Note di Matematica Vol. 19 - n. 2, 219-226 (1999)

GEODESIC GENERATORS OF $\pi_n Spin(n+1)$

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Abstract. *We show for all $n \geq 2$ the Cartan inclusion of S^n in $Spin(n+1)$, as a totally geodesic submanifold of constant sectional curvature, generates a cyclic direct summand of $\pi_n Spin(n+1)$.*

1 Introduction

There is a precise analogy between representing characteristic maps of principal bundles over spheres by embeddings of equators as totally geodesic submanifolds of the group and the consequent representation of the associated classifying maps into suitable grassmannians, also by a totally geodesic sphere. The second process is directly related to non negatively curved Riemannian structures on bundles over spheres ([5], [11]) with pontentialy strong consequences. It seems natural that the same consequence must also ensue from the characteristic maps process.

In [14], Qi-Ming Wang showed that all elements of the stable homotopy group of the infinite orthogonal group (and consequently its classifying space) can be represented by totally geodesic spheres of constant curvature in the infinite orthogonal group O and its classifying space BO, whose image, of course, lies in some finite $O(n)$ or Grassmannian. The analogous statement for the non stable groups is far from being understood and many cases involved are of independ interest, like for example, exotic differential structures on spheres of various dimensions. The basic difference is that, in the non stable case there is lack of enough space.

In this note we are dealing with the opposite question: We know there is a metric with non negative sectional curvature on the principal bundles in question and we show that the corresponding characteristic maps are totally geodesic. Our result deals with $\pi_n Spin(n+1)$ for all n and covers the generators of at least "one direct summand of the group" in a sense made precise in Theorem 1. The other one is taken care of by Wang's theorem, although only in the stable range. In the last section we prove that our map is the Cartan inclusion of S^n in $Spin(n+1)$. As an application we exhibit in detail a generator of $\pi_4 Sp(2)$.

2 Clifford algebras and Spin Groups

We briefly recall from [9] the basic steps of the construction of Clifford algebras to set the notation.

Associated to the Euclidean space $\mathbb{R}^n$, with the canonical scalar product, we can define an associative algebra $C(\mathbb{R}^n)$, its Clifford algebra, taking the quotient $(\sum \otimes^r \mathbb{R}^n)/A(\mathbb{R}^n)$, where $A(\mathbb{R}^n)$ is the ideal in the tensor algebra of $\mathbb{R}^n$ generated by elements of the form $v \otimes v + ||v||^2 \cdot 1$, for v in $\mathbb{R}^n$. If $\{e_1, e_2, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$, $\{1, e_1, \ldots, e_n, e_1e_2, \ldots, e_{n-1}$

$e_n, \ldots, e_1 e_2 \ldots e_n\}$ is a standard basis for the associated Clifford algebra. Note that $e_i^2 = -1$ and $e_i e_j = -e_j e_i$ for $i \neq j$, from which we obtain a multiplication table for $C(\mathbb{R}^n)$. Consider the group $Pin(n) = \{v_1 \ldots v_k, ||v_i|| = 1, v_i \in \mathbb{R}^n\} \subset C(\mathbb{R}^n)$, and its subgroup $Spin(n)$ $= \{v_1 \ldots v_{2s} \; ||v_i|| = 1, \; v_i \in \mathbb{R}^n\}$. $Spin(n)$ is the 1-connected covering groups of $SO(n)$. For $x \in \mathbb{R}^n$ and $u \in S^{n-1}$ the assignement $x \longmapsto uxu = R_u(x)$ is the reflection in $\mathbb{R}^n$ in the hyperplane perpendicular to u. These maps define the double covering map

$$Spin(n) \to SO(n), \quad v_1 \ldots v_{2s} \longmapsto R_{v_1} \cdots R_{v_{2s}},$$

with $R_{v_1} \cdots R_{v_{2s}}(x) = v_1(\ldots(v_{2s} x v_{2s})\ldots)v_1$.

3 A generator of $\pi_n Spin(n+1)$

The Bott periodicity theorem for the orthogonal group [2], [10] states that the stable homotopy groups $\pi_i(O)$ are periodic with period 8 and are isomorphic to the following groups:

$\mathbb{Z}_2$ for $i \equiv 0$ or 1 (mod8),
0 for $i \equiv 2,4,5,$ or 6 (mod8),
$\mathbb{Z}$ for $i \equiv 3$ or 7 (mod8).

The stability is achieved for $\pi_n Spin(n+2)$. To exhibit generators of $\pi_n SO(n+1)$ it seems more adequate to work with $Spin(n+1)$. The group $\pi_n Spin(n+1)$ is isomorphic to

$\mathbb{Z} \oplus \mathbb{Z}$ for $n \equiv 7$ or 3 (mod8)
$\mathbb{Z} \oplus \mathbb{Z}_2$ for $n \equiv 1$ (mod8)
$\mathbb{Z}$ for $n \equiv 5$ (mod8)
$\mathbb{Z}_2 \oplus \mathbb{Z}_2$ for $n \equiv 0$ (mod8)
$\mathbb{Z}_2$ for $n \equiv 2,4$ or 6 (mod8)
$n \neq 1,2,6$ and $\pi_2 S^3 = 0$, $\pi_6 Spin(7) = 0$ [11, pag. 217].

Let $\Theta : S^n \to Spin(n+1)$ be defined by $\Theta(v) := v e_{n+1}$.

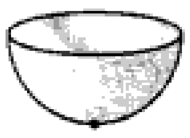
Theorem 1 *The homotopy class $[\Theta]$ generates a non zero cyclic subgroup of $\pi_n Spin(n+1)$, that is isomorphic to $\mathbb{Z}$ when n is odd and to $\mathbb{Z}_2$ when n is even.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccc}
\mathbb{Z}_2 \cdots & Spin(k) & \to & SO(k) \\
 & \vdots & p & \vdots \\
\mathbb{Z}_2 \cdots & Spin(k+1) & \to & SO(k+1) \\
 & \downarrow & & \downarrow \\
 & S^k & = & S^k
\end{array}$$

where $SO(k)$ is included in $SO(k+1)$ as $\begin{pmatrix} SO(k) & 0 \\ 0 & 1 \end{pmatrix}$, $p(v_1, \ldots, v_{2s}) = A$ with $A = R_{v_1} \cdots R_{v_2}$ and the projections on S^k are $A(e_{k+1}) = v_1 \ldots (v_{2s} e_{k+1} v_{2s}) \ldots v_1$, composition of successive reflections.

To establish the boundary operator in the exact sequence of the Spin bundle we will employ the standard diagram of the characteristic map as follows:



$$\begin{array}{ccccc} & & & & Spin(n+1) \\ & S^n = \partial C^{n+1} & & & \\ cell\ C^{n+1} & & \subset S^{n+1} & \overset{\Psi}{\to} & Spin(n+2) \\ & -e_{n+2} & \gamma \searrow & & \downarrow \\ & & & & S^{n+1} \end{array}$$

Let ψ above be

$\psi(v) := ve_{n+2}$ with $\gamma(v) := \pi \circ \psi(v) = (ve_{n+2})e_{n+2}(e_{n+2}v) = -ve_{n+2}v$.

Note that $\gamma(-e_{n+2}) = e_{n+2}$ and $\gamma(v) = -e_{n+2}$ for all v in S^n, since v anticommutes with e_{n+2}. Consequently if γ is restricted to the cell C^{n+1} then $\gamma : (S^{n+1}, S^n) \to (S^{n+1}, -e_{n+1})$ represents the fundamental homology class, generator of $H_{n+1}\ (S^{n+1})$. One could prove that this restriction is a relative homeomorphism by the following argument: let $v =$ $= -\cos(t)e_{n+1} + sen(t)w$, $w \in S^n \subseteq span\{e_1, \ldots, e_{n+1}\}$, $0 < t < \pi/2$, $v \in C^{n+1} - \{S^n \cup -e_{n+2}\}$. One has $\gamma(v) = \cos(2t)e_{n+2} - sen(2t)w$ which shows that γ restricted to this domain is a diffeomorphism between $C^{n+1} - S^n \cup \{-e_{n+2}\}$ and $S^{n+1} - \{e_{n+2}, -e_{n+2}\}$, so γ is a relative homeomorphism as claimed and $[\gamma]$ is a generator of $\pi_{n+1}\ (Spin(n+2), Spin(n+1)) \cong \pi_{n+1}$ $(C^{n+1}, S^n) \cong \mathbb{Z}$, that is, we can suppose $[\gamma] = 1$ in $\mathbb{Z}$.

The relevant part of the exact homotopy sequence of the above bundle is described below taking into account the fact that $\pi_n Spin(n+2)$ is already stable.

	$\pi_{n+1}(S^{n+1})$	$\to$	$\pi_n Spin(n+1)$	$\to$	$\pi_n Spin(n+2)$	$\to$ 0
$n(\text{mod}8)$						
0	$\mathbb{Z}$	$\to$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\to$	$\mathbb{Z}_2$	
1	$\mathbb{Z}$	$\to$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\to$	$\mathbb{Z}_2$	
2	$\mathbb{Z}$	$\to$	$\mathbb{Z}_2$	$\to$	0	
3	$\mathbb{Z}$	$\to$	$\mathbb{Z} \oplus \mathbb{Z}$	$\to$	$\mathbb{Z}$	
4	$\mathbb{Z}$	$\to$	$\mathbb{Z}_2$	$\to$	0	
5	$\mathbb{Z}$	$\to$	$\mathbb{Z}$	$\to$	0	
6	$\mathbb{Z}$	$\to$	$\mathbb{Z}_2$	$\to$	0	
7	$\mathbb{Z}$	$\to$	$\mathbb{Z} \oplus \mathbb{Z}$	$\to$	$\mathbb{Z}$	

It follows immedately that $\partial[\gamma]$ generates an infinite cyclic direct summand for all odd n and a $\mathbb{Z}_2$ direct summand for all even n.

As $\partial[\gamma]$ is equal to the class of ψ restricted to $S^n = \partial C^{n+1}$, it is enough to show that this last one is equal $[\Theta]$. Observe that $\Theta(S^n) \subset \pi^{-1}(e_{n+2})$ while $\psi(S^n) \subset \pi^{-1}\ (-e_{n+2})$. Right translation by $-e_{n+2}e_{n+1}$ is a fiber preserving diffeomorphism and

$$\psi(v)(-e_{n+2}e_{n+1}) = (ve_{n+2})(-e_{n+2}e_{n+1}) = ve_{n+1} = \Theta(v), \text{ for } v \in S^n.$$

$\square$

Remark 2 *As observed in the introduction, a generator of the remaining summand is taken care of by Wang's theorem [14], though only in the stable range. This means that there is a totally geodesic eulidean n-sphere in $\pi_n Spin(n+k)$, $k \geq 2$, that generates $\pi_n Spin$. This k is usually large. However, in the first steps of the Bott periodicity theorem corresponding to $\pi_3 Spin$ and $\pi_7 Spin$, both infinite cyclic, this generator is given by totally geodesic sections of the bundles of Spin frames over S^3 and S^7 obtained by quaternionic (in the case of S^3) and Cayley (in the case of S^7) multiplication by the element of the base [4].*

Theorem 3 *The image $\Theta(S^n)$ is a totally geodesic submanifold of $Spin(n+1)$.*

Proof. Let $\alpha(t) := \cos(t)e_{n+1} + \sin(t)e_k, t \in [\pi/2, 3\pi/2]$ in S^n, with $\alpha(\pi) = -e_{n+1}$ and $\alpha'(\pi) = -e_k$. Compose with Θ to get $d\Theta(-e_k) = -e_k e_{n+1}$ for $k = 1, \ldots, n$ and therefore $d\Theta(v) = ve_{n+1}$ for all $v = a_1 v_1 + \ldots + a_n v_n$, at the point $\alpha(\pi) = -e_{n+1}$. As $\Theta(-e_{n+1}) = 1$ the tangent space of $\Theta(S^n)$ at 1 is $T_1\Theta(S^n) = \{ve_{n+1}, v = a_1 v_1 + \ldots + a_n v_n\}$. The one parameter subgroup corresponding to ve_{n+1} for v unitary in $\mathbb{R}^n$ is $\exp(tve_{n+1}) = \cos(t)1 + \sin(t)ve_{n+1} = [\cos(t)(-e_{n+1}) + \sin(t)v]e_{n+1} = \Theta \circ \alpha(t)$, since $(ve_{n+1})^2 = -1$, for $\alpha(t) = \cos(t)\ (-e_{n+1}) + \sin(t)v$. I.e., $\exp(tve_{n+1})$ is in $\Theta(S^n)$. On the other hand, every element of S^n is in such a curve α. Consequently, $\exp(T_1\Theta(S^n)) = \Theta(S^n)$. To show that $\Theta(S^n)$ is a totally geodesic submanifold we have to prove that $T_1\Theta(S^n)$ is a Lie triple system [8, pg. 224]. In effect,

$[[e_i e_{n+1}, e_k e_{n+1}], e_s e_{n+1}] = [e_i e_{n+1} e_k e_{n+1} - e_k e_{n+1} - e_k e_{n+1} e_i e_{n+1}, e_s e_{n+1}] =$
$[2e_i e_k, e_s e_{n+1}] = 2e_i e_k e_s e_{n+1} - 2e_s e_{n+1} e_i e_k =$

$$2(e_i e_k e_s - e_s e_i e_k)e_{n+1} = \begin{cases} 0 & s \neq i,k \\ 0 & s = i = k \\ 4e_k e_{n+1} & s = i \neq k \\ -4e_i e_{n+1} & s = k \neq i \end{cases}$$

all possible results being elements of $T_1\Theta(S^n)$ that is spanned by all $e_i e_{n+1}, i = 1, \ldots, n$ and therefore it is a Lie triple system. □

A generator of $\pi_n SO(n+1)$ is obtained by projecting $\Theta(S^n)$ to $SO(n+1)$ through the map π that is a local isometry and therefore preserves totally geodesic submanifolds.

4 Algebraic Formulation

There is an algebraic structure that suits the map Θ, which we describe now.

Let $\omega = e_1 \ldots e_n$ in $Pin(n)$. Define the automorphism σ of $Spin(n+1)$ by $\sigma(A) = \omega A \omega^{-1}$. It follows from $\omega^2 = (-1)^{n(n+1)}$ that $\sigma^2 = id$ and we can easily see that $\sigma(A) = A \Leftrightarrow A \in Spin(n)$. We can now define the map

$\tilde{\sigma} : Spin(n+1)/Spin(n) \cong S^n \to Spin(n+1)$ by $\tilde{\sigma}([A]) = A\sigma(A^{-1})$. The following theorem of E. Cartan [5, pg. 77] applies in this case:

Theorem 4 (Cartan). *If G is a compact Lie Group with a bi-invariant metric with an automorphism σ of order 2 having $H \subset G$ as its fixed point subgroup, then the map $\tilde{\sigma} : G/H \to G$ defined by $\tilde{\sigma}([g]) = g\sigma(g^{-1})$ is a well defined totally geodesic embedding and the metric on G/H, induced by $\tilde{\sigma}$ is twice the normal one.*

Theorem 5 *We have* $\Theta(S^n) = \tilde{\sigma}(S^n)$.

Proof. Both manifolds being totally geodesic with common point the identity element of $Spin(n+1)$, it is enough to show that the tangent spaces to both at the identity are equal. Recall that $T_1\Theta(S^n) = span\{e_ie_{n+1}, e = 1,\ldots,n\}$. To calculate $T_1\tilde{\sigma}\ (S^n)$ let $v_i(t)$ and $v_j(t)$ be curves in S^n, such that $v_i(0) = v_j(0) = e_{n+1}$, $v_i'(0) = e_i$ and $v_j'(0) = e_j$, $i \neq j$. Taking the product of these two curves we obtain the curve $v_i(t)v_j(t)$ in $Spin(n+1)$. Let $\beta(t) = \tilde{\sigma}$ $([v_i(t)v_j(t)]) = v_i(t)v_j(t)\omega v_j(t)v_i(t)\omega^{-1}$ and note that $\beta(0) = 1$ and $\beta'(0) \in T_1\tilde{\sigma}(S^n)$, $\beta'(0) = 2\ (e_i - e_j)e_{n+1}$. Repeating this with $v_i'(0) = e_i$, $v_j'(0) = -e_j$, we get $\beta'(0) = 2\ (e_i + e_j)e_{n+1}$. As the vectors of the form $\{(e_i - e_j)e_{n+1}, (e_i + e_j)e_{n+1}\}$ generate $T_1\Theta(S^n)$, the result follows.

Noting that the submersion metric on the sphere is of constant sectional curvature we can restate the **Theorem 3** as

Theorem 6 *The image* $\Theta(S^n)$ *is a totally geodesic submanifold of* $Spin(n+1)$ *with constant sectional curvature.*

5 A special case

It is known that
$Sp(2) = \left\{A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, a,b,c,d \in H, AA^\star = A^\star A = I\right\}$, where H is the algebra of quaternions, is isomorphic to $Spin(5)$. The isomorphism uses the classification of simply connected compat Lie groups or considerations concerning the Cayley algebra [11], [4]. Working with quaternions and the Cartan Theorem we get a totally geodesic submanifold of $Sp(2)$, isometric to the round S^4, which is precisely the generator of $\pi_4 Sp(2) = \mathbb{Z}_2$ described by **Theorem 1**.

Let $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in Sp(2)$ and

$Sp(1) \times Sp(1) = \{A = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, p,q \in Sp(1) = S^3$, the unitary quaternions$\}$.

For $A \in Sp(2), A\Lambda = \Lambda A \Leftrightarrow A \in Sp(1) \times Sp(1)$. Therefore the conjugate orbit through Λ, $O(\Lambda) = \{A\Lambda A^{-1}, A \in Sp(2)\}$ is the symmetric space $Sp(2)/Sp(1) \times Sp(1)$.

Claim. $Sp(2)/Sp(1) \times Sp(1)$ is isometric to S^4 with the euclidean metric of constant sectional curvature.

Proof. Take the Hopf projection
$\begin{pmatrix} ap & cq \\ bp & dq \end{pmatrix} \longmapsto (||a||^2 - ||b||^2, 2a\bar{b})$, where $2a\bar{b}$ is in the disk D^4 and $||a||^2 - ||b||^2$ in $[-1,1]$ so that the vector is in the unit S^4. The metric induced on it by submersion from the Killing - Cartan metric on $Sp(2)$ in the euclidean one as is well known. □

We remark that the metric induced on S^7 as an intermediate step of the above map from $Sp(2)$ is not the euclidean one and this fact has interesting consequences in the geometry of Σ^7, the Gromoll - Meyer exotic 7 - sphere. (Carlos E. Duran, Blaschke structures on an exotic sphere, preprint).

The matrix Λ has order 2, so the map $\sigma : Sp(2) \to Sp(2), A \longmapsto \Lambda A \Lambda$ makes $(Sp(2), Sp(1) \times Sp(1))$ a symmetric pair. By the **Cartan Theorem** we have

$\tilde{\sigma} : S^4 \simeq Sp(2)/Sp(1) \times Sp(1) \to Sp(2)$. As the Cartan embbeding is a right translation of a conjugate orbit and there is only one conjugate orbit that is S^4 [1, pg 103] follows, independently of the identification between $Sp(2)$ and $Spin(5)$, from **Theorem 4**, that $\tilde{\sigma}$ is a generator for $\pi_4 Sp(2) = \mathbb{Z}_2$. In fact, we can get another simple expression for this generator by:

Theorem 7 *Let* $\Phi : S^4 \subseteq \mathbb{R}^5 = \mathbb{R} \times H \to Sp(2), (x,\xi) \longmapsto \begin{pmatrix} x & \bar{\xi} \\ \xi & -x \end{pmatrix}$, *where* $x \in [-1,1]$, $\xi \in H$, $\xi\bar{\xi} + x^2 = 1$. *Then* Φ *is a totally geodesic embedding with constant sectional curvature that generates* $\pi_4 Sp(2)$.

Proof. Let $\Psi : S^4 \cong Sp(2)/Sp(1) \times Sp(1) \to Sp(2), A \longmapsto A\Lambda A^{-1}$. As Ψ is obtained from $\tilde{\sigma}$ by right translation by Λ it follows from **Theorem 4** that Ψ is a totally geodesic embedding with constant sectional curvature. It remains to show that $\Psi = \Phi$. From the relations between the coefficients of matrices in $Sp(2)$ we have, if $a \neq 0$.

$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} ||a|| & \frac{a\bar{b}}{||a||} \\ \frac{b\bar{a}}{||a||} & -||a|| \end{pmatrix} \begin{pmatrix} \frac{a}{||a||} & 0 \\ 0 & -\frac{d}{||d||} \end{pmatrix}$. We can suppose then that A is of the form $\begin{pmatrix} ||a|| & \frac{a\bar{b}}{||a||} \\ \frac{b\bar{a}}{||a||} & -||a|| \end{pmatrix}$ and we have

$\Psi(A) = \begin{pmatrix} ||a||^2 - ||ab||^2 & 2a\bar{b} \\ 2b\bar{a} & ||b||^2 - ||a||^2 \end{pmatrix}$. If $x = ||a||^2 - ||b||^2$, $\xi = 2b\bar{a}$ we get $\Psi(A) = \Phi(x,\xi)$. For $a = 0$, we can suppose $A = \begin{pmatrix} 0 & -c \\ b & 0 \end{pmatrix}$ and $\Psi(A) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \Phi(-1,0)$.

□

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Received August 23, 1996 and in revised form June 30, 1999
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