WEAKLY FUNCTIONALLY θ-NORMAL SPACES, θ-SHRINKING OF COVERS AND PARTITION OF UNITY

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Abstract. Characterizations of weakly functionally θ-normal spaces, similar to that of a normal space, are obtained and used to establish the existence of partition of unity subordinated to certain locally finite open covers.

Key words: θ-closed set, θ-open set, weakly functionally θ-normal space (= \(wf\ θ\)-normal space), \(aθ\)-limit point, θ-continuous function, θ-shrinking, partition of unity.

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1 Introduction

It is of fundamental importance in topology to obtain a factorization of a given topological property in terms of two weaker topological properties. The literature in topology is replete with the results of this nature. Normality is an important topological invariant and hence a decomposition of normality is desirable. First step in this direction was taken by G. Vigino [9], who defined seminormal spaces. Subsequently, Singal and Arya [6] introduced the class of almost normal spaces and proved that a space is normal if and only if it is both a seminormal space and an almost normal space. A search for another decomposition of normality led us to introduce in [4] the class of θ-normal spaces and certain of its variants such as weakly functionally θ-normal (\(wf\ θ\)-normal) spaces. The notion of \(wf\ θ\)-normality serves as a necessary ingredient for a decomposition of normality. In [4], \(wf\ θ\)-normal spaces are defined in terms of the existence of certain continuous real-valued functions. In this paper, in analogy with the normal spaces, we obtain a characterization of \(wf\ θ\)-normal space in terms of separation of certain closed sets by open sets. Moreover, we introduce the notion of a θ-shrinking of an open cover and obtain a characterization of \(wf\ θ\)-normal spaces in terms of θ-shrinking of certain covers. Furthermore, we characterize \(wf\ θ\)-normal spaces in terms of the existence of a partition of unity subordinated to certain locally finite open covers.

Section 2 is devoted to basic definitions and preliminaries. In section 3 we obtain a characterization of \(wf\ θ\)-normal spaces analogous to that of Uryshon Lemma and in section 4 we give a characterization of \(wf\ θ\)-normal spaces in terms of θ-shrinking of θ-open covers and use the same to obtain a characterization of \(wf\ θ\)-normal spaces in terms of the existence of partition of unity subordinated to certain locally finite θ-open covers.

2 Preliminaries and basic definitions

Definition 1 [8]. Let \(X\) be a topological space and let \(A \subseteq X\). A point \(x \in X\) is called a θ-limit point of \(A\) if every closed neighbourhood of \(x\) intersects \(A\). Let \(A_θ\) denote the set of
all $\theta$-limit points of $A$.

The set $A$ is called $\theta$-closed if $A = \overline{A}$.

The complement of a $\theta$-closed set will be referred to as a $\theta$-open set.

**Lemma 2** [4]. A subset $A$ in a topological space $X$ is $\theta$-open if and only if for each $x \in A$ there is an open set $U$ containing $x$ such that $\overline{U} \subset A$.

In general the $\theta$-closure operator is not a Kuratowski closure operator since $\theta$-closure of a set might not be $\theta$-closed (see [3]). However, the following modification in [5] yields a Kuratowski closure operator.

**Definition 3** [5]. Let $X$ be a topological space and let $A \subset X$. A point $x \in X$ is called a $u\theta$-limit point of $A$ if every $\theta$-open set $U$ containing $x$ intersects $A$. Let $A_{u\theta}$ denote the set of all $u\theta$-limit points of $A$.

**Lemma 4** [5]. The correspondence $A \rightarrow A_{u\theta}$ is a Kuratowski closure operator.

It is observed in [5] that the set $A_{u\theta}$ is the smallest $\theta$-closed set containing $A$.

**Definition 5** [2]. A function $f : X \rightarrow Y$ is said to be $\theta$-continuous if for each $x \in X$ and each open set $V$ containing $f(x)$ there exists an open set $U$ containing $x$ such that $f(U) \subset V$.

Every continuous function is $\theta$-continuous but the converse is not true in general. However, a $\theta$-continuous function into a regular space is continuous in a somewhat strong sense.

**Lemma 6** [5]. Let $f : X \rightarrow Y$ be a $\theta$-continuous function and let $U$ be a $\theta$-open set in $Y$. Then $f^{-1}(U)$ is $\theta$-open in $X$.

3 Weakly Functionally $\theta$-Normal Spaces

**Definition 7** [4]. A topological space $X$ is said to be weakly functionally $\theta$-normal ($w.f \ \theta$-normal) if for every pair of disjoint $\theta$-closed sets $A$ and $B$ there exists a continuous function $f : X \rightarrow [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$.

The class of $w.f \ \theta$-normal spaces is much larger than the class of normal space. An example of a $w.f \ \theta$-normal spaces which is not normal is given in [4]. Moreover, the cofinite topology on an infinite set is (vacuously) $w.f \ \theta$-normal but not normal. Similarly, the particular point topology [7, p. 44] and the indiscrete rational (irrational) extension of $\mathbb{R}$ [7, p. 88] are $w.f \ \theta$-normal but are not normal. Furthermore, every finite topological space is $w.f \ \theta$-normal which need not be normal.

**Theorem 8** For a topological space $X$, the following statements are equivalent.

(a) $X$ is $w.f \ \theta$-normal.

(b) Every pair of disjoint $\theta$-closed sets are contained in disjoint $\theta$-open sets.

(c) For every $\theta$-closed set $A$ and every $\theta$-open set $U$ containing $A$ there exists a $\theta$-open set $V$ such that $A \subset V \subset V_{u\theta} \subset U$. 

Proof. To prove the assertion \((a) \Rightarrow (b)\), let \(X\) be a \(w\,f\) \(\vartheta\)-normal spaces and let \(A, B\) be disjoint \(\vartheta\)-closed sets in \(X\). By \(w\,f\) \(\vartheta\)-normality of \(X\) there exists a continuous function \(f: X \to [0, 1]\) such that \(f(A) = 0\) and \(f(B) = 1\). Since \([0, \frac{1}{2})\) and \((\frac{1}{2}, 1]\) are \(\vartheta\)-open sets in \([0, 1]\), by Lemma 6 \(f^{-1}[0, \frac{1}{2})\) and \(f^{-1}(\frac{1}{2}, 1]\) are disjoint \(\vartheta\)-open sets in \(X\) containing \(A\) and \(B\) respectively.

To prove \((b) \Rightarrow (c)\), let \(U\) be a \(\vartheta\)-open set in \(X\) containing a \(\vartheta\)-closed set \(A\). Then \(A\) and \(X - U\) are disjoint \(\vartheta\)-closed sets in \(X\). So there exist disjoint \(\vartheta\)-open sets \(V\) and \(W\) such that \(A \subset V\) and \((X - U) \subset W\). Now, \(A \subset V \subset X - W\). Since \(X - W\) is \(\vartheta\)-closed and since \(V_{\vartheta}\) is the smallest \(\vartheta\)-closed set containing \(V, A \subset V \subset V_{\vartheta} \subset U\).

To prove the implication \((c) \Rightarrow (a)\), let \(A\) and \(B\) be disjoint \(\vartheta\)-closed sets in \(X\). Then \(A \subset X - B = U_1\) (say). Since \(U_1\) is \(\vartheta\)-open, there exists a \(\vartheta\)-open set \(U_{1/2}\) such that \(A \subset U_{1/2} \subset (U_{1/2})_{\vartheta} \subset U_1\). Again, since \((U_{1/2})_{\vartheta}\) is a \(\vartheta\)-closed set contained in the \(\vartheta\)-open set \(U_1\), there exist \(\vartheta\)-open set \(U_{1/4}\) and \(U_{3/4}\) such that \(A \subset U_{1/4} \subset (U_{1/4})_{\vartheta} \subset U_{1/2}\) and \((U_{1/2})_{\vartheta} \subset U_{3/4} \subset (U_{3/4})_{\vartheta} \subset U_1\). Continuing the above process, we obtain for each dyadic rational \(r\), a \(\vartheta\)-open set \(U_r\) satisfying the condition that \(r < s\) implies \((U_s)_{\vartheta} \subset U_r\). Define a mapping \(f: X \to [0, 1]\) by

\[
f(x) = \begin{cases} 
\inf\{r \in X \mid r \in U_r\}, & \text{if } x \text{ belongs to some } U_r \\
1, & \text{if } x \text{ does not belong to any } U_r
\end{cases}
\]

Clearly \(f\) is well defined and \(f(A) = 0\), \(f(B) = 1\). Now it remains to prove that \(f\) is continuous. To this end we first observe that if \(x \in U_r\), then \(f(x) \leq r\). Similarly, \(f(x) \geq r\) if \(x \not\in (U_r)_{\vartheta}\). To prove continuity, let \(x \in X\) and \((a, b)\) be an open interval containing \(f(x)\). Now choose two dyadic rationals \(p\) and \(q\) such that \(a < p < f(x) < q < b\). Let \(U = U_q - (U_p)_{\vartheta}\). Then \(U\) is an open set containing \(x\). Now for \(y \in U\), \(y \in U_q\). So \(f(y) \leq q\). Also as \(y \in U\), \(y \not\in (U_p)_{\vartheta}\). Thus \(f(y) \geq p\). And so \(f(y) \in [p, q]\). Therefore, \(f(U) \subset [p, q] \subset (a, b)\). Hence \(f\) is continuous. \(\square\)

4 \(\vartheta\)-Shrinking of Covers and Partition of Unity

**Definition 9** An open cover \(u = \{U_\alpha : \alpha \in A\}\) of \(X\) is said to be \(\vartheta\)-shrinkable if there exists a \(\vartheta\)-open cover \(v = \{V_\alpha : \alpha \in A\}\) of \(X\) such that \((V_\alpha)_{\vartheta} \subset U_\alpha\) for each \(\alpha \in A\).

Recall that a covering \(u\) of \(X\) is said to be point finite if every \(x \in X\) belongs to only finitely many elements of \(u\).

**Theorem 10** A topological space \(X\) is \(w\,f\) \(\vartheta\)-normal if and only if every point finite \(\vartheta\)-open cover of \(X\) is \(\vartheta\)-shrinkable.

**Proof.** Let \(X\) be a \(w\,f\) \(\vartheta\)-normal spaces and let \(u = \{U_\alpha : \alpha \in \Lambda\}\) be a point finite \(\vartheta\)-open cover of \(X\). Well order the set \(\Lambda\). For convenience we may assume that \(\Lambda = \{1, 2, \ldots, \alpha, \ldots\}\). Now construct \(\{V_\alpha : \alpha \in \Lambda\}\) by transfinite induction as follows. Let \(F_1 = X - \bigcup_{\alpha > 1} U_\alpha\). Then \(F_1\) is a \(\vartheta\)-closed set contained in the \(\vartheta\)-open set \(U_1\). So by Theorem 8 there exists a \(\vartheta\)-open set \(V_1\) such that \(F_1 \subset V_1 \subset (V_1)_{\vartheta} \subset U_1\). Suppose \(V_\beta\) has been defined for each \(\beta < \alpha\). Let \(F_\alpha = X - [(U_{\beta < \alpha} V_\beta) \cup (U_{\beta \geq \alpha} U_\beta)]\). Then \(F_\alpha\) is a \(\vartheta\)-closed set contained in the \(\vartheta\)-open set \(U_\alpha\). So, again, by Theorem 8 there exists a \(\vartheta\)-open set \(V_\alpha\) such that \(F_\alpha \subset V_\alpha \subset (V_\alpha)_{\vartheta} \subset U_\alpha\). Now \(V = \{V_\alpha : \alpha \in \Lambda\}\) is a \(\vartheta\)-shrinking of \(u\) provided it cover \(X\). Let \(x \in X\). Then \(x\) belongs to only
finitely many members of \( u \), say \( U_{\alpha_1}, \ldots, U_{\alpha_k} \). Suppose \( \alpha = \max \{ \alpha_1, \ldots, \alpha_k \} \). Now \( x \) does not belong to \( U_\lambda \) for \( \lambda > \alpha \) and hence if \( x \notin V_\beta \) for \( \beta < \alpha \), then \( x \in F_\alpha \subset V_\alpha \). So in any case \( x \in V_\beta \) for \( \beta \leq \alpha \). Thus \( v \) is a \( \theta \)-shrinking of \( u \).

Conversely, suppose \( A \) and \( B \) are disjoint \( \theta \)-closed subsets of \( X \). Then \( \{ X - A, X - B \} \) is a point finite \( \theta \)-open cover of \( X \). So, by hypothesis there exists a \( \theta \)-shrinking \( \{ U, V \} \) of \( \{ X - A, X - B \} \). Now \( X - (U)_\theta \) and \( X - (V)_\theta \) are disjoint \( \theta \)-open sets containing \( A \) and \( B \), respectively. Again, in view of Theorem 8 \( X \) is w.f \( \theta \)-normal.

Recall that for a continuous real-valued function \( f \) defined on \( X \), the support of \( f \) is the closed set \( \{ x \in X : f(x) \neq 0 \} \).

Definition 11. \( f \) is a family \( \{ f_\alpha : \alpha \in \Lambda \} \) of continuous functions on a space \( X \) to the closed unit interval \( [0, 1] \) is called a partition of unity on \( X \) if the collection \( \{ \text{support } f_\alpha : \alpha \in \Lambda \} \) forms a locally finite closed cover of \( X \) and \( \sum_{\alpha \in \Lambda} f_\alpha(x) = 1 \) for every \( x \in X \).

A partition of unity \( \{ f_\alpha : \alpha \in \Lambda \} \) on a space \( X \) is said to be subordinated to a cover \( u = \{ U_\alpha : \alpha \in \Lambda \} \) of \( X \) if support \( f_\alpha \subset U_\alpha \) for each \( \alpha \in \Lambda \).

Theorem 12. A space \( X \) is w.f \( \theta \)-normal if and only if for every locally finite \( \theta \)-open cover \( u \) of \( X \) there exists a partition of unity subordinated to \( u \).

Proof. Let \( X \) be a w.f \( \theta \)-normal space and let \( u = \{ U_\alpha : \alpha \in \Lambda \} \) be a locally finite \( \theta \)-open cover of \( X \). Since every locally finite collection is point finite, by Theorem 10 choose a \( \theta \)-shrinking \( v = \{ V_\alpha : \alpha \in \Lambda \} \) of \( u \), i.e. \( (V_\alpha)_\theta \subset U_\alpha \) for each \( \alpha \in \Lambda \). Since the collection \( u \) is locally finite, so is the collection \( v \) and thus \( v \) is point finite. Again by Theorem 10 choose a \( \theta \)-shrinking \( w = \{ W_\alpha : \alpha \in \Lambda \} \) of \( v \). The cover \( w \) is locally finite, since \( v \) is locally finite. Since \( X \) is w.f \( \theta \)-normal, for each \( \alpha \in \Lambda \) there exists a continuous function \( \phi_\alpha : X \to [0, 1] \) such that \( \phi_\alpha(W_\alpha)_\theta = 1 \) and \( \phi_\alpha(X - V_\alpha) = 0 \). Since \( \phi_\alpha^{-1}(0, 1) \) is contained in \( V_\alpha \) and since \( V_\alpha \subset (V_\alpha)_\theta \subset U_\alpha \), support \( \phi_\alpha \subset U_\alpha \). Now let \( x \in X \). Again, since \( w \) is locally finite, there exists a neighbourhood \( U_\alpha \) of \( x \) and a finite subset \( \Lambda_0 = \{ \alpha_1, \ldots, \alpha_n \} \) of \( \Lambda \) such that \( \phi_\alpha(x) = 0 \) for all \( \alpha \in \Lambda - \Lambda_0 \). Thus for each \( x \in X \), \( \phi = \sum_{\alpha = 1}^n \phi_\alpha(x) \) is positive. Therefore, we may define, for each \( \alpha \), \( f_\alpha(x) = \phi_\alpha(x)/\phi(x) \). Then the collection \( \{ f_\alpha : \alpha \in \Lambda \} \) is the desired partition of unity subordinated to \( u \).

Conversely, suppose that every locally finite \( \theta \)-open cover of \( X \) has a partition of unity subordinated to it and let \( A \) and \( B \) be any two disjoint \( \theta \)-closed sets in \( X \). Then \( \{ X - A, X - B \} \) constitutes a finite (and hence locally finite) \( \theta \)-open cover of \( X \) and so there exists a partition of unity \( \{ f_1, f_2 \} \) subordinated to it. Suppose that support \( f_1 \subset X - A \). Then support \( f_2 \subset X - B \). Therefore \( A \subset X - f_1^{-1}(0, 1) \subset X - f_1^{-1}(0, 1] \) and \( B \subset X - f_2^{-1}(0, 1] \subset X - f_2^{-1}(0, 1] \). Now define \( h : X \to [0, 1] \) by \( h(x) = \frac{f_1(x)}{f_1(x) + f_2(x)} \). Clearly \( h \) is continuous, \( h(A) = 0 \) and \( h(B) = 1 \). Thus \( X \) is a w.f \( \theta \)-normal space.

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