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The Maximality of the Group of Euclidean Similarities within the Affine Group II

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Abstract. The purpose of this note is to show that for an arbitrary Pythagorean field K the group $O_n^+(K)$ is maximal within $SL_n(K)$ if, and only if, K admits only of Archimedean orderings. Under the same conditions the group of $n-$dimensional Euclidean similarities is maximal within the group of all affine mappings having a determinant of the form $\pm\lambda^n \neq 0$.

Al nostro carissimo amico Adriano Barlotti
per il suo ottantesimo compleanno compiuto il 12 ottobre 2003

Introduction

Investigating a problem in analytical mechanics in 1965 or somewhat earlier Walter Noll was lead to the question whether the group of Euclidean motions is maximal within the unimodular group (the group of matrices of determinant ± 1 with real entries). He got an affirmative answer by himself but posed the question also to a number of other mathematicians. Thus in 1965 two different proofs of this answer appeared in the Archive for Rational Mechanics and Analysis ([3] and [6]), one by Noll himself, the other one by Richard Brauer. As a geometric application one may easily conclude from this that the group of Euclidean similarities is maximal within the affine group so that in the sense of Klein's Erlanger program there is no geometry between the classical Euclidean and the affine geometry.

Questions of maximality of one group in another one have been dealt with in a large number of cases but mainly within three contexts: for finite groups, for Lie groups, and for algebraic groups over some algebraically closed field. In the general case of classical groups over arbitrary fields there remain many open problems. In [7] the question of the maximality of the Euclidean group of similarities within the affine group has been studied for Euclidean fields. These are defined to be ordered fields in which every positive element is a square. In this note we shall extend this study to the case of Pythagorean fields which are defined by the following two properties: i) the element -1 is not a square and ii) the sum of two squares is always again a square (s. Bachmann [1, p. 216], or Weyl [9, p. 13]).

We shall restrict ourselves to quadratic forms which can be reduced to $x_1^2 + x_2^2 + \cdots + x_n^2$ with respect to a suitable basis and to the corresponding orthogonal groups $O_n^+(K)$ and $O_n(K)$. As a consequence of the definition of a Pythagorean field in each $1-$dimensional subspace of the underlying vector space it is possible to find two vectors of length one. Such a vector will always be part of infinitely many orthonormal bases and any two such bases can be moved into one another by suitable elements of the orthogonal group. Thus it follows that in a vector space over a Pythagorean field endowed with the special quadratic form above we get free mobility in the sense of Helmholtz's space problem under the orthogonal group (see [4, p. 89], or [2]). In particular the orthogonal group acts transitively on $1-$dimensional subspaces.

Where possible we shall use standard notation. In more special cases we shall keep the notations of [7]. Thus Λ_n denotes the group of mappings $x \to \lambda x$, $0 \neq \lambda \in K$. We shall make no distinction between matrices and linear transformations defined abstractly. Thus the group of Euclidean similarities consists of mappings of the form $x \to xM + b$ where $M \in \Lambda_n O_n(K)$. We cannot expect this group to be maximal within the affine group. But it may be maximal within the subgroup of affine mappings $x \to xM + b$ such that $\det M = \pm\lambda^n$ for some $\lambda \neq 0$. As in [7] we shall denote this subgroup by $GL_n(K)^*$ and by $SL_n(K)^{\pm}$ the group of all linear mappings of determinant ± 1.

It is not surprising that in dealing with orthogonal groups one is lead to consider pairs s, c of elements in the field K satisfying

$$s^2 + c^2 = 1. \tag{1}$$

We shall denote the set of possible values of s, or equivalently c, by $T(K)$. We shall also call any pair s, c satisfying (1) an admissible pair. The key to our problem is a detailed study of this set of "trigonometric values" and of the subring generated by them. At this point we gratefully acknowledge that we owe the essentials of this study to [1].

Our main result is as follows:

1 Theorem. *For a Pythagorean field K the following statements are equivalent:*

(i) K admits only of Archimedean orderings,

(ii) the subring generated by $T(K)$ coincides with K,

(iii) $\Lambda_2 O_2(K)$ *is maximal within* $GL_2(K)^*$,

(iv) $\Lambda_n O_n(K)$ *is maximal within* $GL_n(K)^*$ *for all* $n \geq 2$,

(v) $O_n(K)$ *is maximal within* $SL_n(K)^{\pm}$ *for all* $n \geq 2$,

(vi) $O_n^+(K)$ *is maximal within* $SL_n(K)$ *for all* $n \geq 2$,

(vii) for all dimensions $n \geq 2$ *the group of Euclidean similarities is maximal within the group of all affine mappings whose determinant is an* n*-th power times* ± 1.

1 The subring generated by $T(K)$

In this section we collect some basic results on the set $T(K)$ and on the subring it generates. These will be needed in later sections.

2 Lemma. *Let* $u, v \in T(K)$. *Then*

(i) $\sqrt{1-u^2}, \sqrt{1-v^2} \in T(K)$.

(ii) $uv \pm \sqrt{1-u^2}\sqrt{1-v^2} \in T(K)$.

(iii) $\sqrt{1-u^2}v \pm u\sqrt{1-v^2} \in T(K)$.

Proof. i) Let $u \in T(K)$. Then $u^2 + u_1^2 = 1$ for some $u_1 \in K$. This implies that $u_1 \in T(K)$ but u_1 is a root of $1 - u^2$. Since $-u_1$ obviously is another solution of equation (1) it follows that $\sqrt{1-u^2} \in T(K)$.
ii) and iii). Let $u, v \in T(K)$. Consider the elements $p = uv - \sqrt{1-u^2}\sqrt{1-v^2}$ and $q = \sqrt{1-u^2}v + u\sqrt{1-v^2}$. (We may think of p and q as $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$.) It is easy to check that $p^2 + q^2 = 1$. $\boxed{QED}$

3 Lemma. *Let* K *be any field admitting an ordering and let* $s \in T(K)$. *Then* $-1 \leq s \leq 1$ *in any ordering of* K.

Proof. Let $s^2 + t^2 = 1$. Then in any ordering $0 \leq s^2, t^2$. Hence $s^2 = 1 - t^2 \leq 1$. This implies $-1 \leq s \leq 1$. $\boxed{QED}$

From now on let us assume that K is Pythagorean.

4 Lemma. $\frac{1}{1+x^2} \in T(K)$ *for any* $x \in K$.

Proof. We need only to show that

$$1 - \left(\frac{1}{1+x^2}\right)^2 = \frac{x^2(x^2+2)}{(1+x^2)^2}$$

is a square. This follows if x^2+2 is a square which is true since K is Pythagorean. $\boxed{QED}$

5 Lemma. *If* c, s *is an admissible pair then so is* $c^2 - s^2, 2cs$.

Proof. $(c^2 - s^2)^2 + 4c^2 s^2 = (c^2 + s^2)^2 = 1$. $\boxed{QED}$

6 Lemma. *If c, s is an admissible pair then so is $\sqrt{\frac{1+c}{2}}, \sqrt{\frac{1-c}{2}}$.*

Proof. If the square roots exist in K then they obviously satisfy equation (1). Therefore it suffices to prove the existence of the roots. If $(1+c)(1-c)=0$ it follows that $1-c=2$ or $1+c=2$ and hence obviously both roots exist. Otherwise from the identity

$$\left(\frac{s}{1+c}\right)^2 = \frac{1-c}{1+c}$$

we may conclude that the elements

$$\frac{1}{1+\frac{1-c}{1+c}} = \frac{1+c}{2}, \quad \text{and} \quad \frac{1}{1+\frac{1+c}{1-c}} = \frac{1-c}{2}$$

are squares. $\boxed{QED}$

An important consequence of 5 and 6 is that $c^2-s^2, 2cs$ runs through the set of all admissible pairs as c, s does.

An element s is said to be totally includable between integers if there exists an integer m such that $-m \leq s \leq m$ in any ordering of K. The elements that are totally includable between integers form a ring. This ring is generated by the elements s such that $-1 \leq s \leq 1$ for any ordering of K (see [1]).

7 Theorem. *Let K be a Pythagorean field. Then the subring S generated by $T(K)$ is the ring of all elements that are totally includable between integers.*

Proof. It is proved in [1] that for a Pythagorean field K the ring of all elements that are totally includable between integers is generated by the elements of the form $1/(1+x^2)$ (see [1, p. 294]). Since these elements belong to $T(K)$ they generate a subring of the ring generated by $T(K)$. On the other hand because of 3 the ring generated by $T(K)$ is contained in the ring of all elements totally includable between integers. This proves the theorem. $\boxed{QED}$

2 The Maximality of $\Lambda_n O_n(K)$ in $GL_n(K)^*$

In this section we assume throughout that K is a Pythagorean field and that $H_n(K)$ is a group of matrices properly containing $\Lambda_n O_n(K)$ and contained in $GL_n(K)^*$. We consider first the case $n=2$.

8 Lemma. *The elements x of K such that $\left(\begin{smallmatrix}1 & 0\\ x & 1\end{smallmatrix}\right)$ belongs to $H_2(K)$ form a subring of K which is distinct from $\{0\}$.*

Proof. By 2.2 and 2.4 in [7] it follows that $H_2(K)$ contains at least one matrix of the form $\left(\begin{smallmatrix}1 & 0\\ q & 1\end{smallmatrix}\right)$ where $q \neq 0$.

Let S denote the subset of all $x \in K$ such that $\left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right)$ belongs to $H_2(K)$. By 2.5 of [7] it follows that if $q \in S$ then for each admissible pair $c^2 + s^2 = 1$ we also have

$$2q(q \cdot c \cdot s - c^2 + s^2) \in S. \tag{2}$$

Taking $c = \frac{1}{2}\sqrt{2}$ and $s = \frac{1}{2}\sqrt{2}$ in (2) it follows that $q^2 \in S$ if $q \in S$. Clearly the set S is an additive group. Therefore with $u, v \in S$ it follows that $(u+v)^2 = u^2 + 2uv + v^2 \in S$. This implies $2uv \in S$.

Taking $c = \frac{1}{4}\sqrt{7}$ and $s = \pm\frac{3}{4}$ in (2) it follows that $s^2 - c^2 = \frac{1}{8}$ and $s \cdot c = \pm\frac{3\sqrt{7}}{16}$. Therefore $2q(\pm q\frac{3\sqrt{7}}{16} + \frac{1}{8}) \in S$ which implies that $\frac{1}{2}q \in S$ when $q \in S$. This proves that S is a ring. $\boxed{QED}$

9 Lemma. *The ring S of the previous lemma satisfies $T(K)S \subseteq S$.*

Proof. If c, s is an admissible pair then so is $c, -s$. As in the proof of the previous lemma we have $2q(\pm q \cdot c \cdot s - c^2 + s^2) \in S$ and hence $4q(-c^2 + s^2) \in S$ when $q \in S$. As $\frac{1}{2}q \in S$ this implies that $(c^2 - s^2)q \in S$. Then 5 and 6 teach us that the pair $c^2 - s^2, 2c \cdot s$ runs through the set of all admissible pairs when c, s does. $\boxed{QED}$

10 Lemma. *If the ring generated by $T(K)$ coincides with K then $H_2(K) = GL_2(K)^*$,* i.e. *$\Lambda_2 O_2(K)$ is maximal in $GL_2(K)^*$.*

Proof. If the hypothesis of 10 is satisfied it follows from $S \neq \{0\}$ and $T(K)S \subseteq S$ that S coincides with K. This means that $H_2(K)$ contains all transvections of the form $\left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right)$ and hence by conjugation with elements of $O_2(K)$ it contains all transvections. Here we have used the fact that $O_2(K)$ acts transitively on 1−dimensional subspaces. Since $SL_2(K)$ is generated by its transvections (see [4, p. 37]) this implies that $H_2(K)$ contains $SL_2(K)$ and therefore $H_2(K) = GL_2(K)^*$. $\boxed{QED}$

Assertion 10 generalizes the corresponding statement 2.6 of [7].

It is stated in [7] that if $\Lambda_2 O_2(K)$ is maximal in $GL_2(K)^*$ for some Pythagorean field K then the analogous statement is true for all $n \geq 2$. The proof given is by induction on n and contains the following gap. On page 37 it is asserted that there is a certain triangular matrix M which belongs to $H_n(K)$ but does not belong to $\Lambda_n O_n(K)$. It is then stated that if the deviations that cause it not to belong to $\Lambda_n O_n(K)$ occur in the first $n-1$ rows one can use the induction hypothesis on the submatrices formed by the first $n-1$ rows and columns. Here however, it was forgotten to check that the submatrix of M formed by omitting the last row and column has a determinant of the form $\pm\lambda^{n-1}$. To bridge this gap we may first make sure that M contains with respect to some ordering of K only positive elements in the main diagonal. This is possible since we may multiply M by a diagonal matrix which has only ones or minus ones in the main diagonal. Then we may raise M to the power $n-1$. It is easy to show by induction on ν that the (i,j) - entry b_{ij}^{ν} of M^{ν} has the form $b_{ij}^{\nu} = a_{ij}p(a_{jj},a_{ii}) + a_{ij+1}q_1 + \cdots + a_{ii-1}q_{i-j-1}$ with suitable $q_h \in K$ where a_{kl} are the entries of M and $p(a_{jj},a_{ii})$ is a polynomial with positive integer coefficients. Hence it follows easily that if M is not contained in $\Lambda_n O_n(K)$ then M^{ν} is not contained in $\Lambda_n O_n(K)$. Now in M^{n-1} the submatrix formed by omitting the last row and column obviously has a determinant of the form $\pm\lambda^{n-1}$ since the elements along the main diagonal of M^{n-1} are a_{ii}^{n-1}, $i = 1, \dots, n$.

3 Subgroups between $O_n^+(K)$ and $SL_n(K)$

In this section let $O_n^+(K) < H_n(K) \le Sl_n(K)$. We shall show by induction on n that $H_n(K)$ contains transvections.

Before going into the details let us note

11 Lemma. *Let $A \in SL_n(K)$ such that $A^{-1}O_n^+(K)A \subseteq O_n^+(K)$. Then $A \in O_n^+(K)$.*

Proof. Let $e_1, e_2 \dots, e_n$ denote the vectors

$$e_1 = (1,0,\dots,0), \qquad e_2 = (0,1,0,\dots,0), \quad \text{etc..}$$

Let $e_jA = v_j$, $j = 1, \dots, n$. Then for any pair of indices $i < j$ the subgroup $A^{-1}O_n^+(K)A$ contains elements mapping v_i to v_j, v_j to $-v_i$, and v_k to v_k for $k \ne i, j$. Since these elements belong to $O_n^+(K)$ it follows that v_i and v_j have equal length and are orthogonal to each other. This implies that $A \in \Lambda_n O_n(K)$. Since $A \in SL_n(K)$ it follows that $A \in O_n^+(K)$. $\boxed{QED}$

12 Lemma. *$H_2(K)$ contains transvections.*

Proof. The group $H_2(K)$ can be written as product $O_2^+(K)U$ where U is the stabilizer of the subspace $K(1,0)$ and consists of matrices $\left(\begin{smallmatrix} a & 0 \\ t & a^{-1} \end{smallmatrix}\right)$. Commutators of any two such matrices are either transvections or equal to the identity. If there are no transvections then U is Abelian. But $O_2^+(K)$ is Abelian too and hence according to Itô's theorem $H_2(K)$ is metabelian (see [5]) hence soluble. Then by a theorem of Mal'cev (see [8, p. 45]) $H_2(K)$ contains a subgroup F of finite index which is triangulizable in an extension field where all the occurring eigenvalues exist. Hence any element of the commutator subgroup F' which is distinct from the identity is a transvection. Thus if $F' \ne 1$ we are finished. Otherwise F would be Abelian and $F \cap U = \{1\}$. But U contains at least one matrix of the form $\left(\begin{smallmatrix} a & 0 \\ t & a^{-1} \end{smallmatrix}\right)$ where $a \ne \pm 1$. From this it follows that U is infinite. This is a contradiction since F has finite index. $\boxed{QED}$

13 Lemma. *$H_n(K)$ contains transvections.*

PROOF. We use an idea of Richard Brauer [3] and proceed by induction on n. For $n = 2$ this has been shown above. Thus we may assume $n \geq 3$.

Let M denote the set of mappings in $O_n^+(K)$ that have two eigenvalues -1 and $n-2$ eigenvalues 1. Then M generates $O_n^+(K)$ (cf. Dieudonné [4, p. 50]). Let $G \in H_n(K) - O_n^+(K)$. It follows from 11 that there exists $T \in M$ such that $T_1 = G^{-1}TG \notin O_n^+(K)$. We can find a basis $v_1, v_2, v_3, \dots, v_n$ such that T_1 maps v_1 to $-v_1$, v_2 to $-v_2$, and v_j to v_j for $j = 3, \dots, n$. We may assume here that v_1, v_2 form an orthonormal basis of the subspace $\langle v_1, v_2\rangle$ and that $v_3, \dots, v_n$ form an orthonormal basis of $\langle v_3, \dots, v_n\rangle$. Since the intersection of the orthogonal complement of $\langle v_1, v_2\rangle$ with the subspace $\langle v_3, \dots, v_n\rangle$ has dimension at least $n-4$ we may also assume that $v_5, \dots, v_n$ belong to the orthogonal complement of $\langle v_1, v_2, v_3, v_4\rangle$.

We may replace the basis elements v_3, v_4 by elements w_3, w_4 so that the basis v_1, v_2, w_3, w_4, $v_5, \dots, v_n$ becomes orthonormal. Moreover we may do this in such a way that $w_3 \in \langle v_1, v_2, v_3\rangle$ and $w_4 \in \langle v_1, v_2, v_3, v_4\rangle$. Let W denote the subspace $\langle v_1, v_2, w_3, w_4\rangle$ so that $W^\top := \langle v_5, \dots, v_n\rangle$ is the the orthogonal complement of W. (Here of course the vectors w_4 and $v_5, \dots, v_n$ are only present if $n \geq 4$.) In the following it will be sufficient to consider elements of $H_n(K)$ which like T_1, preserve W and act as the identity on $W^\top$, and which we can therefore identify with their matrices with respect to the orthonormal basis chosen in W.

With respect to this basis the matrix of T_1 takes the form

$$\begin{pmatrix} -1 & & & \\ 0 & -1 & & \\ \alpha_1 & \alpha_2 & \delta_1 & \\ \beta_1 & \beta_2 & \gamma & \delta_2 \end{pmatrix}.$$

Since T_1 is an involution and has determinant 1 it is easy to see that $\delta_1 = \delta_2 = \pm 1$. Looking at the entries of T_1^2 below the diagonal we see that $\gamma = 0$ and that $\delta_1 = \delta_2 = 1$. Since T_1 does not belong to $O_n^+(K)$ not all of the coefficients α_1, α_2, β_1, β_2 may vanish simultaneously. If α_1 and α_2 are both zero then T_1 is a transvection and we are finished. But if $\alpha_1 = 0$, $\alpha_2 \neq 0$ we may interchange v_1 and v_2 and we get $\alpha_1 \neq 0$. Thus we may assume $\alpha_1 \neq 0$.

Let $n = 3$. If $S = \operatorname{diag}(-1, -1, 1)$ then ST_1 obviously is a transvection.

Let $n \geq 4$. Then $R = \operatorname{diag}(1, -1, -1, 1)$ belongs to $O_n^+(K)$. We form the commutator $RT_1R^{-1}T_1^{-1}$ and verify that

$$T_2 = RT_1R^{-1}T_1^{-1} = \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ 2\alpha_1 & 0 & 1 & \\ 0 & 2\beta_2 & 0 & 1 \end{pmatrix}$$

Next conjugating T_2 with the element R_1 of $O_n^+(K)$ which maps v_1 to v_1, v_2 to w_3, w_3 to $-v_2$, and w_4 to w_4 we get

$$T_3 = R_1 T_2 R_1^{-1} = \begin{pmatrix} 1 & & & \\ 2\alpha_1 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & -2\beta_2 & 1 \end{pmatrix}.$$

Let U denote the subgroup of $O_n^+(K)$ which keeps the vectors w_3, w_4 and all vectors of $W^\top$ fixed. Then the conjugate $T_3 U T_3^{-1}$ shares this property and because of 11 it contains elements not in U. Thus we have reduced this case to the case $n = 2$. $\boxed{QED}$

We add as a remark that in [3] Brauer used a somewhat similar argument to show inductively that if $O_2^+(K)$ is maximal in $SL_2(K)$ then the same is true in all dimensions. He stated this only for the field of real numbers but his argument is valid for any Pythagorean field.

14 Lemma. *The transvections of $H_n(K)$ together with $O_n^+(K)$ generate a subgroup M which is invariant under involutions of $O_n(K) - O_n^+(K)$.*

Proof. Let $t \in H_n(K)$ be a transvection. Then by an element h of $O_n^+(K)$ we may conjugate it into the form $(I_{n-1}, q, 1)$ (see [7] for this notation). Let j denote the element $(I_{n-1}, 0, -1)$ of $O_n(K)$. Then $jh^{-1}thj = jh^{-1}jjtjjhj = (I_{n-1}, -q, 1) \in H_n(K)$. Since $jhj \in O_n^+(K)$ this implies $jtj \in H_n(K)$. $\boxed{QED}$

4 Proof of Theorem 1

i) implies ii). We have shown in theorem 7 that for a Pythagorean field K the subring generated by $T(K)$ consists of all elements which are totally includable between integers. If all orderings of K are Archimedian then this set of elements equals the whole of K.

$\neg$ i) implies $\neg$ ii). If K has an ordering which is not Archimedean then in this ordering there exist elements which are greater than any integer. Such elements are certainly not totally includable between integers. Hence the subring generated by $T(K)$ is a proper subring of K.

ii) implies iii). Consider a subgroup H of $GL_2(K)^*$ such that $\Lambda_2 O_2(K) < H \leq GL_2(K)^*$. By 8 the elements x occurring in transvections of the form $\left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right)$ of H form a subring S of K which is distinct from 0 and by 9 satisfies $T(K)S \subseteq S$. If $T(K)$ generates K as a ring this implies $KS \subseteq S$ whence $S = K$. This implies that H contains all transvections and therefore it contains $SL_2(K)$ (see [4, p. 37]). Since $\Lambda_2 SL_2(K) = GL_2(K)^*$ it follows that $H = GL_2(K)^*$.

$\neg$ ii) implies $\neg$ iii). If the ring S generated by $T(K)$ is a proper subring of K then the group $SL_2(S)$ of 2×2 matrices with entries from the ring S and determinant one is a subgroup H between $O_2^+(K)$ and $SL_2(K)$. Since $H = SL_2(S)$ is normalized by $O_2(K)$ it follows that $O_2(K) < H_1 < SL_2(K)^\pm$ where $H_1 = H \cup Hg$ for any $g \in O_2(K) - O_2^+(K)$. Then $\Lambda_2 O_2(K) < \Lambda_2 H_1 < GL_2(K)^*$.

By [7] we see that (iii) and (iv) are equivalent.

iv) and v) are equivalent. Since Λ_n is a normal subgroup of $\Lambda_n O_n(K)$ and of $\Lambda_n SL_n(K)^\pm = GL_n(K)^*$ it follows that the subgroups lying in between $O_n(K)$ and $SL_n(K)^\pm$ are in one-to-one correspondence with the subgroups in between $\Lambda_n O_n(K)$ and $\Lambda_n SL_n(K)^\pm = GL_n(K)^*$.

$\neg$ vi) implies $\neg$ v). If $O_n^+(K) < H_n(K) < SL_n(K)$ then as we have shown in 13 the subgroup $H_n(K)$ contains transvections. Let H denote the subgroup generated by $O_n^+(K)$ and all the transvections of $H_n(K)$. Then $O_n^+(K) < H < SL_n(K)$ and because of 14 the subgroup H is normalized by all elements of $O_n(K)$. Let g denote an involution in $O_n(K) - O_n^+(K)$. It follows that $H_1 = H \cup Hg$ is a group such that $O_n(K) < H_1 < SL_n(K)^\pm$.

vi) implies v). Let $O_n(K) < H_1 \leq SL_n(K)^\pm$ and consider an element $g \in H_1 - O_n(K)$. If $\det g = -1$ we may multiply by an element of $O_n(K) - O_n^+(K)$ and obtain an element g_1 in $H_1 - O_n(K)$ such that $\det g_1 = 1$. This shows that $H = H_1 \cap SL_n(K)$ properly contains $O_n^+(K)$. From vi) it follows $H = SL_n(K)$ and this implies $H_1 = SL_n(K)^\pm$.

iv) and vii) are equivalent. $\Lambda_n O_n(K)$ is isomorphic to the group of Euclidean similarities modulo the translation group. $GL_n(K)^*$ is isomorphic to the group of all affine mappings whose determinant is an n-th power times ± 1 modulo the translation group. Therefore the groups in between $\Lambda_n O_n(K)$ and $GL_n(K)^*$ are in one-to-one correspondence with the groups in between the group of Euclidean similarities and the group of affine mappings whose determinant is an n-th power times ± 1. This completes the proof of the theorem.

References

[1] Friedrich Bachmann: Aufbau der Geometrie aus dem Spiegelungsbegriff, Springer, Berlin, Göttingen, Heidelberg (1959).

[2] Reinhold Baer: *Free mobility and orthogonality*, Trans. Amer. Math. Soc. **68** (1951), 439–460.

[3] Richard Brauer: *On the Relation between the Orthogonal and the Unimodular Group*, Arch. Rat. Mech. and Analysis **18** (1965), 97–99.

[4] Jean Dieudonné: La géométrie des groupes classiques, Springer 2nd edition (1963).

[5] Noboru Itô: *Über das Produkt von zwei abelschen Gruppen*, Math. Z. **62** (1955), 400–401.

[6] Walter Noll: *Proof of the Maximality of the Orthogonal Group in the Unimodular Group*, Arch. Rat. Mech. and Analysis **18** (1965), 100–102.

[7] Adolf Schleiermacher: *The Maximality of the Group of Euclidean Similarities within the Affine Group*, Note di Matematica Vol. 19, n. 1 (1999), 33–42.

[8] Bertram A. F. Wehrfritz: Infinite Linear Groups, Springer (1973).

[9] Hermann Weyl: The Classical Groups, Their Invariants and Representations, Princeton Landmarks in Mathematics, Princeton, (1997).