Note di Matematica **21,** n. 1, 2002, 113–125.

Generalizations of Fibonacci and Lucas sequences

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Received: 9 September 2001; accepted: 20 March 2002.

Abstract. In this paper, we consider the Hecke groups $H(\sqrt{q})$, $q \geq 5$ prime number, and we find an interesting number sequence which is denoted by d_n. For $q = 5$, we get $d_{2n} = L_{2n+1}$ and $d_{2n+1} = \sqrt{5}F_{2n+2}$ where L_{2n+1} is $(2n+1)$th Lucas number and F_{2n+2} is $(2n+2)$th Fibonacci number. From this sequence, we obtain two new sequences which are, in a sense, generalizations of Fibonacci and Lucas sequences.

Keywords: Hecke groups, Fibonacci numbers, Lucas numbers.

MSC 2000 classification: 11B39, 20H10.

Introduction

Hecke groups $H(\lambda)$ are, in some sense, the generalizations of the well-known modular group

$$PSL(2,\mathbb{Z}) = \left\{ \frac{az+b}{cz+d} \mid a,b,c,d \in \mathbb{Z},\ ad-bc=1 \right\}.$$

They are the discrete subgroups of $PSL(2,\mathbb{R})$ (the group of orientation preserving isometries of the upper half plane U) generated by two linear fractional transformations

$$R(z) = -\frac{1}{z} \text{ and } T(z) = z + \lambda$$

where λ is a fixed positive real number. Hecke groups $H(\lambda)$ have been introduced by Erich Hecke, [4]. Hecke asked the question that for what values of λ these groups are discrete. In answering this question, he proved that $H(\lambda)$ has a fundamental region iff $\lambda \geq 2$ or $\lambda = \lambda_q = 2\cos(\frac{\pi}{q})$, $q \in \mathbb{N}$, $q \geq 3$. Therefore $H(\lambda)$ is discrete only for these values of λ.

The most interesting and worked Hecke group is the modular group $H(\lambda_3) = PSL(2,\mathbb{Z})$ obtained for $q = 3$. In this case all coefficients of the elements of $H(\lambda_3)$ are integers. The next two important Hecke groups are $H(\sqrt{2})$ and $H(\sqrt{3})$ obtained for $q = 4$ and 6, respectively. For these groups, the underlying fields are the quadratic extensions of $\mathbb{Q}$ by the algebraic numbers $\sqrt{2}$ and $\sqrt{3}$, that is,

$\mathbb{Q}\,(\sqrt{2})$ and $\mathbb{Q}\,(\sqrt{3})$. It is known that $H(\lambda_q)$ is isomorphic to the free product of two finite cyclic groups of orders 2 and q, i.e. $H(\lambda_q) \cong C_2 * C_q$, [1].

Here we are going to be interested in the case $\lambda \geq 2$. In this case, the element $S = RT$ is parabolic when $\lambda = 2$, or hyperbolic when $\lambda > 2$. It is known that when $\lambda \geq 2$, $H(\lambda)$ is a free product of two cyclic groups of orders 2 and infinity, [8], so all such $H(\lambda)$ have the same algebraic structure, i.e.

$$H(\lambda) \cong C_2 * \mathbb{Z}$$

so that the signature of $H(\lambda)$ is $(0; 2, \infty; 1)$. In particular, we deal with the case where $\lambda = \sqrt{q},\ q \geq 5$ prime number, and denote the group by $H(\sqrt{q})$.

On the other hand, Edouard Lucas (1842–1891) made a deep study of sequences which is called "generalized Fibonacci sequences", that begin with any two positive integers, each number thereafter being the sum of the preceding two. The simplest such series, $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$, is called the Fibonacci sequence by Lucas. The next simplest series, $2, 1, 3, 4, 7, 11, 18, \ldots$, is then called the Lucas numbers in his honor. The Fibonacci rule of adding the latest two to get the next is kept, but here we begin with 2 and 1 (in this order). The position of each number in this sequence is traditionally indicated by a subscript, so that $F_0 = 0,\ F_1 = 1,\ F_2 = 1,\ F_3 = 2$, and so on. F_n refers to nth Fibonacci number. Explicitly the Fibonacci sequence can be defined as follows:

$$F_n = F_{n-1} + F_{n-2},\ n \geq 2 \tag{1}$$

with two boundary conditions: $F_0 = 0,\ F_1 = 1$.

The Lucas sequence is defined as follows where we write its members as L_n, for Lucas:

$$\begin{aligned} L_n &= L_{n-1} + L_{n-2},\ n > 1 \\ L_0 &= 2 \\ L_1 &= 1. \end{aligned} \tag{2}$$

The nth Fibonacci number is in the following formula:

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]. \tag{3}$$

The formula that gives the nth number of the Lucas sequence exactly is

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n. \tag{4}$$

Lucas numbers have lots of properties similar to those of Fibonacci numbers and the Lucas numbers often occur in various formulas for the Fibonacci numbers. For example: The nth Lucas number is equal to $F_{n-1}+F_{n+1}$, i.e.

$$L_n = F_{n-1} + F_{n+1} \tag{5}$$

for all integers n. The product of F_n and L_n is equal to F_{2n}. Another recurrence relations for the Fibonacci and Lucas numbers are

$$F_{n+4} = 3F_{n+2} - F_n, \tag{6}$$

and

$$L_{n+4} = 3L_{n+2} - L_n. \tag{7}$$

For more details about the Fibonacci and Lucas sequences, see [2], [3], [5] and [7].

We came across an interesting number sequence which is denoted by d_n, when we were studying the principal congruence subgroups of the Hecke groups $H(\sqrt{q})$, $q \geq 5$ prime number. The sequence d_n is defined as follows:

$$\begin{aligned} d_n &= \sqrt{q}d_{n-1} - d_{n-2},\ n \geq 2 \\ d_0 &= 1,\ d_1 = \sqrt{q}. \end{aligned} \tag{8}$$

For $q = 5$, we get the sequence

$$\begin{aligned} d_n &= \sqrt{5}d_{n-1} - d_{n-2},\ n \geq 2 \\ d_0 &= 1,\ d_1 = \sqrt{5}. \end{aligned}$$

It is surprising that $d_{2n} = L_{2n+1}$ and $d_{2n+1} = \sqrt{5}F_{2n+2}$, and each d_n (for all q) has similar properties L_n and F_n. In some sense d_n's are generalizations of L_n and F_n including both of them for $q = 5$. In this paper we will introduce this sequence and get two new sequences from this. Firstly we will give a brief information about the Hecke group $H(\sqrt{q})$.

1 The Hecke Group $H(\sqrt{q})$

In the case $\lambda = \sqrt{q}$, $q \geq 5$ prime, the underlying field is a quadratic extension of $\mathbb{Q}$ by $\sqrt{q}$, i.e. $\mathbb{Q}(\sqrt{q})$. A presentation of $H(\sqrt{q})$ is

$$H(\sqrt{q}) = \langle R, S;\ R^2 = S^\infty = (RS)^\infty = 1 \rangle$$

where $S = RT$. By identifying the transformation $\frac{az+b}{cz+d}$ with the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $H(\sqrt{q})$ may be regarded as a multiplicative group of 2×2 matrices in which a matrix is identified with its negative. R and S have matrix representations

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -1 \\ 1 & \sqrt{q} \end{pmatrix},$$

respectively. All elements of $H(\sqrt{q})$ are of the following two types:

$$(i)\ \begin{pmatrix} a & b\sqrt{q} \\ c\sqrt{q} & d \end{pmatrix};\ a,b,c,d \in \mathbb{Z},\ ad - qbc = 1$$

$$(ii)\ \begin{pmatrix} a\sqrt{q} & b \\ c & d\sqrt{q} \end{pmatrix};\ a,b,c,d \in \mathbb{Z},\ qad - bc = 1.$$

But the converse is not true. That is, all elements of type (i) or (ii) need not be in $H(\sqrt{q})$ (see [6]). Those of type (i) are called even while those of type (ii) are called odd. The set of all even elements form a subgroup of index 2 called the even subgroup. It is denoted by $H_e(\sqrt{q})$, [9].

When we were determining the group structure of principal congruence subgroups of $H(\sqrt{q})$, we needed the powers of the transformation S which is one of the generators of $H(\sqrt{q})$. We were looking for the answer of the question that for what values of n satisfy

$$S^n \equiv \pm I \pmod{p},\ p \text{ is an odd prime},$$

we needed to compute nth power of S, for every integer n. Notice that the transformation S is hyperbolic with fixed points

$$z_1 = \frac{\sqrt{q-4} - \sqrt{q}}{2} \text{ and } z_2 = \frac{-\sqrt{q-4} - \sqrt{q}}{2} \tag{9}$$

and therefore of infinite order. It is hard to compute S^n easily. The next section we will try to compute nth power of S. Note that S is an odd element of $H(\sqrt{q})$, so odd powers of S are odd and even powers of S are even elements.

2 Powers of the Transformation S

In this section we will prove that

$$S^n = \begin{pmatrix} -d_{n-2} & -d_{n-1} \\ d_{n-1} & d_n \end{pmatrix}$$

by induction. Therefore the problem to compute S^n is reduced to compute the explicit formula of d_n.

Proposition 1. *The n th power of S is*

$$S^n = \begin{pmatrix} -d_{n-2} & -d_{n-1} \\ d_{n-1} & d_n \end{pmatrix}$$

where $d_0 = 1,\ d_1 = \sqrt{q}$ *and* $d_n = \sqrt{q}d_{n-1} - d_{n-2},$ *for* $n \geq 2.$

Proof. First we prove

$$S^n = \begin{pmatrix} b_{n-1} & -d_{n-1} \\ d_{n-1} & \sqrt{q}d_{n-1} + b_{n-1} \end{pmatrix}.$$

Remember that $S = \begin{pmatrix} 0 & -1 \\ 1 & \sqrt{q} \end{pmatrix}$. Let us write

$$S = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \quad \text{and} \quad S^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}.$$

Then we have

$$S^2 = \begin{pmatrix} 0 & -1 \\ 1 & \sqrt{q} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & \sqrt{q} \end{pmatrix} = \begin{pmatrix} -1 & -\sqrt{q} \\ \sqrt{q} & q-1 \end{pmatrix} = \begin{pmatrix} b_1 & -d_1 \\ d_1 & \sqrt{q}d_1 + b_1 \end{pmatrix}.$$

Hence assertion is true for $n = 2$. Let us assume that

$$S^{n-1} = \begin{pmatrix} b_{n-2} & -d_{n-2} \\ d_{n-2} & \sqrt{q}d_{n-2} + b_{n-2} \end{pmatrix}.$$

Then we have

$$\begin{aligned} S^n &= \begin{pmatrix} b_{n-2} & -d_{n-2} \\ d_{n-2} & \sqrt{q}d_{n-2} + b_{n-2} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & \sqrt{q} \end{pmatrix} \\ &= \begin{pmatrix} -d_{n-2} & -(\sqrt{q}d_{n-2} + b_{n-2}) \\ \sqrt{q}d_{n-2} + b_{n-2} & \sqrt{q}(\sqrt{q}d_{n-2} + b_{n-2}) - d_{n-2} \end{pmatrix} \\ &= \begin{pmatrix} b_{n-1} & -d_{n-1} \\ d_{n-1} & \sqrt{q}d_{n-1} + b_{n-1} \end{pmatrix}. \end{aligned}$$

Notice that $b_2 = -d_1,\ b_{n-1} = -d_{n-2}$ and $b_n = -d_{n-1}$. If we write $d_0 = 1$, we get $b_1 = -d_0$ and hence we proved that

$$S^n = \begin{pmatrix} -d_{n-2} & -d_{n-1} \\ d_{n-1} & \sqrt{q}d_{n-1} - d_{n-2} \end{pmatrix}.$$

$\boxed{QED}$

Thus we have a real number sequence which can be defined as

$$\begin{aligned} d_n &= \sqrt{q}d_{n-1} - d_{n-2},\ n \geq 2 \\ d_0 &= 1,\ d_1 = \sqrt{q}. \end{aligned} \tag{10}$$

Now we have to compute d_n.

Proposition 2. *d_n is in the following formula for all n:*

$$d_n = \frac{1}{\sqrt{q-4}}\left[\left(\frac{\sqrt{q}+\sqrt{q-4}}{2}\right)^{n+1} - \left(\frac{\sqrt{q}-\sqrt{q-4}}{2}\right)^{n+1}\right]. \tag{11}$$

Proof. To solve the equation (10), let d_n to be a characteristic polynomial r^n, which appears to quickly reduce (10) to

$$r^n = \sqrt{q}r^{n-1} - r^{n-2} \Rightarrow r^2 - \sqrt{q}r + 1 = 0$$

with the roots

$$r_{1,2} = \frac{\sqrt{q} \pm \sqrt{q-4}}{2}.$$

The general solution of the equation $d_n = r^n$ will be all possible combinations of roots r_1 and r_2. Let us write

$$d_n = A\left(\frac{\sqrt{q}+\sqrt{q-4}}{2}\right)^n + B\left(\frac{\sqrt{q}-\sqrt{q-4}}{2}\right)^n.$$

Constants A and B can be found from the boundary conditions $d_0 = 1$ and $d_1 = \sqrt{q}$. We have

$$\begin{aligned} d_0 &= 1 = A + B, \\ d_1 &= \sqrt{q} = A\left(\frac{\sqrt{q}+\sqrt{q-4}}{2}\right) + B\left(\frac{\sqrt{q}-\sqrt{q-4}}{2}\right) \end{aligned}$$

and so

$$2\sqrt{q} = A(\sqrt{q}+\sqrt{q-4}) + (1-A)(\sqrt{q}-\sqrt{q-4}).$$

Hence we find

$$A = \frac{\sqrt{q}+\sqrt{q-4}}{2\sqrt{q-4}} \text{ and } B = \frac{\sqrt{q-4}-\sqrt{q}}{2\sqrt{q-4}}.$$

Then we get the formula of d_n as follows:

$$\begin{aligned} d_n &= \left(\frac{\sqrt{q}+\sqrt{q-4}}{2\sqrt{q-4}}\right)\left(\frac{\sqrt{q}+\sqrt{q-4}}{2}\right)^n + \\ &\qquad + \left(\frac{\sqrt{q-4}-\sqrt{q}}{2\sqrt{q-4}}\right)\left(\frac{\sqrt{q}-\sqrt{q-4}}{2}\right)^n \\ &= \frac{1}{\sqrt{q-4}}\left[\left(\frac{\sqrt{q}+\sqrt{q-4}}{2}\right)^{n+1} - \left(\frac{\sqrt{q}-\sqrt{q-4}}{2}\right)^{n}\right]. \end{aligned}$$

$\boxed{QED}$

For $q=5$, we arrive at familiar expressions for Fibonacci and Lucas numbers. We have $d_{2n}=L_{2n+1}$, $(2n+1)$th Lucas number, and $d_{2n+1}=\sqrt{5}F_{2n+2}$, $(2n+2)$th Fibonacci number. Indeed, from (11) we get the sequence

$$d_n=\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{\sqrt{5}-1}{2}\right)^{n+1}.$$

If n is even, then $n+1$ is odd and hence we can write

$$d_n=\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}+\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}=L_{n+1}.$$

If n is odd, then $n+1$ is even and we can write

$$d_n=\sqrt{5}\left[\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)\right]=\sqrt{5}F_{n+1}.$$

So we get

$$S^{2n}=\begin{pmatrix}-L_{2n-1} & -F_{2n}\sqrt{5}\\ F_{2n}\sqrt{5} & L_{2n+1}\end{pmatrix}$$

and

$$S^{2n+1}=\begin{pmatrix}-F_{2n}\sqrt{5} & -L_{2n+1}\\ L_{2n+1} & F_{2n+2}\sqrt{5}\end{pmatrix}$$

in the Hecke group $H(\sqrt{q})$.

In general, the sequence d_n has similar properties of F_n and L_n for all q.

Proposition 3. *We have*

$$d_{2n}=(q-2)d_{2n-2}-d_{2n-4} \tag{12}$$

and

$$d_{2n+1}=(q-2)d_{2n-1}-d_{2n-3}. \tag{13}$$

Proof. By (10), we get $d_{2n-3}=\sqrt{q}d_{2n-4}-d_{2n-5}$ and so $d_{2n-3}+d_{2n-5}=$

$\sqrt{q}d_{2n-4}$. Using this last equation and (10), we obtain

$$\begin{aligned}
d_{2n} &= \sqrt{q}d_{2n-1} - d_{2n-2} \\
&= \sqrt{q}(\sqrt{q}d_{2n-2} - d_{2n-3}) - d_{2n-2} \\
&= (q-1)d_{2n-2} - \sqrt{q}d_{2n-3} \\
&= (q-1)d_{2n-2} - \sqrt{q}(\sqrt{q}d_{2n-4} - d_{2n-5}) \\
&= (q-1)d_{2n-2} - qd_{2n-4} + \sqrt{q}d_{2n-5} \\
&= (q-2)d_{2n-2} + (\sqrt{q}d_{2n-3} - d_{2n-4}) - qd_{2n-4} + \sqrt{q}d_{2n-5} \\
&= (q-2)d_{2n-2} - d_{2n-4} - qd_{2n-4} + \sqrt{q}(d_{2n-3} + d_{2n-5}) \\
&= (q-2)d_{2n-2} - d_{2n-4} - qd_{2n-4} + \sqrt{q}(\sqrt{q}d_{2n-4}) \\
&= (q-2)d_{2n-2} - d_{2n-4}.
\end{aligned}$$

Similarly, from (10), we find $d_{2n-2} + d_{2n-4} = \sqrt{q}d_{2n-3}$ and then

$$\begin{aligned}
d_{2n+1} &= \sqrt{q}d_{2n} - d_{2n-1} \\
&= \sqrt{q}(\sqrt{q}d_{2n-1} - d_{2n-2}) - d_{2n-1} \\
&= (q-1)d_{2n-1} - \sqrt{q}d_{2n-2} \\
&= (q-1)d_{2n-1} - \sqrt{q}(\sqrt{q}d_{2n-3} - d_{2n-4}) \\
&= (q-1)d_{2n-1} - qd_{2n-3} + \sqrt{q}d_{2n-4} \\
&= (q-2)d_{2n-1} + (\sqrt{q}d_{2n-2} - d_{2n-3}) - qd_{2n-3} + \sqrt{q}d_{2n-4} \\
&= (q-2)d_{2n-1} - d_{2n-3} - qd_{2n-3} + \sqrt{q}(d_{2n-2} + d_{2n-4}) \\
&= (q-2)d_{2n-1} - d_{2n-3} - qd_{2n-3} + \sqrt{q}(\sqrt{q}d_{2n-3}) \\
&= (q-2)d_{2n-1} - d_{2n-3}.
\end{aligned}$$

$\boxed{QED}$

Note that for $q = 5$, we have

$$d_{2n} = 3d_{2n-2} - d_{2n-4}$$

and

$$d_{2n+1} = 3d_{2n-1} - d_{2n-3},$$

that is, we have $L_{2n+1} = 3L_{2n-1} - L_{2n-3}$ and $\sqrt{5}F_{2n+2} = 3\sqrt{5}F_{2n} - \sqrt{5}F_{2n-2}$, so $F_{2n+2} = 3F_{2n} - F_{2n-2}$ which are coincide (7) and (6), respectively.

From d_n we have two subsequences. Indeed, let us consider the polynomials

$$\begin{aligned}
u_0 &= 1,\ u_1 = q-2 \\
u_n &= (q-2)u_{n-1} - u_{n-2},\ n \geq 2.
\end{aligned} \tag{14}$$

Then we have

$$d_{2n+1} = u_n\sqrt{q}. \tag{15}$$

Now we will prove this by induction. As $d_1 = u_0\sqrt{q},\ d_3 = u_1\sqrt{q}$, assume that $d_{2n-1} = d_{2(n-1)+1} = u_{n-1}\sqrt{q}$. From (13), we have

$$\begin{aligned} d_{2n+1} &= (q-2)d_{2n-1} - d_{2n-3} = (q-2)d_{2(n-1)+1} - d_{2(n-2)+1} \\ &= (q-2)u_{n-1}\sqrt{q} - u_{n-2}\sqrt{q} = [(q-2)u_{n-1} - u_{n-2}]\sqrt{q} \\ &= u_n\sqrt{q}. \end{aligned}$$

Similarly, if we consider the polynomials

$$\begin{aligned} v_0 &= 1,\ v_1 = q-1 \\ v_n &= (q-2)v_{n-1} - v_{n-2},\ n \geq 2, \end{aligned} \tag{16}$$

we have

$$d_{2n} = v_n. \tag{17}$$

Indeed, as we have $d_0 = v_0,\ d_2 = v_1$, assume that $d_{2n-2} = d_{2(n-1)} = v_{n-1}$. From (12), we have

$$\begin{aligned} d_{2n} &= (q-2)d_{2n-2} - d_{2n-4} = (q-2)d_{2(n-1)} - d_{2(n-2)} \\ &= (q-2)v_{n-1} - v_{n-2} = v_n. \end{aligned}$$

Note that for $q = 5$, v_n's are Lucas numbers with odd index and u_n's are Fibonacci numbers with even index. Now we want to generalize the sequences u_n and v_n. First consider the sequence u_n and generalize into a sequence of which elements of even index are u_n's. Let us define

$$\mathcal{U}_{2n+2} = u_n$$

and compute $\mathcal{U}_n$ using the equalities

$$\begin{aligned} \mathcal{U}_{2n} &= \mathcal{U}_{2n-1} + \mathcal{U}_{2n-2} \\ \mathcal{U}_{2n+1} &= (q-2)\mathcal{U}_{2n-1} - \mathcal{U}_{2n-3}. \end{aligned}$$

Then we have

$$\begin{aligned} \mathcal{U}_0 &= 0 \\ \mathcal{U}_1 &= 1 \\ \mathcal{U}_2 &= 1 \\ \mathcal{U}_3 &= q-3 \\ \mathcal{U}_4 &= q-2 \\ \mathcal{U}_5 &= q^2-5q+5 \\ \mathcal{U}_6 &= q^2-4q+3 \\ &\cdots \end{aligned}$$

For even indexed elements of the sequence, the condition

$$\mathcal{U}_{2n} = \mathcal{U}_{2n-1} + \mathcal{U}_{2n-2} \tag{18}$$

is hold. We will investigate whether this condition holds for odd indexed elements or not. From $\mathcal{U}_{2n+2} = \mathcal{U}_{2n+1} + \mathcal{U}_{2n}$, we have

$$\begin{aligned}
\mathcal{U}_{2n+1} &= \mathcal{U}_{2n+2} - \mathcal{U}_{2n} \\
&= (q-2)\mathcal{U}_{2n} - \mathcal{U}_{2n-2} - \mathcal{U}_{2n} \\
&= (q-3)\mathcal{U}_{2n} - \mathcal{U}_{2n-2} = (q-3)(\mathcal{U}_{2n-1} + \mathcal{U}_{2n-2}) - \mathcal{U}_{2n-2} \\
&= (q-3)\mathcal{U}_{2n-1} + (q-4)\mathcal{U}_{2n-2} \\
&= \mathcal{U}_{2n-1} + (q-4)(\mathcal{U}_{2n-1} + \mathcal{U}_{2n-2}) \\
&= \mathcal{U}_{2n-1} + (q-4)\mathcal{U}_{2n}.
\end{aligned}$$

Therefore only for $q = 5$, we have the Fibonacci sequence.

Let us now generalize v_n into a sequence of which elements of odd index are v_n's. Similarly, let us define

$$\mathcal{V}_{2n+1} = v_n$$

and compute $\mathcal{V}_n$ using the equalities

$$\begin{aligned}
\mathcal{V}_{2n+1} &= \mathcal{V}_{2n} + \mathcal{V}_{2n-1} \\
\mathcal{V}_{2n} &= (q-2)\mathcal{V}_{2n-2} - \mathcal{V}_{2n-4}.
\end{aligned}$$

Then we have

$$\begin{aligned}
\mathcal{V}_0 &= 2 \\
\mathcal{V}_1 &= 1 \\
\mathcal{V}_2 &= q-2 \\
\mathcal{V}_3 &= q-1 \\
\mathcal{V}_4 &= q^2 - 4q + 2 \\
\mathcal{V}_5 &= q^2 - 3q + 1 \\
&\dots
\end{aligned}$$

For odd indexed elements of the sequence, the condition

$$\mathcal{V}_{2n+1} = \mathcal{V}_{2n} + \mathcal{V}_{2n-1} \tag{19}$$

is hold. Again similarly an easy calculation shows that

$$\mathcal{V}_{2n} = (q-4)\mathcal{V}_{2n-1} + \mathcal{V}_{2n-2}.$$

So only for $q = 5$, we get the Lucas sequence.

Therefore we get two new sequences, $\mathcal{U}_n$ and $\mathcal{V}_n$, which are not generalized Fibonacci sequences except $q = 5$. Explicitly $\mathcal{U}_n$ can be defined as follows:

$$\mathcal{U}_n = (q-2)\mathcal{U}_{n-2} - \mathcal{U}_{n-4}, n \geq 4 \tag{20}$$

with the boundary conditions: $\mathcal{U}_0 = 0, \mathcal{U}_1 = 1, \mathcal{U}_2 = 1$ and $\mathcal{U}_3 = q-3$. $\mathcal{V}_n$ can be defined as follows:

$$\mathcal{V}_n = (q-2)\mathcal{V}_{n-2} - \mathcal{V}_{n-4}, n \geq 4 \tag{21}$$

with the boundary conditions: $\mathcal{V}_0 = 2, \mathcal{V}_1 = 1, \mathcal{V}_2 = q-2$ and $\mathcal{V}_3 = q-1$. For $q = 5$, we have polynomial representations of Fibonacci and Lucas numbers. In a sense, $\mathcal{U}_n$ is a generalization of F_n and $\mathcal{V}_n$ is a generalization of L_n. Furthermore, the sequence d_n defined in (10), contains both of even indexed elements of $\mathcal{U}_n$ and of odd indexed elements of $\mathcal{V}_n$. The sequences $\mathcal{U}_n$ and $\mathcal{V}_n$ have similar properties of Fibonacci and Lucas numbers F_n and L_n. For example we have the following propositions:

Proposition 4.

$$\mathcal{V}_n = \mathcal{U}_{n+1} + \mathcal{U}_{n-1}. \tag{22}$$

Proof. Since $\mathcal{V}_1 = \mathcal{U}_0 + \mathcal{U}_2 = 0+1 = 1$ and $\mathcal{V}_2 = \mathcal{U}_1 + \mathcal{U}_3 = 1+q-3 = q-2$, assume that the identity is true for $n = 1, 2, 3, \ldots, k-1$ and show that it holds for $n = k$. We have

$$\begin{aligned}
\mathcal{V}_k &= (q-2)\mathcal{V}_{k-2} - \mathcal{V}_{k-4} = (q-2)(\mathcal{U}_{k-1} + \mathcal{U}_{k-3}) - (\mathcal{U}_{k-5} + \mathcal{U}_{k-3}) \\
&= (q-2)\mathcal{U}_{k-1} - \mathcal{U}_{k-3} + (q-2)\mathcal{U}_{k-3} - \mathcal{U}_{k-5} \\
&= \mathcal{U}_{k+1} + \mathcal{U}_{k-1}.
\end{aligned}$$

$\boxed{QED}$

Proposition 5. *The product of $\mathcal{U}_n$ and $\mathcal{V}_n$ is equal to $\mathcal{U}_{2n}$, that is,*

$$\mathcal{U}_{2n} = \mathcal{U}_n \mathcal{V}_n. \tag{23}$$

Proof. As we have $\mathcal{U}_2 = 1 = \mathcal{U}_1\mathcal{V}_1$ and $\mathcal{U}_4 = q-2 = \mathcal{U}_2\mathcal{V}_2$, assume that $\mathcal{U}_{2n-2} = \mathcal{U}_{n-1}\mathcal{V}_{n-1}$. Now we can compute $\mathcal{U}_n\mathcal{V}_n$. From (22), we have $\mathcal{U}_n\mathcal{V}_n = \mathcal{U}_n(\mathcal{U}_{n+1} + \mathcal{U}_{n-1})$ and using definitions we find

$$\begin{aligned}
\mathcal{U}_n\mathcal{V}_n &= \mathcal{U}_n((q-2)\mathcal{U}_{n-1} - \mathcal{U}_{n-3}) + \mathcal{U}_{n-1}((q-2)\mathcal{U}_{n-2} - \mathcal{U}_{n-4}) \\
&= (q-2)\mathcal{U}_n\mathcal{U}_{n-1} + (q-2)\mathcal{U}_{n-1}\mathcal{U}_{n-2} - \mathcal{U}_n\mathcal{U}_{n-3} - \mathcal{U}_{n-1}\mathcal{U}_{n-4} \\
&= (q-2)\mathcal{U}_{n-1}(\mathcal{U}_n + \mathcal{U}_{n-2}) - \mathcal{U}_n\mathcal{U}_{n-3} - \mathcal{U}_{n-1}\mathcal{U}_{n-4} \\
&= (q-2)\mathcal{U}_{n-1}\mathcal{V}_{n-1} - \mathcal{U}_n\mathcal{U}_{n-3} - \mathcal{U}_{n-1}\mathcal{U}_{n-4}.
\end{aligned}$$

If n is even, we know that $\mathcal{U}_n = \mathcal{U}_{n-1} + \mathcal{U}_{n-2}$. Using this and $n-2$ is even, we have

$$\begin{aligned}
\mathcal{U}_n\mathcal{V}_n &= (q-2)\mathcal{U}_{n-1}\mathcal{V}_{n-1} - \mathcal{U}_{n-3}(\mathcal{U}_{n-1}+\mathcal{U}_{n-2}) - \mathcal{U}_{n-1}(\mathcal{U}_{n-2}-\mathcal{U}_{n-3}) \\
&= (q-2)\mathcal{U}_{n-1}\mathcal{V}_{n-1} - \mathcal{U}_{n-3}\mathcal{U}_{n-1} - \mathcal{U}_{n-2}(\mathcal{U}_{n-3}+\mathcal{U}_{n-1}) + \mathcal{U}_{n-3}\mathcal{U}_{n-1} \\
&= (q-2)\mathcal{U}_{n-1}\mathcal{V}_{n-1} - \mathcal{U}_{n-2}\mathcal{V}_{n-2}.
\end{aligned}$$

By the assumption, we get that

$$\mathcal{U}_n\mathcal{V}_n = (q-2)\mathcal{U}_{2n-2} - \mathcal{U}_{2n-4} = \mathcal{U}_{2n}.$$

If n is odd, we know that $\mathcal{U}_n = (q-4)\mathcal{U}_{n-1} + \mathcal{U}_{n-2}$. Since $n-2$ is odd, we have $\mathcal{U}_{n-2} = (q-4)\mathcal{U}_{n-3} + \mathcal{U}_{n-4}$. Hence, we obtain

$$\begin{aligned}
\mathcal{U}_n\mathcal{V}_n &= (q-2)\mathcal{U}_{n-1}\mathcal{V}_{n-1} - \mathcal{U}_{n-3}((q-4)\mathcal{U}_{n-1}+\mathcal{U}_{n-2}) \\
&\qquad - \mathcal{U}_{n-1}(\mathcal{U}_{n-2}-(q-4)\mathcal{U}_{n-3}) \\
&= (q-2)\mathcal{U}_{n-1}\mathcal{V}_{n-1} - \mathcal{U}_{n-2}(\mathcal{U}_{n-3}+\mathcal{U}_{n-1}) - \mathcal{U}_{n-3}\mathcal{U}_{n-1}(q-4) \\
&\qquad + \mathcal{U}_{n-3}\mathcal{U}_{n-1}(q-4) \\
&= (q-2)\mathcal{U}_{n-1}\mathcal{V}_{n-1} - \mathcal{U}_{n-2}\mathcal{V}_{n-2}.
\end{aligned}$$

Then by assumption, we have

$$\mathcal{U}_n\mathcal{V}_n = (q-2)\mathcal{U}_{2n-2} - \mathcal{U}_{2n-4} = \mathcal{U}_{2n}.$$

Therefore assertion is true for all n. $\boxed{QED}$

Finally we have the following theorem:

Theorem 1. *In the Hecke group $H(\sqrt{q})$, $q \geq 5$ prime, the nth powers of S are obtained as the following forms:*

$$S^{2n} = \begin{pmatrix} -\mathcal{V}_{2n-1} & -\mathcal{U}_{2n}\sqrt{q} \\ \mathcal{U}_{2n}\sqrt{q} & \mathcal{V}_{2n+1} \end{pmatrix}$$

and

$$S^{2n+1} = \begin{pmatrix} -\mathcal{U}_{2n}\sqrt{q} & -\mathcal{V}_{2n+1} \\ \mathcal{V}_{2n+1} & \mathcal{U}_{2n+2}\sqrt{q} \end{pmatrix}.$$

Proof. From Proposition 1, we know that

$$S^n = \begin{pmatrix} -d_{n-2} & -d_{n-1} \\ d_{n-1} & d_n \end{pmatrix}.$$

By the definitions $\mathcal{U}_n$ and $\mathcal{V}_n$ and the equalities (15) and (17), we have

$$d_{2n+1} = u_n\sqrt{q}, \text{ so } d_{2n+1} = \mathcal{U}_{2n+2}\sqrt{q}$$

and

$$d_{2n} = v_n, \text{ so } d_{2n} = \mathcal{V}_{2n+1}.$$

Thus we have $d_{2n-2} = v_{n-1} = \mathcal{V}_{2n-1}$ and $d_{2n-1} = u_{n-1}\sqrt{q} = \mathcal{U}_{2n}\sqrt{q}$. Therefore, as we have

$$S^{2n} = \begin{pmatrix} -d_{2n-2} & -d_{2n-1} \\ d_{2n-1} & d_{2n} \end{pmatrix},$$

we find

$$S^{2n} = \begin{pmatrix} -\mathcal{V}_{2n-1} & -\mathcal{U}_{2n}\sqrt{q} \\ \mathcal{U}_{2n}\sqrt{q} & \mathcal{V}_{2n+1} \end{pmatrix}.$$

Similarly, we have

$$S^{2n+1} = \begin{pmatrix} -d_{2n-1} & -d_{2n} \\ d_{2n} & d_{2n+1} \end{pmatrix}$$

and so,

$$S^{2n+1} = \begin{pmatrix} -\mathcal{U}_{2n}\sqrt{q} & -\mathcal{V}_{2n+1} \\ \mathcal{V}_{2n+1} & \mathcal{U}_{2n+2}\sqrt{q} \end{pmatrix}.$$

$\boxed{QED}$

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