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FUNDAMENTAL GROUPS OF FAT-GRAPHS

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Abstract. *Let Γ be a fat-graph of genus g, with n faces. In this note, we describe a combinatorial algorithm, leading to a presentation of its fundamental group by $2g+n$ generators satisfying the surface relation.*

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1 Introduction

Let Γ be a finite and connected graph, and consider its fundamental group. This is a free group with $h(\Gamma) = a(\Gamma) - s(\Gamma) + 1$ generators, where $a(\Gamma)$ and $s(\Gamma)$ are respectively the number of geometrical edges and vertices of the graph. The way to get a set of generators using a maximal tree is well-known [9].

Assume now that Γ is a fat-graph, or map, in the sense of [1], [5], or [7]. Then we can realize Γ in a canonical way as a deformation retract of an orientable compact surface punctured by a finite number of points. The fundamental group of Γ is thus isomorphic to that of the punctured surface.

Let $\Pi_{g,n}$ the group defined by generators and relations:

$$\Pi_{g,n} =< \{A_1,B_1,\ldots,A_g,B_g,C_1,\ldots,C_n\}/\prod_{i=1}^{g}[A_i,B_i]\prod_{k=1}^{n}C_k = 1 > .$$

The topological classification of compact orientable surfaces (as described in [2] or [10], by reduction to the normal form) and Van Kampen's theorem furnish a non canonical isomorphism between the fundamental group of a n-punctured compact orientable surface and some $\Pi_{g,n}$; the integer g is the genus of the surface. Thus the fundamental group of a fat-graph Γ is also isomorphic to some $\Pi_{g,n}$.

Let Γ be a fat-graph of genus g (defined by Euler-Poincare formula) with n faces. The aim of this note is, starting from the algebraic definition of a fat-graph, to establish a combinatorial algorithm leading to an isomorphism between $\pi_1(\Gamma)$ and $\Pi_{g,n}$ (theorem 12).

For this, we use three operations on fat-graphs, Whitehead's moves, duality, and cut and paste, and we interpret them at the level of their fundamental groups.

This result is motivated by the combinatorial study of moduli spaces $\mathcal{M}_{g,n}$ of compact Riemann surfaces punctured at n points (see [3], [6], [8]), and more generally the study of ramified coverings between compact Riemann surfaces, and of their moduli spaces (called Hurwitz spaces, see [4]).

To make the texte readable for everyone, we recall in the next paragraph some basic notions on graphs. In the third one, we define fat-graphs and combinatorial operations on them. In the last one, we state and prove the result of this note.

2 Graphs

We recall some elementary facts on graphs and their fundamental groups, with the book [9] of J.P. Serre as reference.

Definition 1 *A graph Γ is made of a set of vertices $S(\Gamma)$ and of a set of oriented edges $A(\Gamma)$, together with two maps:*

$$A(\Gamma) \to S(\Gamma) \times S(\Gamma) \text{ and } A(\Gamma) \to A(\Gamma)$$

$$y \longmapsto (y(0), y(1)) \qquad y \longmapsto \bar{y}$$

such that the second one is an involution without fixed points, and such that $y(0) = \bar{y}(1)$.

If $y \in A(\Gamma)$, then $\bar{y}$ is its opposite edge. We denote by $y(0)$ the origin of y, and by $y(1)$ its end. If $y(0) = y(1)$, the edge is called a loop. The couple $\{y, \bar{y}\}$ forms a geometric edge. We denote by $A_g(\Gamma)$, the set of geometric edges. We set $s(\Gamma) = \#S(\Gamma)$ and $a(\Gamma) = \#A_g(\Gamma)$, so that $\#A(\Gamma) = 2a(\Gamma)$.
There is an evident notion of sub-graph that we do not develop here.

Definition 2 *An oriented path c in a graph Γ is a finite collection of successive oriented edges: $a_1, \ldots, a_n$, with $a_i(1) = a_{i+1}(0)$ for $i \in \{1, \ldots, n-1\}$. We denote by $o(c)$ the origin of the path, and by $t(c)$ its end. If $o(c) = t(c)$, then the oriented path is called an oriented loop based at $o(c)$.*

In this note, all the graphs will be finite (i.e. $\#A(\Gamma)$ and $\#S(\Gamma)$ are finite), and connected (i.e. two any vertices are linked by a path).

Recall that a tree is a graph such that for every vertex s, there is no loop based at s. The following fact is classical:

Lemma 3 *Let Γ be a graph. There exists a subgraph T of Γ which is a tree, whose set of vertices is $S(\Gamma)$, and with $a(T) = s(\Gamma) - 1$.*

The proof is by induction on the number of vertices of the graph. Such a tree is called a maximal tree, it is not unique.

We now recall the combinatorial definition of the fundamental group of a graph. If $\alpha = (a_1, \ldots, a_n)$ and $\beta = (b_1, \ldots, b_m)$ are two oriented paths in a graph Γ with $a_n(1) = b_1(0)$, then we define a new path $\alpha\beta = (a_1, \ldots, a_n, b_1, \ldots, b_m)$.

Two oriented paths α and β are elementary homotopic if $\alpha = ca\bar{a}d$ and $\beta = cd$, where a is an oriented edge, c and d are two oriented paths (or if the same fact holds reversing the role of α and β). They are homotopic if they differ by a sequel of elementary homotopies. The notation is $\alpha \sim \beta$.

Two homotopic paths possess same origins and ends. Thus the homotopy classes of oriented loops based at a vertex p of Γ, with the composition law described above, make up a group $\pi_1(\Gamma, p)$ called the fundamental group of Γ based at p.

If q is another vertex of Γ, then we have the following isomorphism of groups:

$$\begin{array}{rcl} h_{p,q} : \pi_1(\Gamma, p) & \longrightarrow & \pi_1(\Gamma, q) \\ \gamma & \longmapsto & \alpha\gamma\overline{\alpha} \end{array}$$

where α is an oriented path with $o(\alpha) = p$ and $t(\alpha) = q$.

It is rather intuitive that $\pi_1(\Gamma, p)$ is isomorphic to the first homotopy group of a geometrical realization $|\Gamma|$ of Γ.

The classical result on the fundamental group of a graph is:

Theorem 4 *Let Γ be a graph. Then $\pi_1(\Gamma, p)$ is a free group in $h(\Gamma)$ generators, where $h(\Gamma) = a(\Gamma) - s(\Gamma) + 1$.*

The number $h(\Gamma)$ is called the cyclotomic number of the graph. See [9] for a proof.

3 Fat-graphs

We recall definitions and basic properties of fat-graphs. Our references for fat-graphs and their applications are [1], [5] and [7].

Imagine a graph embedded in an oriented smooth surface. Look at oriented tangent planes at each vertex of the graph. Project neighborhoods of the vertices on tangent planes. We obtain a cyclic ordering of edges incident on a vertex, and since the surface is oriented, the cyclic ordering is the same at each vertex. Hence to embed a graph in a surface gives an additional data. This leads to the following definition:

Definition 5

- *A* **fat-graph** *Γ is given by a finite set $A(\Gamma)$ (of oriented edges) and by two permutations σ_0 and σ_1 of this set, where σ_1 is an involution without fixed points.*

- *The subgroup $C(\Gamma)$ of the group of bijections of $A(\Gamma)$ generated by σ_0 and σ_1 is called the* **cartographical group** *of Γ.*

The geometric edges are the orbits (disjoint cycles) of $\sigma_1(\Gamma)$, and the vertices are those of $\sigma_0(\Gamma)$. We denote by $A_g(\Gamma)$ and $S(\Gamma)$ there sets, and by $a(\Gamma)$ and $s(\Gamma)$ their cardinals.

Note that Γ is a graph. We take the convention that $a(1)$ is the σ_0-orbit of a and $a(0)$ is the σ_0-orbit of $\sigma_1(a)$. For convenience, we often put $\overline{a} = \sigma_1(a)$. The connectivity amounts to assume the transitivity of $C(\Gamma)$ on the set of oriented edges.

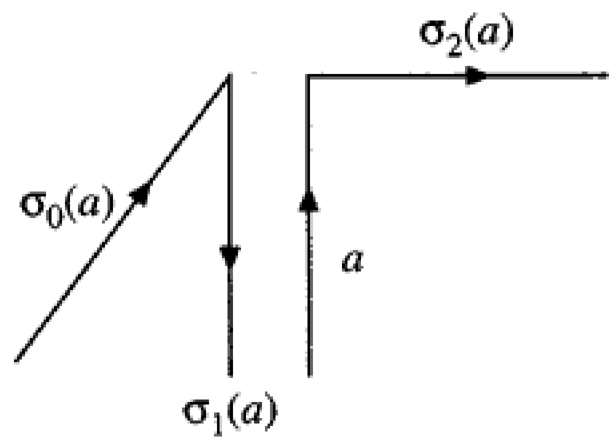

Figure 1: The action of the cartographical group

We define $\sigma_2(\Gamma) = \sigma_1(\Gamma)\sigma_0(\Gamma)^{-1}$, such that $\sigma_0\sigma_1\sigma_2 = 1$. The orbits of $\sigma_2(\Gamma)$ are caled the **faces** of Γ. We denote the set of faces by $F(\Gamma)$ and its cardinal by $f(\Gamma)$. The length of a cycle of $\sigma_0(\Gamma)$ (resp. $\sigma_2(\Gamma)$) is the valence of the corresponding vertex (resp. face).
Every face is an oriented loop, for, if $b = \sigma_2(a)$ then $b(0) = a(1)$.

The **genus** $g(\Gamma)$ of the fat-graph Γ is then defined by the Euler-Poincaré formula:

$$2 - 2g(\Gamma) = s(\Gamma) - a(\Gamma) + f(\Gamma).$$

We can see a fat-graph Γ as a collection of oriented polygons (given by $\sigma_2(\Gamma)$) glued with $\sigma_1(\Gamma)$. This point of view is a good one for fat-graphs with only one face, such as the fundamental polygons of compact orientable surfaces (see figure 2).
Moreover, we can realize the faces of Γ as oriented polygons is an oriented plane, and fill in by punctured disks. Then gluing them with σ_1 gives an orientable compact surface $F(\Gamma)$ minus $f(\Gamma)$ points (one for each face), together with an embedding $i : |\Gamma| \hookrightarrow F(\Gamma)$ such that $i(|\Gamma|)$ is a retract by deformation of $F(\Gamma)$.

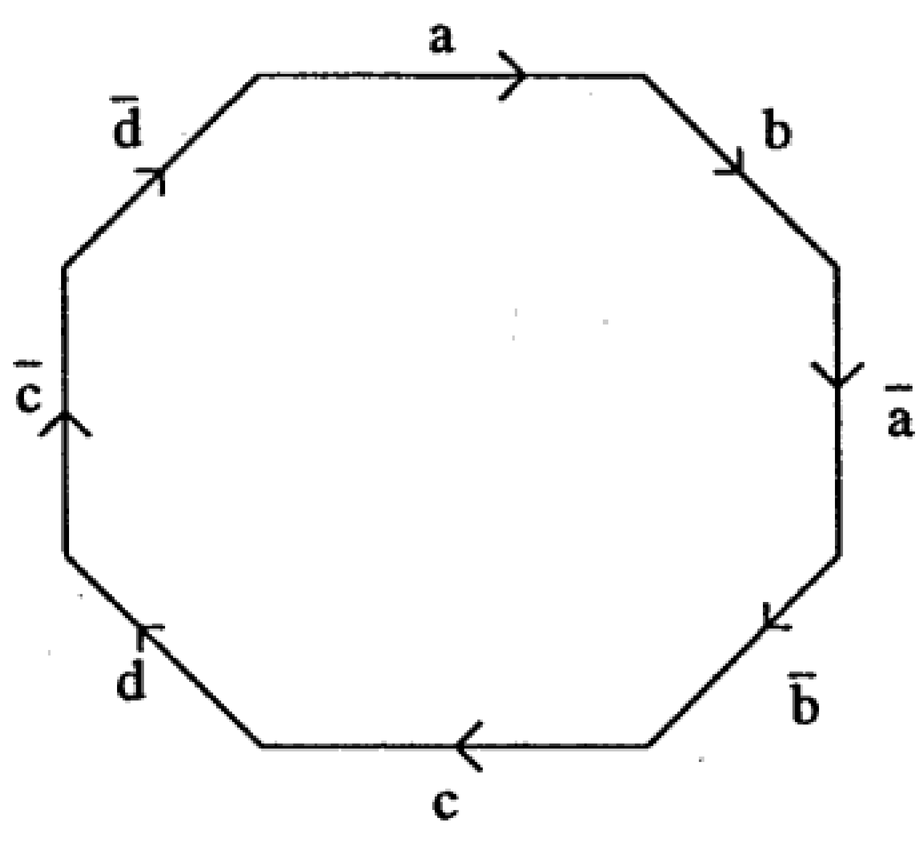

Figure 2: Fundamental polygon of a genus two surface

From the induced isomorphism between first homotopy groups $\pi_1(|\Gamma|)$ and $\pi_1(F(\Gamma))$, we deduce that $g(\Gamma) = g(F(\Gamma))$. For, on one hand, $\pi_1(F(\Gamma))$ is a free group of rank $2g(F(\Gamma)) + f(\Gamma) - 1$, and on the other hand, $\pi_1(|\Gamma|)$ is isomorphic to the fundamental group of Γ which is free of rank $a(\Gamma) - s(\Gamma) + 1 = 2g(\Gamma) + f(\Gamma) - 1$.
In fact the theorem 12 may be used to define $g(F(\Gamma)) := g(\Gamma)$.

We describe now elementary operations on fat-graphs and their invariants.

Definition 6 *A morphism $f : \Gamma \to \Gamma'$ between two fat-graphs is a map $f : A(\Gamma) \to A(\Gamma')$ which satisfies to $f \circ \sigma_i = \sigma'_i \circ f$ for $i \in \{0,1,2\}$. If f is bijective, then f is an isomorphism. The automorphism group is the centralizer of $C(\Gamma)$ into the group of all permutations of $A(\Gamma)$.*

The transitivity implies $f(A(\Gamma)) = A(\Gamma')$. Such a morphism induces a morphism between the underlying graphs. An isomorphism induces an isomorphism between their corresponding fundamental groups.

Definition 7 *Let Γ be a fat-graph, and $e = (a,\overline{a})$ a geometrical edge which is not a loop $(a(0) \neq a(1))$. Then the Whitehead's move on Γ along e consists in removing e and identifying vertices $a(0)$ and $a(1)$, with the induced cyclic ordering. We denote by $W_e(\Gamma)$ the new fat-graph obtained in this way. See the figure 3.*

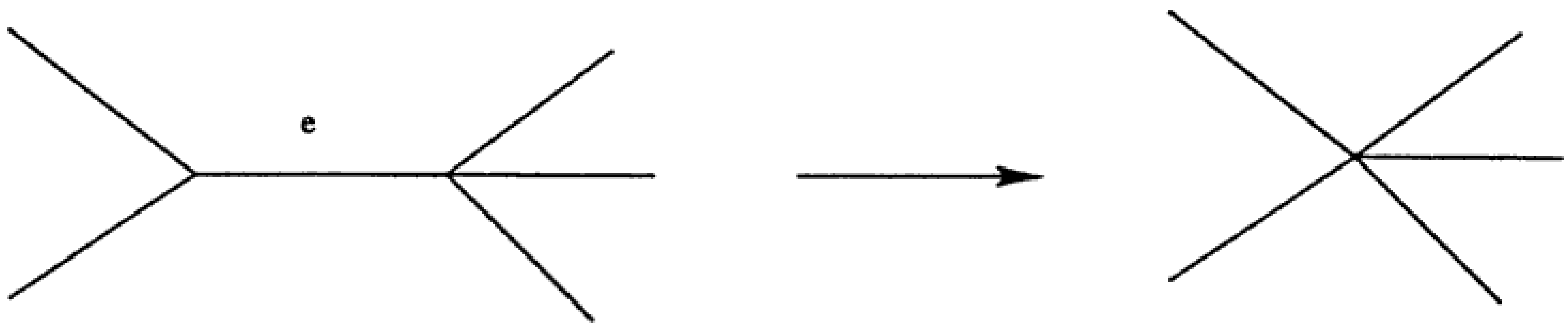

Figure 3: Whitehead's move.

Let us describe the cyclic ordering of edges around the new vertex. Let $(a_1,\ldots,a_p = \overline{a})$ and $(a = b_1,\ldots,b_q)$ be the orbits of $\overline{a}$ and a under $\sigma_0(\Gamma)$. Then the new orbit is $(a_1,\ldots,a_{p-1}, b_2, \ldots, b_q)$.
Since the only mono-valent faces are given by loops, the number of faces keeps constant under Whitehead's moves. Since the number of edges and vertices both decrease by one, we deduce that $g(\Gamma) = g(W_e(\Gamma))$.
Furthermore, the reader can easily check that $W_e(W_f(\Gamma))$ is isomorphic to $W_f(W_e(\Gamma))$.

Another classical operation is the duality, which consists in exchanging the role of vertices and faces (see the figure 5).

Definition 8 *Let Γ be a fat-graph described by its cartographical group $C(\Gamma) = <\sigma_0,\sigma_1,\sigma_2>$. Then the dual fat-graph Γ^* is defined by $\sigma_0^* = \sigma_2$, $\sigma_1^* = \sigma_1$.*

Then we have $\sigma_2^* = \sigma_1\sigma_0\sigma_1$, and $g(\Gamma^*) = g(\Gamma)$, $\mathrm{Aut}(\Gamma^*) = \mathrm{Aut}(\Gamma)$. Note that Γ^{**} is isomorphic (but not equal) to Γ, by conjugation by σ_1.

We now restrict ourselves to fat-graphs Γ with only face and one vertex. By Euler-Poincare, we have $a(\Gamma) = 2g(\Gamma)$. Such a fat-graph may be seen as an oriented polygon with $2a = 4g$ edges, with a pairing which induces only one vertex. We can label the oriented edges $b_1,\ldots,b_{2a}$ and assume that $\sigma_2 = b_1 \cdots b_{2a}$.

We can also look at Γ as a symbol made of $4g$ letters shared in two sets by $\sigma_1 : \{a_1,\ldots,a_{2g}\}$, and $\{\overline{a_1},\ldots,\overline{a}_{2g}\}$. Note that this symbol do not contain something like $\cdots a\overline{a}\cdots$, since this gives one more mono-valent vertex.

As a well-known example, we have the so-called normal symbol: $a_1a_2\overline{a}_1\overline{a}_2\cdots a_{2g-1}$ $a_{2g}\overline{a}_{2g-1}\overline{a}_{2g}$, for which the reader can easily check that σ_0 is a $4g$-cycle.
We also have the symbol $a_1\cdots a_{2g}\overline{a}_1\cdots\overline{a}_{2g}$, for which $\sigma_1 = \sigma_2^{2g}$, hence $\sigma_0 = \sigma_2^{2g-1}$ is a $4g$-cycle.

Let us introduce now the notion of cut and paste, classically used in the classification of compact orientable surfaces (see [2] or [10]).

Definition 9 *Take a symbol of* $2a$ *letters like above which represents a fat-graph with one face:* $a\alpha b\overline{a}\beta$*, where* a *and* b *are oriented edges,* α *and* β *are oriented paths. The operation* **cut** *along* b *and* **paste** *along* a *consists in forming the new symbol* $cb\alpha\overline{c}\beta$ *where* c *is an oriented edge and considering the associated fat-graph.*

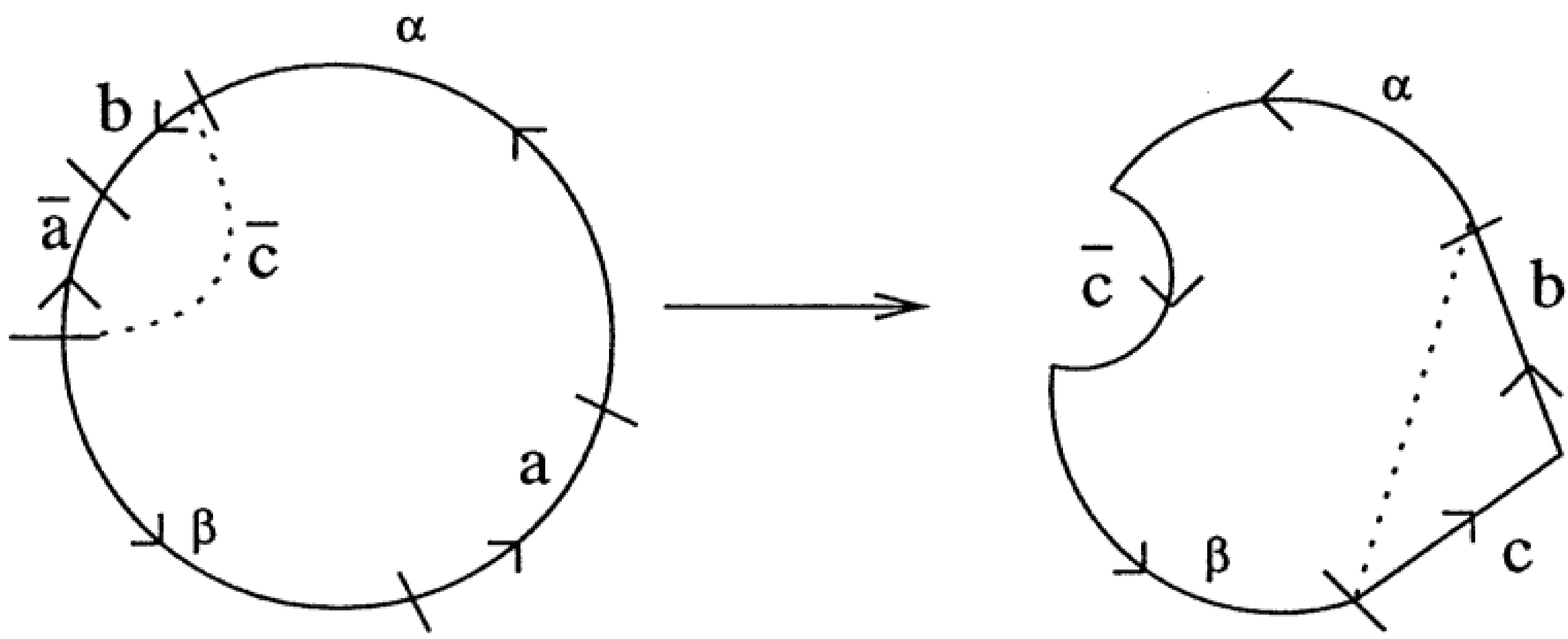

Figure 4: Cut and paste on a one face fat-graph.

The pictorial translation of cut and paste is given on the figure 4.

If we perform many cut and paste, we are able to alter the symbol $a\beta\gamma\overline{a}\delta\sigma$ into the symbol $c\gamma\beta\overline{c}\sigma\delta$.

The following lemma is crucial in the fourth paragraph.

Lemma 10 *Let* $a\alpha b\overline{a}\beta$ *be a symbol representing a fat-graph* Γ *with one face and one vertex. Then the fat-graph* Δ *associated to the symbol* $cb\alpha\overline{c}\beta$ *obtained by cut analog* b *and paste along* a *possess again one face and one vertex.*

Proof. Denote by $(\sigma_0,\sigma_1,\sigma_2)$ the permutations describing Γ, and $(\delta_0,\delta_1,\delta_2)$ those which describe Δ. We have to show that δ_0 is a $4g$-cycle (if g is the genus of Γ).
We first cut the words α and β into letters (oriented edges): $\alpha = a_1\cdots a_p$, $\beta = b_1\cdots b_q$. Then we have to distinguish four cases according to the position of $\overline{b}$ into α or β.

<u>First case:</u> $\overline{b} = a_l$
Then $l \neq 1$ and $l \neq p$. Indeed, if $l = p$ this refutes the fact that σ_0 is a $4g$-cycle. Again if $l = 1$ then we have $\sigma_0(a) = b$ and $\sigma_0(b) = a$ which refutes our hypothesis.
Set $\Lambda = (a_1,\ldots,a_{l-1},a_{l+1},\ldots,a_p,b_1,\ldots,b_q)$.
Look at the respective actions of permutations σ_0 and δ_0 on the whole set of oriented edges. Note that if $x \in \Lambda$ then $\sigma_0(x) \in \Lambda$ except if $x \in (\overline{a}_1,\overline{a}_{l+1},\overline{b}_1)$, since we have $\sigma_0(\overline{a}_1) = a$, $\sigma_0(\overline{a}_{l+1}) = \overline{b}$ and $\sigma_0(\overline{b}_1) = \overline{a}$. Furthermore if x and $y = \sigma_0(x)$ belong to Λ, then $y = \delta_0(x)$.
Iterate the permutations σ_0 and δ_0 starting from a for σ_0 and from b for δ_0. We have $\sigma_0(a) = b$, $\sigma_0(b) = a_{l-1} \in \Lambda$ and $\delta_0(b) = a_{l-1}$. So, δ_0 is in late.
The permutation σ_0 can not go away from Λ by the letter $\overline{a}_1$ since $\sigma_0(\overline{a}_1) = a$ and $\overline{b}$ is not yet reached. Here, σ_0 goes away from the set Λ with $\overline{b}_1$ or $\overline{a}_{l+1}$. Assume that this is with $\overline{b}_1$ (the conclusion is the same in the other case). We have $\sigma_0(\overline{b}_1) = \overline{a}$, $\sigma_0(\overline{a}) = b_q$ and $\delta_0(\overline{b}_1) = \overline{c}$,

$\delta_0(\overline{c}) = b_q$. We are again in Λ, and we can leave it only with $\overline{a}_{l+1}$ (the obstruction for $\overline{a_1}$ being always valuable). Then we have $\sigma_0(\overline{a}_{l+1}) = \overline{b}$, $\sigma_0(\overline{b}) = a_p$ on one hand, and $\delta_0(\overline{a}_{l+1}) = \overline{b}$, $\delta_0(\overline{b}) = \overline{c}$, $\delta_0(\overline{c}) = a_p$ on the other hand. Now δ_0 has caught again σ_0.
Back in Λ, the only exist is by $\overline{a}_1$ with $\sigma_0(\overline{a_1}) = a$ and $\delta_0(\overline{a_1}) = b$.
We have covered the $4g$ oriented edges for both σ_0 and δ_0 in $4g$ "times", and this proves that δ_0 is a $4g$-cycle.

Second case: $\overline{b} = b_j (1 < j < q)$
Set $\Lambda = (a_1,\ldots,a_p,b_1,\ldots,b_{j-1},b_{j+1},\ldots,b_q)$. The permutation σ_0 coincide with δ_0 on Λ and σ_0 leaves it only in $\overline{a}_1$, $\overline{b}_1$ or $\overline{b}_{j+1}$. We play again to the preceding game, starting from a (resp. b) for σ_0 (resp. δ_0).
We have $\sigma_0(a) = b$, $\sigma_0(b) = b_{j-1} \in \Lambda$ and $\delta_0(b) = b_{j-1}$. For the same reasons than in the first case, δ_0 can leave Λ only by $\overline{b}_1$ or $\overline{b}_{j+1}$. Take for example (the other case is similar) $\overline{b}_{j+1}$. Then $\sigma_0(\overline{b}_{j+1}) = \overline{b}$, $\sigma_0(\overline{b}) = a_p \in \Lambda$ and $\delta_0(\overline{b}_{j+1}) = \overline{b}$, $\delta_0(\overline{b}) = c$, $\delta_0(c) = a_p$. Back in Λ we have the sole possibility to go away by $\overline{b}_1 : \sigma_0\ (\overline{b}_1) = \overline{a}$, $\sigma_0(\overline{a}) = b_q \in \Lambda$ on one hand, and $\delta_0(\overline{b}_1) = \overline{c}$, $\delta_0(\overline{c}) = b_q$ on the other hand. We leave Λ for the last time by $\overline{a}_1$, then $\sigma_0(\overline{a}_1) = a$ and $\delta_0(\overline{a}_1) = b$.
We can conclude that δ_0 is a $4g$-cycle.

Third case: $\overline{b} = b_1$
Playing again with $\Lambda = (a_1,\ldots,a_p,b_2,\ldots,b_q)$, σ_0 have to reach $\overline{a}_1$ or $\overline{b}_2$ to be out of Λ at the next time.
We have $\sigma_0(a) = b$, $\sigma_0(b) = \overline{a}$, $\sigma_0(\overline{a}) = b_q \in \Lambda$ and $\delta_0(b) = \overline{c}$, $\delta_0(\overline{c}) = b_q$. Like in the preceding cases, we can not leave by $\overline{a}_1$ since $\sigma_0(\overline{a}_1) = a$ and $\overline{b} = b_1$ is not yet reached by σ_0 which is a $4g$-cycle. Thus we have $\sigma_0(\overline{b}_2) = \overline{b}$, $\sigma_0(\overline{b}) = a_p$ and $\delta_0(\overline{b}_2) = \overline{b}$, $\delta_0(\overline{b}) = c$, $\delta_0(c) = a_p$. Back in Λ we can leave it only by $\overline{a}_1$ and we end the game by $\sigma_0(\overline{a}_1) = a$ and $\delta_0(\overline{a}_1) = b$.
We can conclude that δ_0 is a $4g$-cycle.

Fourth case: $\overline{b} = b_q$
Here $\Lambda = (a_1,\ldots,a_p,b_1,\ldots,b_{q-1})$. We have $\sigma_0(a) = b$, $\sigma_0(b) = b_{q-1} \in \Lambda$ and $\delta_0(b) = b_{q-1}$. Always for the same reasons σ_0 can leave Λ only by $\overline{b}_1$. This leads to $\sigma_0(\overline{b}_1) = \overline{a}, \sigma_0(\overline{a}) = \overline{b}, \sigma_0(\overline{b}) = a_p$ and $\delta_0(\overline{b}_1) = \overline{c}$, $\delta_0(\overline{c}) = \overline{b}$, $\delta_0(\overline{b}) = c$, $\delta_0(c) = a_p$. The final leaving of Λ is by $\overline{a}_1$, so we end this game by $\sigma_0(\overline{a}_1) = a$ and $\delta_0(\overline{a}_1) = b$.
We can conclude that δ_0 is a $4g$-cycle. □

4 Fundamental group of a fat-graph

We have to describe an algorithm to find a presentation for the fundamental group of a fat-graph Γ with $2g(\Gamma) + f(\Gamma)$ generators, satisfying only one relation (the so-called surface relation).

The first $2g$ loops give the homological loops of the associated surface $F(\Gamma)$; the other ones give the loops around the punctures and they come from the faces.

Indeed, faces furnish privileged conjugacy classes of $\pi_1(\Gamma, p)$. Let $(a_1,\ldots,a_l)$ be a face of Γ viewed as an orbit of $\sigma_2(\Gamma)$. Join $a_1(0)$ to the base point p by an oriented path α, and consider the homotopy class of $\gamma = \alpha a_1 \ldots a_l \overline{\alpha}$. Another choice for α leads to a conjugate of γ. If the face starts from a_i with $i > 1$, then we get the homotopy class of $\beta a_i \ldots a_{i-1} \overline{\beta}$ for some β, which is homotopic to $\beta a_i \ldots a_l a_1 \ldots a_{i-1} a_i \ldots a_l \overline{a}_l \ldots \overline{a}_i \overline{\beta}$, a conjugate of γ.

Thus, to each face is associated a well-defined conjugacy class in $\pi_1(\Gamma)$.

Definition 11 *Every element of the conjugacy class of $\pi_1(\Gamma)$ associated to a face is called a* **loop-face**.

Using only Whitehead's move, cut and paste, and duality, we want to prove the following theorem, a refinement of the classical theorem which classify by their genus the orientable compact surfaces (see [2] or [10]).

Theorem 12 *Let Γ be a fat-graph of genus g, with n faces. There exists a combinatorial algorithm to find loop-faces $C_1,\dots,C_n$ and $2g$ other oriented loops $A_1,B_1,\dots,A_g,B_g$ which generate $\pi_1(\Gamma)$, and are submitted to the sole relation:*

$$[A_1,B_1]\cdots[A_g,B_g]C_1\cdots C_n = 1.$$

We call such a presentation of $\pi_1(\Gamma)$ a topological presentation. To show the theorem, the key-point is the interpretation of operations on Γ at the level of its fundamental group. Firstly, we have the following result (classical for graphs):

Lemma 13 *Let Γ be a fat-graph, and $W_e(\Gamma)$ the fat-graph obtained by Whitehead's move along the geometric edge $e=(a,\overline{a})$. Set $v=a(0)$, $w=a(1)$ and let x be the new vertex of $W_e(\Gamma)$. Then there exists a canonical isomorphism between $\pi_1(\Gamma,w)$ and $\pi_1\ (W_e(\Gamma),x)$.*

Proof. Recall that the fundamental groupoïd of a graph consists in homotopy classes of oriented paths with the law (not always defined) of successive paths. We define a morphism ι from the fundamental groupoïd of $W_e(\Gamma)$ to that of Γ.
If $b\in A(\Gamma)\backslash(a,\overline{a})$, set b^* the corresponding oriented edge of $W_e(\Gamma)$ such that $\overline{b^*}=\overline{b}^*$. If $b(1)=v$ then define $\iota(b^*)=b\overline{a}$, if $b(0)=v$ then $\iota(b^*)=ab$, otherwise $\iota(b^*)=b$.
Since $\iota(b^*\overline{b^*})=1\forall b^*\in A(W_e(\Gamma))$, we have a well-defined map

$$\begin{array}{ccc}\pi_1(W_e(\Gamma),x) & \longrightarrow & \pi_1(\Gamma,w)\\ \gamma^*=a_1^*\cdots a_n^* & \longmapsto & \gamma=\iota(a_1^*\cdots a_n^*)\end{array}$$

This is a surjective morphism between two free groups of same rank $(a(\Gamma)-s(\Gamma)+1)$, hence this is an isomorphism. □

Lemma 14 *Let Γ be a fat-graph with $s(\Gamma)\geq 2$. Choose an edge $e=(a,\overline{a})$ with $a(0)\neq a(1)$, for example in a maximal tree of Γ. If the theorem is true for $W_e(\Gamma)$, then it is also true for Γ.*

Proof. Assume that a and $\overline{a}$ belong to the same face, and let $ae_1\cdots e_r\overline{a}f_1\cdots f_s$ be an associated loop-face. Since $a(1)=w=e_1(0)$, we have $\iota(e_1^*)=ae_1$. And $\iota(e_r^*)=e_r\overline{a}$ because $e_r(1)=w$. If for $1<j<r$, $e_j(1)=w$, then $e_{j+1}(0)=w$ so that $\iota(e_j^*e_{j+1}^*)=e_je_{j+1}$. Idem for the f_j. Thus the image by ι of the loop-face $C_k^*=e_1^*\cdots e_r^*f_1^*\cdots f_s^*$ of $W_e(\Gamma)$ is the loop-face $C_k=ae_1\cdots e_r\overline{a}f_1\cdots f_s$ of Γ.
This means that somewhere before or after C_k^* there is $\overline{e_1}^*$ to cancel e_1^*. But $\overline{e_1}(1)=e_1(0)$, hence $\iota(\overline{e_1}^*)=\overline{e_1}\overline{a}$ cancelling ae_1 of C_k. Similarly $\overline{e_r}(0)=w$ implies $\iota(\overline{e_r}^*)=a\overline{e_r}$ which cancels $e_r\overline{a}$.

We can conclude that $[A_1,B_1]\cdots[A_g,B_g]C_1\cdots C_k\cdots C_n = 1$.
If a and $\overline{a}$ belong to two different faces, then the same kind of arguments shows that the good presentation for $W_e(\Gamma)$ gives the good one for Γ. □

This lemma reduces the problem to the case of fat-graphs with one vertex. We also want to reduce it to the case of fat-graphs with one face.

Lemma 15 *Let Γ be a fat-graph with $f(\Gamma) \geq 2$. There exists a geometric edge $(a,\overline{a})$ separating two distinct faces.*

Proof. If not, then for every $(a,\overline{a})$, a and $\overline{a}$ belong to the same face F. This implies that F, which is already an orbit of $\sigma_2(\Gamma)$ is stable under $\sigma_1(\Gamma)$. Hence F is stable under the action of the whole cartographical group, which is not compatible with $f(\Gamma) \leq 2$ and the connectivity of Γ. □

Then consider some Γ with at least two faces, and take among them F_1 and F_2 separated by $e = (a,\overline{a})$. We define $\Gamma\# = \{W_{e^*}(\Gamma^*)\}^*$ (see definitions 7 and 8). This is nothing but removing e for gluing F_1 and F_2 (see figure 5). Precisely, if $F_1 = (a_1,\dots,a_r = a)$ and $F_2 = (\overline{a} = b_1,\dots,b_s)$, then the new face is $F_1 \# F_2 = (a_1,\dots,a_{r-1},b_2,\dots,b_s)$. Other faces remain unchanged.
We have $s(\Gamma\#) = s(\Gamma)$, $a(\Gamma\#) = a(\Gamma) - 1$, and $f(\Gamma\#) = f(\Gamma) - 1$. Thus $g(\Gamma\#) = g(\Gamma)$.
The graph is in a natural way a subgraph of Γ. So, we have an injective morphism $\iota_\# : \pi_1(\Gamma\#) \hookrightarrow \pi_1(\Gamma)$.

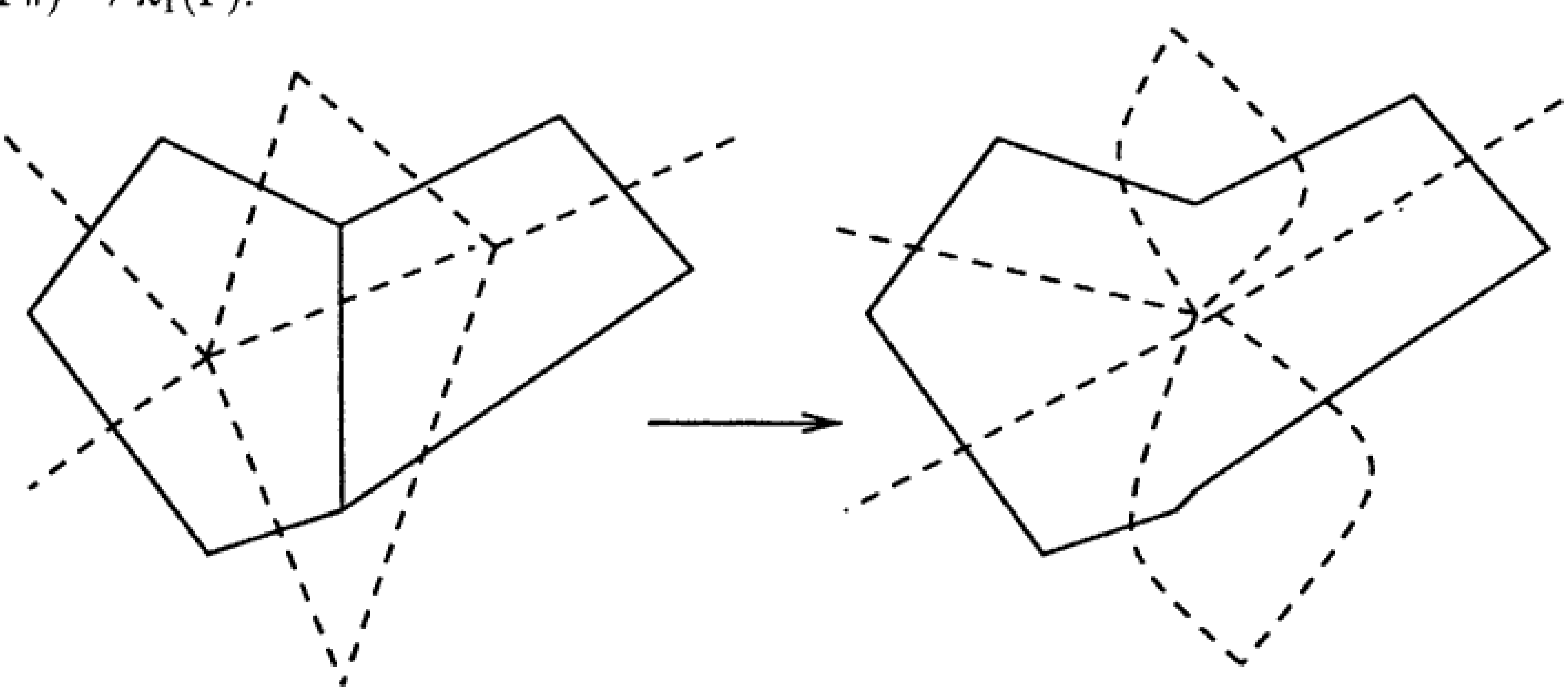

Figure 5: Whitehead's move on the dual fat-graph

Lemma 16 *Let Γ be a fat-graph with $f(\Gamma) \geq 2$, and e separating two faces F_1 and F_2. Set $\Gamma\# = \{W_{e^*}(\Gamma^*)\}^*$. Then a topological presentation for $\Gamma\#$ gives a topological presentation for Γ.*

Proof. The image by $\iota_\#$ of a loop-face associated to a face $F_j (j \geq 3)$ is a loop-face Γ. If $\gamma_{1,2}$ is the loop-face associated to the face $F_1\#F_2$, then $\iota_\#(\gamma_{1,2}) = \gamma_1\gamma_2$, and $\Pi[A_i,B_i]\gamma_{1,2}\gamma_3\cdots\gamma_n = 1$ implies $\Pi[A_i,B_i]\,\gamma_1\gamma_2\gamma_3\cdots\gamma_n = 1$. □

Proof of the theorem:

Using lemma 16, we can suppose that $f(\Gamma) = 1$. If $g(\Gamma) = 0$, then we are done: Γ is a tree with trivial fundamental group.

Now, $g(\Gamma) \geq 1$. Using lemma 14, we can furthermore retract all the edges of a maximal tree, and assume that $s(\Gamma) = 1$.

We represent Γ by a $4g$ cycle, as in the §2.

We can write this cycle $a\alpha b\beta\overline{a}\gamma\overline{b}\delta$, with $\alpha, \beta, \gamma, \delta$ some words associated to oriented paths (we say that b separates a from $\overline{a}$). Let us prove this.

Let $(\sigma_i)_{i=0,1,2}$ be the permutations describing vertices, edges and faces of Γ. Write the symbol $a_1 a_2 \cdots a_i \cdots$ with $a = a_1$ and $\overline{a} = a_i$ for $i > 1$. Then, if for all j between 1 and $i, \overline{a_j} = a_k$ for another index k between 1 and i, the set $(a_1, \ldots, a_{i-1})$ is an orbit for σ_0, which contradicts $s(\Gamma) = 1$ under hypothesis of connectivity. Indeed, if $a_k \neq \overline{a_2}$ for $1 < k < i$, then $\sigma_0(a_k) = a_l$ for $1 < l < i$. Since $\sigma_0(\overline{a_2}) = a_1$, $\sigma_0(a_1) = a_{i-1}$, our set is an orbit of σ_0, and we have proved that we can separate a from $\overline{a}$.

We are in position to conclude. The key-point is that a cut an paste has an interpretation as an operation on words in $\pi_1(\Gamma)$. For example, $a\alpha b\overline{a}\beta \to cb\alpha\overline{c}\beta$ consists in setting $\overline{c} = b\overline{a}$ or, up to homotopy, $a = cb$ (see figure 4).

Start with a symbol $b\beta\overline{a}\gamma\overline{b}\delta a\alpha$. Then perform a cut and paste along $(b, \overline{b})$; this leads to $c\overline{a}\gamma\beta\overline{c}\alpha\delta a$, or equivalently $ac\overline{a}\phi\overline{c}\psi$ if $\phi = \gamma\beta$ and $\psi = \alpha\delta$. By lemma 10, the new fat-graph has again one vertex, and this allows us to continue. In $\pi_1(\Gamma)$, the loop-face $a\alpha b\beta\overline{a}\gamma\overline{b}\delta$ becomes $ac\overline{a}\phi\overline{c}\psi$. Note that $c \in \pi_1(\Gamma)$.

Perform another cut and paste along $(a, \overline{a})$. Then we pass from $ac\overline{a}\phi\overline{c}\psi$ to $dc\overline{d}\overline{c}\psi\phi$ with $\overline{d} = \overline{a}\phi$.

The end of the proof, which mimes the procedure used in the classification of compact orientable surfaces, is an induction on the number of commutators already present in the writing of the loop-face. We have done the first step.

Assume that the loop-face of our fat-graph writes down $[a_1, b_1] \cdots [a_p, b_p]\ c_1 \cdots c_m$, with $a_i, b_i \in \pi_1(\Gamma)$ for $i = 1, \ldots, p$ and $p < 2g$.

We have seen that there exists an oriented edge $c_i (i > 1)$ which separates c_1 from $\overline{c}_1$. Set $\zeta = \prod_{j=1}^{p} [a_j, b_j]$, $d = c_i$ et $c = c_1$. Our symbol, or element of the fundamental group becomes $\zeta c\alpha d\beta\overline{c}\gamma\overline{d}\delta$, and $d\beta\overline{c}\gamma\overline{d}\delta\zeta c\alpha$ up to homotopy.

We cut and paste along $(d, \overline{d})$ which leads to $e\overline{c}\gamma\beta\overline{e}\alpha\delta\zeta c$, or $\overline{e}\alpha\delta\zeta ce\overline{c}\gamma\beta$ with $e = \alpha d\beta$.

Then another cut and paste along $(\overline{e}, e)$ gives $\overline{c}\overline{f}c\alpha\delta\zeta f\gamma\beta$ if we set $\overline{f} = \gamma\beta\overline{e}\alpha\delta\zeta$.

A last cut and paste along $(c, \overline{c})$ leads to $\overline{f}gf\gamma\beta\alpha\delta\zeta\overline{g}$ where $g = c\alpha\delta\zeta$, or equivalently $\zeta[a_{p+1}, b_{p+1}]\ \gamma\beta\alpha\delta$ where $a_{p+1} = \overline{g}$, $b_{p+1} = \overline{f} \in \pi_1(\Gamma)$.

The theorem 12 is thus proved. □

Example: Consider the symbol $abcd\overline{a}\overline{b}\overline{c}\overline{d}$ which represents a fat-graph of genus two with one face and one vertex. This symbol also represents a loop-face as an element of the fundamental group $\pi_1(\Gamma)$.

Write it $bcd\overline{a}\overline{b}\overline{c}\overline{d}a$ and perform a cut and paste along $(b, \overline{b})$, setting $e = bcd$. Hence this loop-face writes down $e\overline{a}cd\overline{e}\overline{c}\overline{d}a$ or $ae\overline{a}cd\overline{e}\overline{c}\overline{d}$. Then set $f = \overline{a}cd$ and cut and paste along $(a, \overline{a})$ gives $fe\overline{f}\overline{e}c\overline{d}cd$.

We have found the topological presentation of this fat-graph, his homological loops are $A_1 = \overline{a}cd$, $B_1 = bcd$, $A_2 = \overline{c}$, and $B_2 = \overline{d}$.

Remark: Using the fundamental groups $\pi_1(\Gamma)$ of fat-graphs Γ, it is not difficult to obtain a combinatorial version of Riemann's existence theorem, i.e. a correspondence between coverings of Γ and homomorphisms from $\pi_1(\Gamma)$ to symmetric groups. The images of loop-faces give the monodromy (see [4]).
One can also describe a fat-graph as the quotient of its universal covering (an infinite fat-tree) by the action of its fundamental group.

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