q)\} \) where the translation plane is represented with respect to \( x = 0, y = 0 \) in for the form
\[
\{(x_1, x_2, x_3, y_1, y_2, y_3); x_1, y_1 \in GF(q), \text{ for } i = 1, 2, 3\}.
\]

Hence, we may represent elements of the \( p \)-group fixing \( \pi_{\pi} \) pointwise in the form
\[
\text{diag} \begin{bmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ d & e & 1 \end{bmatrix}.
\]
We first assert that for the group of order \( q \) the set of elements \( e \) in the \((3,1)\)-entries is \( GF(q) \).

Suppose there are two elements with the same \((3-1)\)-entries \( e \) say \( \text{diag} \begin{bmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ d & e & 1 \end{bmatrix} \) and

\[
\text{diag} \begin{bmatrix} 1 & 0 & 0 \\ c^* & 1 & 0 \\ d^* & e^* & 1 \end{bmatrix}.
\]

Then, we obtain an element of the form

\[
\text{diag} \begin{bmatrix} 1 & 0 & 0 \\ c^* & 1 & 0 \\ d^* & e = e^* & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ d & e & 1 \end{bmatrix} = \text{diag} \begin{bmatrix} 1 & 0 & 0 \\ c - c^* & 1 & 0 \\ d - d^* & 0 & 1 \end{bmatrix}
\]

as \( \text{diag} \begin{bmatrix} 1 & 0 & 0 \\ c^* & 1 & 0 \\ d^* & e & 1 \end{bmatrix}^{-1} = \text{diag} \begin{bmatrix} 1 & 0 & 0 \\ -c^* & 1 & 0 \\ -d^* + c^*e^* & -e^* = -e & 1 \end{bmatrix} \).

This element fixes a point on \( x = 0 \) say \((x_1, x_2, x_3)\) if and only if

\[
(c - c^*)x_2 + (d - d^*)x_3 = 0.
\]

If \((c - c^*)(d - d^*) \neq 0\) then there exist additional fixed points on \( x = 0 \) which cannot occur. If \( c - c^* = 0 \) but \( d - d^* \neq 0 \) then \((x_1, x_2, 0)\) is fixed pointwise. Hence, \( c - c^* = 0 \) if and only if \( d - d^* = 0 \).

The above argument is basically symmetric so that it follows that the set of \((2,1)\)-entries is \( GF(q) \). It thus follows that \( c = f(e) \) for all \( e \in GF(q) \) where \( f \) is a \( 1-1 \) function on \( GF(q) \) which must be additive since the \( p \)-group is elementary Abelian. Furthermore, it similarly follows that \( d = g(e) \) where \( g \) is a function on \( GF(q) \).

Since the group is commutative, it is direct to verify that we obtain the following conditions on the functions:

\[
g(a) + ag(b) + g(b) = g(b) + bf(a) + g(a) = g(a + b)
\]

and so

\[
bf(a) = af(b)
\]

for all \( a, b \in GF(q) \).

The second equation implies that \( f(a) = af \) for some non-zero element \( f \) of \( GF(q) \).

Hence,

\[
g(a) + g(a) + af = g(a + b)
\]

Represent \( g \) as follows:

\[
g(a) = \sum_{i=0}^{q-1} g_i a^i \text{ for } g_i \in GF(q).
\]

Let \( a = b \) in the above equation so that

\[
2g(a) = a^2 f = g(2a).
\]
Hence,
\[
2g_0 + 2g_1 a + (2g_2 + f)a^2 + \sum_{i=3}^{q-1} (2g_i) a^i \\
= g_2 + 2g_1 a + 4g_2 a^2 + \sum_{i=3}^{q-1} (g_i 2^i) a^i
\]
for all \( a \in GF(q) \).

Hence, we obtain
\[
g_0 = 0, f = 2g_2, g_i = 0 \text{ for all } i \geq 3.
\]

Hence,
\[
g(a) = g_1 a + f a^2/2 \\
\text{for all } a \in GF(q).
\]

Now since \( H \) fixes three subplanes, we may choose a basis without alternating the form for \( S_p \) so that element of \( H \) have the form
\[
\text{diag} \begin{bmatrix} 1 & 0 & 0 \\
0 & \nu & 0 \\
0 & 0 & m(b) \end{bmatrix}
\]
for the order of \( b \) dividing \((q-1)/2\) and \( m \) a function on \( GF(q) \).

Since \( S_p \) is normal then
\[
diag \begin{bmatrix} 1 & 0 & 0 \\
0 & b & 0 \\
0 & 0 & m(b) \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\
af & 1 & 0 \\
g(a) & a & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\
0 & b & 0 \\
0 & 0 & m(b) \end{bmatrix}^{-1}
\]
\[
diag \begin{bmatrix} 1 & 0 & 0 \\
0 & b & 0 \\
0 & 0 & m(b) \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\
af & 1 & 0 \\
g(a) & a & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\
0 & b & 0 \\
0 & 0 & m(b) \end{bmatrix}^{-1}
\]
\[
diag \begin{bmatrix} 1 & 0 & 0 \\
0 & b & 0 \\
0 & 0 & m(b) \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\
af & 1 & 0 \\
g(a) & a & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\
0 & b & 0 \\
0 & 0 & m(b) \end{bmatrix}^{-1}
\]

Hence, this implies that
\[
bm(b)^{-1} = b^{-1} \text{ and hence } m(b) = b^2.
\]

This also provides
\[
b^{-2} g(a) = g(b^{-1} a) \text{ for all } a \text{ for all } b \text{ of order dividing } (q-1)/2.
\]

Thus, we obtain
\[
c^2(g_1 a + f a^2/2) = g_1 c a + f c^2 a^2/2
\]
so that \( c^2 g_1 = g_1 c \) for all \( c \) of order dividing \((q-1)/2\). Hence, either \( q = 3 \) or \( g_1 = 0 \).

Thus, for \( q > 3 \),

We may represent the group \( G_{[\kappa_o]} \) as follows:
\[
\begin{bmatrix} 1 & 0 & 0 \\
fa & 1 & 0 \\
f a^2/2 & a & 1 \end{bmatrix} ; a \in GF(q)
\]

\[
\begin{bmatrix} 1 & 0 & 0 \\
0 & b & 0 \\
0 & 0 & b^2 \end{bmatrix} ; \text{ order of } b \text{ divides } (q-1)/2
\]

However, two of the 2-dimensional \( GF(q) \)-subspaces are subplanes of \( N_{\kappa_o} \) and hence are fixed by the group generated by the homologies. It follows that the third 2-space is then
fixed. However, this means that the subplane contains the centers and axes of all of the involutory homologies. That is, the subplane in \( N_{\pi_0} \). Let Then, since we now have a direct product of three subplanes lying in the net it is possible to use the Krull-Schmidt to show all are isomorphic to \( \mathcal{E} \)-modules where \( \mathcal{E} \) is the enveloping algebra of the net so that by Liebler [19], it follows that the net \( N_{\pi_0} \) is a regulus net. The centralizer within \( GL(6,q) \) of \( G_{[\pi_0]} \) is isomorphic to \( GL(2,q) \) and has the form \( \left\langle \begin{bmatrix} \alpha I_3 & \beta I_3 \\ \delta I_3 & \gamma I_3 \end{bmatrix} : \alpha \gamma - \delta \beta \neq 0 \right\rangle \). The group generated by the homologies commutes with \( G_{[\pi_0]} \) and thus, we see that this group is faithful on \( \pi_0 \).

**Theorem 13** Theorem. If a translation plane \( \pi \) of odd order \( q^3 \) with spread in \( PG(5,q) \) admits a solvable collineation group \( G \) in the linear translation complement which fixes a subplane \( \pi_0 \) of order \( q \) and acts transitively on the components of \( \pi_0 \) and transitively on the components of \( \pi - \pi_0 \) then

1. \( q \equiv -1 \mod 4 \),
2. the net \( N_{\Delta} \) defined by the components of \( \pi_0 \) is a regulus net (corresponds to a regulus \( n \) \( PG(5,q) \)),
3. \( G \) is the direct product of two groups \( F \) and \( N \) such that \( F \) fixes \( \pi_0 \) pointwise and has order \( q(q - 1)/2 \), and \( N \) has order \( 2(q^2 - 1) \).

Furthermore, if \( q > 3 \) then bases may be chosen so that the group \( F \) has the following form:

\[
\left\langle \text{diag} \begin{bmatrix} 1 & 0 & 0 \\ fa & 1 & 0 \\ fa^2/2 & a & 1 \end{bmatrix} \right\rangle \left\langle \text{diag} \begin{bmatrix} 1 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b^2 \end{bmatrix} \right\rangle
\]

where the order of \( b \) divides \( (q - 1)/2 \) and for all \( a \) in \( GF(q) \).

The group \( N \) acting on \( \pi_0 \) is faithful and has the following form:

\[
\sigma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad g = \begin{bmatrix} \delta & \gamma \theta \\ \gamma & \delta \theta \end{bmatrix}
\]

where \( g \) has order \( q^2 - 1 \) and \( \delta, \gamma \) in \( GF(q) \) and \( \theta \) is a non-square in \( GF(q) \).

4. Let \( M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 1 & 0 & 0 \end{bmatrix} \) and take the component \( y = xM \).

The image of \( y = xM \) under \( FN \) is

\[
y = x\left( \begin{bmatrix} 1 & 0 & 0 \\ -b^{-1}fa & b^{-1} & 0 \\ b^{-2}fa^2/2 & -b^{-1}a & b^{-2} \end{bmatrix} M \begin{bmatrix} 1 & 0 & 0 \\ fa & b & 0 \\ fa^2/2 & ba & b^2 \end{bmatrix} \right)
\]

\[
xM_{a,b} \text{ by } G_{[\pi_0]} \text{ then } y = xM_{a,b} \text{ onto } y = x(\delta + \gamma(\pm M_{a,b})^{-1}(\gamma \theta + (\pm M_{a,b}))).
\]

5. Hence, the spread is

\[
x = 0, y = x\beta I, y = x(\delta + \gamma(\pm M_{a,b})^{-1}(\gamma \theta + (\pm M_{a,b})))
\]

for all \( \beta, \delta, \gamma \in GF(q) \).

6. The spread is a union of a set of \( q(q - 1)/2 \) Desarguesian spreads sharing the regulus net of degree \( q + 1 \).
Proof. By 3-transitivity choose an involutory homology $\rho$ with axis $y = x$ such that the group generated by the homologies is $\langle \sigma, \rho \rangle$.

$\rho$ has the form $\begin{bmatrix} \alpha & 1 + \alpha \\ 1 - \alpha & -\alpha \end{bmatrix}$ and $\rho \sigma$ is $\begin{bmatrix} \alpha & -1 - \alpha \\ 1 - \alpha & \alpha \end{bmatrix}$.

The group generated by the homologies has order divisible by $2(q+1)$ and contains $-I$ in the kernel group of order $q - 1$ and has two orbits of length $(q + 1)/2$ on the infinite points of $\pi_a$. The order of $\rho \sigma$ is $(q + 1)$. The centralizer of this element is

$$\left\{ \begin{bmatrix} \delta & \gamma g \\ \gamma \delta & \gamma \end{bmatrix} : \gamma, \delta \in GF(q) - \{0\} \right\}$$

so this is the field of order $q^2 - 1$ where $(\alpha - 1)/(\alpha + 1) = \theta^{-1}$.

Let $M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 1 & 0 & 0 \end{bmatrix}$ and take the component $y = xM$.

In order to have a spread we need that the group $\langle \sigma = \begin{bmatrix} I & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rangle$ is mapped onto $y = xM$ by $G_{[9]}$ then

$y = x \begin{bmatrix} 1 & 0 & 0 \\ -b^{-1}fa & b^{-1} & 0 \\ b^{-2}fa^2/2 & -b^{-2}a & b^{-2} \end{bmatrix} M \begin{bmatrix} 1 & 0 & 0 \\ fa & b & 0 \\ fa^2/2 & ba & b^2 \end{bmatrix}$

Hence, it suffices to have a spread that

$0 = (\delta + \gamma(\pm M_{a,b})^{-1})(\gamma g + \delta(\pm M_{a,b})),
(\delta + \gamma(\pm M_{a,b})^{-1})(\gamma g + \delta(\pm M_{a,b})) + \rho I,
(\delta + \gamma(\pm M_{a,b})^{-1})(\gamma g + \delta(\pm M_{a,b}))) - M$ and

$M + \tau I$ are all nonsingular.

Note that if $M + \tau I$ is nonsingular for all $\tau \in GF(q)$ then

$0 = \begin{bmatrix} 1 & 0 & 0 \\ -b^{-1}fa & b^{-1} & 0 \\ b^{-2}fa^2/2 & -b^{-2}a & b^{-2} \end{bmatrix} (M + \tau I) \begin{bmatrix} 1 & 0 & 0 \\ fa & b & 0 \\ fa^2/2 & ba & b^2 \end{bmatrix} = 0$

$M_{a,b} + \tau I$ is nonsingular so the $\beta + \delta(\pm M_{a,b})$ is nonsingular.

Furthermore, in this latter case, $0 = (\delta + \gamma(\pm M_{a,b})^{-1})(\gamma g + \delta(\pm M_{a,b}))) + \rho I$ is nonsingular unless $\delta \rho = -\gamma g$ and $\rho g = -\delta$ which is valid if and only if $\delta^2 = -\gamma^2 g$, a contradiction.

Hence, a spread is obtained if and only if

$M + \tau I$ and $(\gamma g + \delta(\pm M_{a,b}))M$ are nonsingular or identically zero for all $\tau, \delta \gamma \in GF(q)$.

If these two sets are nonsingular or identically zero, a spread is obtained with components

$x = 0, y = xy, y = x(\delta + \gamma(\pm M_{a,b})^{-1})(\gamma g + \delta(\pm M_{a,b})))$.

If $M + \tau I$ for all $\tau \in GF(q)$ is non-singular then $M$ has no eigenvalues and since $M$ satisfies its characteristic polynomial of degree 3, it follows that the characteristic polynomial
is irreducible. Thus, whenever $M + \tau I$ is non-singular for all $\tau \in GF(q)$, then the module
generated by $M$ over $GF(q)$ is a field of order $q^3 - 1$.

Furthermore, it then also follows that $\langle M_{a,b}, GF(q) \rangle$ is a field of order $q^3 - 1$. Note that
the group $(x, y) \mapsto (x, y)$ is a collineation of each such spread.

Hence, we have a set of Desarguesian partial spreads

$$S_{a,b} = \{ y = x(\delta + \gamma(\pm M_{a,b})^{-1}(\gamma y + \delta(\pm M_{a,b})) ; \delta, \gamma \in GF(q) - \{(0,0)\} \}$$
each of degree $2(q+1)$. □

**Remark 14** When $q = 3$, for example, $q(q - 1)/2 = 3$ and there is a planar element of order
3 that acts on one (all) of the above Desarguesian spreads. Hence, the Desarguesian plane
of order $3^3$ provides an example of a translation plane that admits the group indicated.

However, we have used GAP to show that there are no possible examples when $q = 7$.

4 Cubic extensions of order $q^3$ with arbitrary kernels, $q \equiv -1 \mod 4$

In this section, we assume that the associated translation plane of order $q^3$ does not admit
affine involutory homologies. In this case, our arguments may be generalized and do not
require an assumption regarding the kernel.

We begin with a fundamental lemma:

**Lemma 15** Assume that $q$ is not 3. If $G^* = G|_{\pi_0}$ is a solvable transitive subgroup of $\Gamma L(2, q)$
then any 2-group of $G^*$ which fixes a component and which is linear is in the kernel of $\pi_0$.
Furthermore, the involution is the kernel involution of the superplane.

**Proof.** First assume that $q^2 - 1$ has a $p$-primitive divisor $u$ and let $g$ be an element of order
$u$. Then we assert that $g$ is linear. Assume that $u | r$. Then $u$ divides $(p^{u-1} - 1, p^{u-1} - 1) = (p^{u-1}, 2r) - 1$ which implies that $(u - 1, 2r) = 2r$ since $u$ is $p$-primitive so that if $u | r$ we
have a contradiction.

Now we assert that $< gZ >$ is normal in $GL(2, q)/Z$ where $Z$ is the center since $G^* / Z$
is a dihedral group of order dividing 2($q+1$) or $A_4$, or $S_4$. In the latter case $u = 3$. Since the 2-
group of $G$ is faithful on $\pi_0$, it follows that $(q^2 - 1)_2$ is at least 8 so that the group must be $S_4$.
So, $(q + 1)/(r, q + 1) \leq 24$ so that the only possibilities are when $q = 3, 3^2, 3^3, 3^4, 5, 5^2, 7, 11, 13, 17, 19, 23$. However, the only survivors of $u = 3$ and the order of the Sylow 2-subgroup
having order 8 are $q = 5$ and $q = 11$.

Hence, we may assume that $G^* \cap GL(2, q)$ modulo $Z$ is a subgroup of a dihedral group
of order 2($q+1$) and there is a characteristic subgroup of order $u$. Hence, there is a normal
subgroup $< g > Z$ in $G^*$. Since $u$ does not divide the order of $Z$ then $< g >$ is normal and
characteristic in $G^*$ since there cannot be two distinct $u$-subgroups in a solvable subgroup.
Moreover, $G^*$ acts irreducibly on $\pi_0$ and $< g >$ is a cyclic normal subgroup. There is a
kernel involution of the plane which we may assume is $G^*$. Thus, there is a 2-group fixing a
component $\delta$ of order 4. The linear 2-group in $\Gamma L(1, q^2)$ has order $(q^2 - 1)_2$ so the 2-groups
stabilizing components have orders 4. Moreover, there is an unique involution which fixes a
component of $\pi_0$ pointwise in $\Gamma L(1, q^2)$. The 2-group of the kernel of $\pi_0$ has order 2. Hence,
there must be an involution which fixes $\pi_0$ pointwise which does not occur. □
Theorem 16 Let π be a translation plane of order \(q^3\) which admits a collineation group \(G\) that fixes a subplane \(\pi_o\) of order \(q\) and is a \((q+1, q^3-q)\)-transitive group. Assume that there does not exist involutory homologies with axes in \(\pi_o\).

If \(q \equiv -1 \mod 4\) and \(q\) is not 3 then \(G\) is nonsolvable.

Furthermore, the subplane is Desarguesian and \(G|\pi_o\) contains \(SL(2, q)\).

Proof. If the group is nonsolvable acting on \(\pi_o\) then the previous argument shows that the Hering plane or order 27 does not occur.

We note that \((q-1)_2 = 2\). Assume that \(G|\pi_o\) is solvable. If the plane is not Desarguesian then either \(q = 9\) or the group induced on the subplane is a subgroup of \(\Gamma L(1, q^2)\). If there exists a planar \(p\)-group \(S\) of order \(q\) where \(q = p^r\) then by Foulser [7], the subplane \(\pi_o\) is Desarguesian. Assume that \(p\) is not 3. If \(\pi_o\) is non-Desarguesian then the \(p\)-group fixing \(\pi_o\) pointwise has order less than or equal \(\sqrt{q}\). Let \(p^{r/l}\) denote the order of the largest possible proper subfield of \(GF(q)\). Then there is a faithful \(p\)-group of order at least \(p^{r-l}/l\) which is larger than \(r_p\). Hence, there is an elation group of \(\pi_o\) on each axis which generates a non-solvable group except in the case when \(p = 3\) and \(SL(2, 3)\) is generated. In this case, \(p^{r-l}/l > 3p_3\) so that \(SL(2, 3^2 > 3)\) is generated. Thus, in all cases, \(\pi_o\) is Desarguesian.

Now assume that there is not a \(p\)-primitive divisor so that \(q = p\) and \(p + 1 = 2^a\) for some integer \(a\). It follows that the group induced in \(PGL(2, q)\) is a subgroup of a dihedral group of order \(2(p+1)\) or \(2(p-1)\) or \(A_4\) or \(S_4\). We see that \((p+1)/2\) must divide the order of the group so in the second possibility we can only have \(2a = 2\) or 4. Hence, \(p = q = 3\). In the latter case, \((p+1)/2\) must divide 8. Hence, \(2a = 2, 4, 8\) or 16. However, \(2a - 1 = 3\) or 7. If \(q = 3\) then \((3^2 - 1)/2 = 8\) and there must be an element of order 3 induced. An element of order 3 on a Desarguesian plane must be an elation so that the group generated is \(SL(2, 3)\) which is possible. However, we have excluded the case when \(q = 3\). Assume that \(q = 7\). Then the order of a Sylow 2-subgroup of \(G\) is at least 16. We shall come back to this order.

Hence, the induced group is a subgroup of a dihedral group of order \(2(p+1)\). Assume that the intersection with the kernel with the full group is trivial. Hence, there is a dihedral group of order \(2(p+1)\) in \(G^*\). Hence, there is an elementary Abelian group of order 4 acting linearly on a Desarguesian affine plane \(\pi_o\) and there are no Baer involutions in \(G^*\) (Kallaher and Foulser [8]). Hence, there must also be a kernel in the group in the generated group.

Thus, we must have a kernel involution acting in \(G^*\) which leads to a contradiction exactly as previously since there are no involutory homologies.

Now assume that \(q = 5\) or \(11\) and the induced group in \(PGL(2, q)\) is \(S_4\). First assume that \((q^2 - 1)\) divides the order of \(G^*\). When \(q = 5\) then \(q^2 - 1 = 24\) and this is possible. When \(q = 11\) then \((q^2 - 1) = 120 = 5 \cdot 24\) so perhaps this is possible also.

Since \(q = 5\) is not congruent to \(-1 \mod 4\), we are left to consider \(q = 7\) and \(q = 23\). The only problem arises when there is no kernel involution induced on \(\pi_o\), for if a kernel involution is induced then it must be kernel of the plane and this implies by the transitivity requirement that there is an extra 2-element which, in turn, implies that there is a planar 2-element whose fixed points contains those of \(\pi_o\) which cannot occur.

If \(q = 7\) and there is no kernel involution induced then 16 must divide 24 as the only solvable such group would necessarily be \(S_4\) or \(A_4\).

If \(q = 11\) and there is no kernel involution induced then it is possible that \(S_4\) is induced as a subgroup of \(\pi_o\). However, there is an elementary Abelian subgroup of order 4 which
contains no Baer involutions so that there must be a generated kernel involution on $\pi_\circ$. □

5 Final Conclusions

Our results tend to imply that it might be possible to classify the generalized Desarguesian planes directly by the fact that they are cubic extensions of flag-transitive planes. However, it may be possible that even order planes of order $q^3$ admit $S_3(\sqrt{q})$. For example, consider the Lüneburg-Tits planes of order $h^6$ admitting $S_3(h^2)$. If $h = 8$, we obtain a cubic extension. We leave open the following problem:

**Completely determine the cubic extensions of order $q^3$ admitting a collineation group isomorphic to $S_3(q)$.**

We also now can essentially complete the general theory of solvable extensions when there are no involutory homologies.

Combining our general classification result mentioned in the introduction together with our results on cubic extensions and the results on even order [12], we obtain:

**Corollary 17** Let $\pi$ be a solvable extension of order $q^n, q > 4$ of a proper flag-transitive plane of order $q$.

If the spread for $\pi$ is in $PG(2n - 1, q)$ and $q \equiv 1 \mod 4$ then $\pi$ is the Hall plane or order $q^2$ or the derived likeable Walker plane of order 25.

**Corollary 18** Let $\pi$ be a solvable extension of order $q^n, q > 4$ of a proper flag-transitive plane of order $q$.

If $q \equiv -1 \mod 4$ and $\pi$ contains no involutory homologies then $\pi$ is the Hall plane or order $q^2$.

We note that when $q \equiv -1$, we also have a possibly class of translation planes where we have completely determined the equations for the components. Furthermore, while there is at least one example in this class, and GAP shows that no examples are possible when $q = 7$, the question of existence for other orders is completely open.
References


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