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Infinite Maximal Partial Spreads

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Abstract. The concept of 'critical deficiency' of a finite net is generalized to the infinite case and a variety of infinite maximal partial spreads are constructed.

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Introduction

Let N be a finite net of order q^2 and degree $q^2 - q$. Such a net is said to have 'critical deficiency'. In [8], Ostrom showed that there are at most two extensions of a net of critical deficiency to an affine plane and if there are two then these affine planes are derivable and one plane is the derivate of the other.

Of course, infinite derivable affine planes are very much of interest so, in this note, we explore what might be a definition of 'critical deficiency' of infinite nets. If there are two extensions, we would want these affine planes to share all lines but lines on a net capable of containing a Baer subplane.

We recall that a subplane of a projective plane is said to be a 'point-Baer subplane' if and only if every point of the plane is incident with a line of the subplane. Similarly, a subplane is said to be a 'line-Baer subplane' if and only if every line of the plane is incident with a point of the subplane. An affine subplane of an affine plane is point-Baer or line-Baer if and only if the corresponding projective extension of the subplane is point-Baer or line-Baer in the projective extension of the plane.

In the finite case, we may define a net of 'critical deficiency' without assuming that the net may be embedded in an affine plane let alone two affine planes. But, when there are two extensions, there is an associated Baer subplane of either extension which extends the net. In the infinite case, we essentially require the existence of at least a point-Baer subplane to come to grips with the notion of critical deficiency.

Hence, we may arrive at the following definition:

1 Definition. Let N be an arbitrary net such that there exists an affine plane π_1 which extends it. If there exists a point-Baer subplane π_o of π_1 of $\pi_1 - N$, such that each line of π either intersects the projective closure of π_o or of N, we shall say that the net N is of 'critical deficiency'.

In the finite case, there is no distinction between point-Baer and line-Baer subplanes and, in such a case, the order of N must be a square q^2. Furthermore, it follows immediately that the net N has degree $q^2 - q$ and is of critical deficiency.

In Johnson [6], it is noted that there exist derivable nets in non-derivable affine planes. Furthermore, it is shown there that any derivable net may be extended to an affine plane. If the planes are non-derivable, the planes are necessarily infinite, so the natural conclusions on nets of critical deficiency become more complicated in the infinite case. We separate the questions as follows:

(1) If M is a derivable net that is extended to two distinct affine planes π_1 and π_2, are the planes derivable?

(2) Are two planes extending a net of critical deficiency derivable and derivates of each other?

In this note, we are able to deal with question (2) when the two planes comes from spreads in $PG(3, K)$ for K a skewfield.

More generally, we have considered the following question in the finite case in a recent work on finite nets which are transversal-free (see Jha and Johnson [1]):

Let π_1 be a translation plane with point-Baer subplane π_o. Let N denote the net defined by the components of π_o and let S_{π_o} denote the set of all Baer subplanes of the net N. Then does $M = (\pi_1 - N) \cup S_{\pi_o}$ correspond to a maximal partial spread?

We note that it is not even clear that we obtain a corresponding net in this case as we might not obtain a partial spread let alone a maximal partial spread.

In previous work (see *e.g.* Jha and Johnson [3]), the authors note that it is possible to consider maximal partial spreads, called quasifibrations, which are always infinite. In this note, we show that Bruen's construction for maximal partial spreads holds in the case that the spreads are in $PG(3, K)$ for K a skewfield and answer the question above in this setting. So, we show that if there is one point-Baer subplane then any other extensions by K-subspaces are also point-Baer subplanes.

1 The Main Structure Theorem

2 Theorem. *Let N be a net of critical deficiency arising from a partial spread in $PG(3,K)$, K a skewfield, and π_1 is a translation plane with spread in $PG(3,K)$ extending N where the associated point-Baer subplane π_o is a 2-dimensional left K-subspace.*

Then any extension net N^+ meeting the lines of π_1 in exactly N and defined by a partial spread in $PG(3,K)$ is obtained by adjoining Baer subplanes of $\pi_1 - N$ to N.

If N^+ is an affine plane then K is a field and the adjoined net defines a regulus in $PG(3,K)$ so that $N^+ = \pi_2$ and π_1 are derivates of each other.

PROOF. Let π_o be a point-Baer subplane of π_o as in the definition of critical deficiency and let M denote the net of π_1 containing π_o. Let ℓ_2 be any 2-dimensional K-subspace which extends N. We claim ℓ_2 shares all components of M. To see this, we require a vector based argument.

Since π_o is a 2-dimensional (left) K-subspace, we may choose bases for the spread of π_1 such that the net M contains components $x = 0, y = 0, y = x$, and $\ell_2 = \langle (0,1,0,0), (0,0,0,1) \rangle$. It then follows that the net M has components $x = 0, y = x \begin{bmatrix} u & f(u,w) \\ 0 & w \end{bmatrix}$ for all $w \in K$ where $u \in K$ and f is a function on $K \times K$.

Now ℓ_2 is a 2-dimensional subspace so, by Johnson [6], ℓ_2 is a line-Baer subplane of π_1. Hence, the subplane intersects M at least three components. We re-choose the basis so that the two subplanes π_o and ℓ_2 share $x = 0, y = 0$ and $y = x$ and since the subplanes are 2-dimensional K-subspaces, it follows that ℓ_2 has the form $T_{s,t} = \langle (s,t,0,0), (0,0,s,t) \rangle$ where not both s and t are zero. We consider the 1-dimensional subspace $T_{s,t}^v = \langle (s,t,sv,tv) \rangle$ for fixed $v \in K$. Note that $T_{s,t}^v$ must lie on some subset of the components of M which have the form $x = 0, y = x \begin{bmatrix} u & f(u,w) \\ 0 & w \end{bmatrix}$. We may assume that $s \neq 0$. Hence, we must have the intersection component defined by $(s,t) \begin{bmatrix} u & f(u,w) \\ 0 & w \end{bmatrix} = (sv, tv)$.

Since s is non-zero, it follows that $u = v$. Furthermore, $sf(u,w) + tw = tv$. First assume that $t = 0$. Then $sf(u,w) = 0$ so that $f(u,w) = 0$.

Without loss of generality, we may choose a second subplane to have basis such that $\ell_2 = T_{s,0}$ without alternating our basic coordinatization. Hence, the coordinates that $T_{s,0}$ intersects have the general form $x = 0, y = x \begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix}$ for all $v \in K$ and for certain $w's$ in K. To show that $T_{s,0}$ intersects all components of M, we need to show that w takes on all elements of K. Note that since we obtain a partial spread, it follows that $w = g(v)$ for some function g of K and since the difference of matrices is non-singular, it follows that g is also $1-1$. Suppose that there is a component of M which ℓ_2 does not intersect. Then there is such a component of the form $y = x \begin{bmatrix} u & f(u,w) \\ 0 & w \end{bmatrix}$. But, ℓ_2 intersects $y = x \begin{bmatrix} u & 0 \\ 0 & g(u) \end{bmatrix}$ and the difference of the associated matrics is $\begin{bmatrix} 0 & f(u,w) \\ 0 & w-g(w) \end{bmatrix}$ which must be non-singular. Thus, $w = g(w)$ and $f(u,w) = 0$. So, it follows that ℓ_2 and π_o share all of their components as subplanes. It then follows immediately from Jha and Johnson [4], Prop. 43, that these line-Baer subplanes are also point-Baer subplanes; the subplanes are Baer subplanes of the associated translation plane π_1.

Now assume that N is contained in two translation planes π_1 and π_2 with spreads in $PG(3,K)$.

Now suppose that there are exactly two components in M. Then every point on $x=0$ of M is in the union of two 1-dimensional subspaces. Since $x=0$ is a 2-dimensional subspace, this is impossible.

Hence, there are at least three subplanes of M.

Since $\pi_o = T_{0,1}$ and $\ell_2 = T_{1,0}$ share all of their components, we may assume that the three subplanes share at least three components which we take as $x=0$, $y=0$, $y=x$. It then follows easily that any third subplane must have the form $T_{s,t}$ for $st \neq 0$. Hence, we must have that $(s,t) \begin{bmatrix} u & 0 \\ 0 & g(u) \end{bmatrix} = (sv, tv)$ which implies that $u = v = g(u)$ so that $T_{s,t}$ intersects all of the components of M and furthermore that M has the form $x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$ for all $u \in K$.

Hence, it follows (see *e.g.* Johnson [7]) that M is a derivable net and the components of $\pi_2 - \pi_1$ are the Baer subplanes of M which we have pointed out are also Baer subplanes of π_1. Hence, π_1 is derivable and the derivation of M produces π_2. Since the derivable net is also a regulus, it follows that K is a field. This proves all parts of the theorem. QED

2 Maximal Partial Spreads

Applying the result of the previous section to the construction of maximal partial spreads in $PG(3,K)$, we obtain:

3 Theorem. *Let N be a net of critical deficiency in a translation plane π with spread in $PG(3,K)$, where the defining point-Baer subplane π_o is a left 2-dimensional K-subspace and let S_{π_o} denote the set of Baer subplanes of $\pi - N$*

incident with the zero vector which are left 2-dimensional K-subspaces.

Then $(\pi - N) \cup S_{\pi_o}$ is a maximal partial spread in $PG(3, K)$.

PROOF. This follows immediately from the previous theorem. QED

In the finite case, we know that there when there is one Baer subplane of a net defined by the components of the subplane, then there are either $1, 2$ or $q+1$ Baer subplanes incident with the zero vector which are $K = GF(q)$-subspaces. In the finite case, when there are $q + 1$ subplanes, the net is a regulus. In the infinite case, we have more variety as when K is a skewfield which is not a field, even when the net is a pseudo-regulus net since it is not the case that all Baer subplanes are K-subspaces when we restrict ourselves to having say 'left' K-subspaces.

For pseudo-regulus nets over a skewfield K, the set of Baer subplanes which are are 2-dimensional left K-subspaces corresponds to the projective line over $Z(K)$.

4 Theorem. *Let π be a translation plane with spread in $PG(3, K)$ where K is a skewfield. Assume that π_o is a point-Baer subplane of π which is a left K-subspace. Let M denote the set defined by the components of π_o. Let S_{π_o} denote the set of point-Baer subplanes of M which are left K-subspaces.*

Then S_{π_o} consists of either 1 or 2 point-Baer subplanes or corresponds to $PG(1, Z(K))$ and, in all cases, $(\pi - M) \cup S_{\pi_o}$ is a maximal partial spread in $PG(3, K)$.

3 Examples

5 Definition. If S_{π_o} has i-point-Baer subplanes which are K-subspaces, we shall say that the corresponding maximal partial spread constructed as above is of 'type i'.

We note that K could be of finite characteristic p and conceivably $Z(K)$ could be finite. In this case, the corresponding maximal partial spreads are of type $1 + |Z(K)|$.

In particular, it is possible that there are maximal partial spreads of type $1 + p^j$ for any j and prime p.

In Jha and Johnson [2], the authors construct examples of what are called 'skew Hall' planes whose spreads are in $PG(3, K)$, for K a skewfield which is not a field, and which admit a variety of derivable nets all of which produce maximal partial spreads of type $1 + |Z(K)|$.

When K is a field which admits a quadratic field extension F, there is an extension procedure called 'lifting' which produces spreads in $PG(3, F)$ from spreads in $PG(3, K)$. In the infinite case, there is a more general geometric object of related interest termed a 'quasifibration' which is either a spread or a proper maximal partial spread. It is noted in Johnson [5] that quasifibrations may be 'lifted'.

6 Theorem (Johnson [5]). *Let π be translation net with quasifibration S in $PG(3,F)$, for F a field. Assume that there is a quadratic extension field K with basis $\{1,\theta\}$ such that $\theta^2 = \theta\alpha + \beta$ for $\alpha, \beta \in F$. Choose any quasiquasifield (coordinate structure for the translation net associated with the quasifibration) and write the quasifibration as follows:*

$$x = 0, y = x \begin{bmatrix} g(t,u) & h(t,u) - \alpha g(t,u) = f(t,u) \\ t & u \end{bmatrix} \forall t, u \in F$$

where g, f and unique functions on $F \times F$ and h is defined as noted via α.

Define $F(\theta t + u) = -g(t,u)\theta + h(t,u)$.

Then

$$x = 0, y = x \begin{bmatrix} \theta t + u & F(\theta s + v) \\ \theta s + v & (\theta t + u)^\sigma \end{bmatrix} \forall t, u, s, v \in F$$

is a quasibration S^L in $PG(3,q^2)$ called the quasifibration 'algebraically lifted' from S.

Furthermore, S^L is a spread if and only if S is a spread.

We note that there is a derivable net

$$x = 0, y = x \begin{bmatrix} w & 0 \\ 0 & w^\sigma \end{bmatrix} \forall w \in K$$
$$\text{and } \sigma \text{ the involution in } GalF_K$$

which contains exactly two Baer subplanes which are K-subspaces.

Hence, we obtain:

7 Theorem. *Let π be any translation plane with spread (resp. quasifibration) in $PG(3,K)$ for K a field which admits a quadratic field extension F. Then, any spread-set representation of the spread (resp. quasifibration) of π may be lifted to a spread (resp. quasifibration) in $PG(3,K)$ which admits a derivable net D such that there are exactly two Baer subplanes of the net D that are K-subspaces. This construction produces maximal partial spreads (resp. quasifibrations) of type 2 for any field K admitting a quadratic field extension.*

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