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Derivable pseudo-nets

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Abstract. It is shown that a derivable pre pseudo-net is a derivable net.

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1 Pseudo-Nets.

When considering the derivation of affine planes, the main issue is the location of a derivable net within the plane. Considered apart from any particular plane containing it, a study of the incidence geometry of a derivable net reveals that such a net can always be embedded into some affine plane but it may be the case that the affine plane is not itself derivable. That this is possible is considered in the second author's works [2] and [3]. A non-derivable affine plane containing a derivable net does not derive to an affine plane, but it does derive to a linear space with parallelism; a Sperner space. In fact, any infinite derivable net may be embedded into a non-derivable dual translation plane and upon derivation, there is a corresponding new Sperner space.

Similarly, in the first author's article [1], a generalization of derivation is given that applies potentially to arbitrary affine spaces. In particular, there are new constructions of Sperner spaces.

In both of the above constructions of linear spaces with parallelism, the constructed incidence geometries have lines of different parallel classes that do not always intersect. However, lines from different parallel classes of the original derived derivable net do, in fact, intersect. This brings up the following question:

Is there a derivation process involving partial linear spaces with parallelisms where lines of different parallel classes of the derivable substructure may not intersect? This gives rise to the possibility of a derivation procedure using what might be called 'derivable pseudo-nets', structures where lines from distinct parallel classes may not intersect.

This note concerns the above question or alternatively considers if the requirements for derivation could be simplified or reduced.

In this note, we make strict distinction between structures, 'nets', which do assume that two non-parallel lines must intersect and the more general structures, 'partial linear spaces with parallelism', which do not. We also adopt some terminology for such general incidence structures more in keeping with the spirit of net theory.

Definition 1. We adopt the term 'pre pseudo-net' for the general incidence structure of points and lines such that

(i) two points are incident with at most one line,

(ii) each line (respectively, point) has at least one point (respectively, line) incident with it and

(iii) there is a parallelism on the set of lines such that each point is incident with exactly one line of each parallel class. If (α) is a parallel class and p is a point, we shall adopt the notation that the unique line of the parallel class (α) incident with p is denoted by $p\alpha$.

A 'pseudo-net' is a pre pseudo-net such that every line is incident with at least two points and every point is incident with at least two lines. A pseudo-net is also called a 'partial linear space with parallelism'.

A 'closed (pre) pseudo-net' is a (pre) pseudo-net such that any two distinct points are incident with a unique line. Hence, a closed pseudo-net is also a 'linear space with parallelism'.

A 'net' is a pseudo-net such that two non-parallel lines intersect.

Remark 1. Therefore, we may speak of 'derivable pre pseudo-nets' as pre pseudo-nets equipped with a set of Baer subplanes such any two distinct collinear points are incident with a Baer subplane containing them. In this case, we retain the definition of 'Baer subplane' of a pre pseudo-net as a non-trivial affine subplane of the incidence structure which is both 'point-Baer' (every point is incident with a line of the subplane) and 'line-Baer' (every line is incident with a point of the projective extension of the subplane), the latter of which is immediate if we assume that the parallel classes are exactly those of each Baer subplane.

Specifically, we make our definitions concrete as follows.

Definition 2. A 'replaceable (pre) pseudo-net' is a (pre) pseudo-net N such that there exists a corresponding pseudo-net N^* defined on the points of N such that points p and q are incident in N if and only if p and q are incident in N^*. N^* is called a 'replacement' for N.

A 'derivable pre pseudo-net' is a replaceable pre pseudo-net N with a corresponding replacement pre pseudo-net whose lines consist of Baer subplanes of N; distinct points p and q incident in N are incident with a Baer subplane $B(p,q)$ of N.

Theorem 1. *Let D be a pseudo-net containing a replaceable pseudo-net N with replacement pseudo-net N^*. Then, $D^* = (D - N) \cup N^*$ is a pseudo-net.*

Proof. We may define a parallelism for D^* as follows: Given two lines Z and R, they are parallel if and only if Z and R are either both in $D - N$ or in N^* and are parallel in D or N^* respectively. To show that we end up with a pseudo-net, we need to show that every point is incident with a unique line of each parallel class, which follows immediately. $\boxed{QED}$

The question then becomes: Are there derivable pre pseudo-nets which are not, in fact, derivable nets?

Is any derivable pre pseudo-net a (derivable) net?

The main point of this note is to show that these two concepts of derivable nets and derivable pre pseudo-nets are completely equivalent. Our following main result establishes this result.

Theorem 2. *Let $\mathcal{D} =(P,L,C,B)$ be an incidence structure of nonempty sets of 'points' P, 'lines' L (each of which is incident with a nonempty set of points), nonempty sets of 'parallel classes' C and 'Baer subplanes' B (each of which is a non-degenerate affine plane) each with parallel class set C such that*

(1) two distinct points are incident with at most one line,

(2) each line is incident with at least one point,

(3) each point is incident with a unique line of each parallel class, and

(4) each pair of distinct collinear points p and q are incident with a Baer subplane (non-degenerate affine plane) $B(p,q)$.

Statements (1) through (4) simply state that $\mathcal{D}$ is a derivable pre pseudo-net.

Then any two lines of distinct parallel classes intersect in a unique point and there are at least two points per line.

A derivable pre pseudo-net is a (derivable) net.

We shall structure our proof as a series of lemmas. In the following lemmas, we assume the hypothesis in the statement of the theorem.

Lemma 1. *Let (α) be any parallel class of C and let t be any point. If t is a point of a Baer subplane π_o then the unique line $t\alpha$ in the parallel class (α) and incident with t is a line of π_o.*

Proof. Since the parallel class set of π_o is also C, it follows that if t is a point of π_o then $t\alpha$ is a line of π_o. $\boxed{QED}$

Lemma 2. *Every line J of L is incident with at least two points.*

Proof. We have assumed that J contains a point t. Let J be in the parallel class (α). Since the set B of Baer subplanes is non-empty, let π_o be an element of B. Then, since π_o is Baer, there is a line N of π_o incident with t. Since N is a line of a non-degenerate affine plane, there are at least two distinct points s, t incident with N. By our hypotheses, there is a Baer subplane $B(s,t)$ containing the points s and t. By lemma 1, $t\alpha$ is a line of $B(s,t)$ and since $B(s,t)$ is non-degenerate, $t\alpha = J$ is incident with at least two points.

Lemma 3. *Let M_i be lines from distinct parallel classes (α_i), respectively, for $i = 1, 2$, and assume that M_1 and M_2 do not intersect in a point. Let p_i, q_i be points of M_i and let $B(p_i, q_i)$ be Baer subplanes containing p_i, q_i, for $i = 1, 2$.*

Then, any line N_j of $B(p_j, q_j)$ incident with p_i or q_i is a line of $B(p_i, q_i)$ parallel to M_j, for $i \neq j$, $i, j = 1, 2$.

$\boxed{QED}$

Proof. Without loss of generality, assume that a line N_2 of $B(p_2, q_2)$ is incident with p_1. Note that if N_2 is in the parallel class (δ_2) then $N_2 = p_1\delta_2$, implying that N_2 is a line of $B(p_1, q_1)$ by lemma 1. Now suppose that p_1 is a point of $B(p_2, q_2)$. Then $M_1 = p_1\alpha_1$ and M_2 are lines of an affine plane and hence must intersect or are parallel. Since M_1 and M_2 do not intersect and are not parallel then p_1 is not a point of $B(p_2, q_2)$ and there is a unique such line N_2. Suppose that N_2 is not parallel to M_2. Then N_2 intersects M_2 in an point w_2 in $B(p_2, q_2)$, as both lines are lines of this subplane. Since now w_2 and p_1 are collinear on N_2, form the Baer subplane $B(w_2, p_1)$. Then the lines M_1 and M_2 are lines of $B(w_2, p_1)$, as $M_2 = w_2\alpha_2$ and $M_1 = p_1\alpha_1$, by lemma 1. Hence, M_1 and M_2 intersect in a point, which is contrary to our assumptions. Thus, N_2 is parallel to M_2, which completes the proof of the lemma. $\boxed{QED}$

Lemma 4. *Under the assumptions of lemma 3, if, for either $j = 1$ or 2, there is a point z_j in $M_j - B(p_j, q_j)$, then the line N_j is a line of both $B(p_j, q_j)$ and $B(z_j, p_j)$, for $i, j = 1, 2$. Hence, we have two distinct Baer subplanes sharing a common point t_j and containing two distinct parallel lines N_j and M_j.*

Proof. Assume, without loss of generality, that there is a point z_2 in $M_2 - B(p_2, q_2)$.

Now from lemma 3, N_2 is a line of $B(p_2, q_2)$ and $B(p_1, q_1)$, which is parallel to M_2, and we consider N_2 incident with p_1. Since $z_2p_2 = M_2$, we may consider the Baer subplane $B(z_2, p_2)$. Let N_2^* be a line of $B(z_2, p_2)$ that is incident with p_1. Then, by lemma 3 applied to $B(z_2, p_2)$, N_2^* is parallel to M_2. But, N_2 and N_2^* are lines incident with p_1 and parallel to M_2 so that $N_j = N_j^*$. Hence, we have two Baer subplanes $B(p_2, q_2)$ and $B(z_2, p_2)$ sharing two parallel lines N_2 and M_2 and having a common point p_2. These two subplane are distinct since z_2 is not in $B(p_2, q_2)$. This proves the lemma. $\boxed{QED}$

Lemma 5. *Two Baer subplanes that contain a common point and two distinct parallel lines are identical.*

Proof. Since the Baer subplanes π_o and π_1 are (non-degenerate) affine planes, there are at least three parallel classes, say (α_i) for $i = 1, 2, 3$. Let the common parallel lines M and N be incident with a parallel class labeled (α_1). Two lines belonging to a Baer subplane must intersect or are parallel. Let the common point be denoted by t. Hence, the parallel lines $t\alpha_i$ of (α_i) are lines common to both π_o and π_1 for $i = 1, 2, 3$. Two of these lines cannot be parallel to N and must intersect N in at least two distinct points. First assume that t is incident with N and hence not incident with M. Then $t\alpha_i$ intersects M in exactly two distinct points, for $i = 2, 3$, say f and g respectively. It now follows that since M and N are lines common to both subplanes, these points of intersection must be in both subplanes. Also, the point $f\alpha_3 \cap N = e$ is distinct from t since otherwise $f\alpha_3 = t\alpha_3 = g\alpha_3$.

Now assume that t is not incident with either N or M. Then $t\alpha_i$ for $i = 2, 3$ intersect N in distinct points t' and e respectively and intersect M in distinct points f and g, respectively.

In either of the two above cases, the points of intersection are common points of the two Baer subplanes π_o and π_1. Hence, in any case, it also follows that π_o and π_1 share at least four distinct points, say t, e, f, g and the four points contain a triangle, say t, e, f.

Let r be a point of π_o and, without loss of generality, assume that r is not incident with te. Then t and r are collinear so that $tr = t\delta$ for some parallel class (δ) and e and r are collinear so that $er = e\rho$ for some parallel class (ρ). But, by lemma 1, it follows that tr and er are lines of both Baer subplanes so that r is also a point of π_1. Hence, the two subplanes are equal, thus completing the proof of our lemma. $\boxed{QED}$

Lemma 6. *Under the assumptions of lemma 3, we have:*

(i) All points of M_i are points of $B(p_i, q_i)$, for $i = 1, 2$.

(ii) Furthermore, all lines of $B(p_j, q_j)$ that are incident with points of M_i are lines of $B(p_i, q_i)$ that are parallel to M_j, for $i \neq j$, $i, j = 1, 2$.

Proof. Assume that z_2 is a point of $M_2 - B(p_2, q_2)$. Then, from lemma 4, we have two distinct Baer subplanes sharing a point and two distinct parallel lines, which is contrary to lemma 5. This proves (i). Since the points of M_i are points of a Baer subplane $B(p_i, q_i)$, any line on one of these points is a line of that subplane by lemma 1. $\boxed{QED}$

We are now able to give a proof to our main theorem.

Proof. We now claim that the two Baer subplanes $B(p_1, q_1)$ and $B(p_2, q_2)$ share two common parallel lines and a common point and hence are identical by lemma 5. To see this, note there are at least two points of M_1 and any line N_2 of $B(p_2, q_2)$ incident with one of these points is parallel to M_2 and is a line common to both subplanes by lemma 1. Hence, we have at least two common parallel lines of both subplanes on (α_1) and on (α_2). However, the lines from distinct parallel classes that are in both Baer subplanes must intersect so we have a common point of the two subplanes. Hence, we have verified the assertion. But, lemma 6 shows that the two Baer subplanes are identical, so that M_1 and M_2 are lines in distinct parallel classes of the same Baer subplane; the two lines intersect in a point.

This completes the proof of the theorem. $\boxed{QED}$

It now follows that the definition of a derivable 'net' may be considerable strengthened.

Remark 2. So, to define a 'derivable net' it is not required that the Baer subplanes in question actually sit in a net, a derivable pre pseudo-net will suffice.

In the definition of a net and a derivable net in Johnson [3], the phrase 'each parallel class forms a cover of the points' was intended to implicitly indicate that two lines from distinct parallel classes must intersect. However, in the explicit interpretation of 'cover', the problem then remains whether any two such lines must, in fact, intersect. The word 'cover' also implicitly indicates that there are no lines of a parallel class without at least two incident points. That is, strictly speaking, a line might not have incident points so that there could be no intersections. So, when one would use the word 'cover', it is assumed that lines are defined by their points. Hence, our results show that even a strict interpretation of the terms 'cover', etc. would lead to the same geometric structures when considering derivable structures.

References

[1] A. Blunck: *A new approach to derivation*, (submitted).

[2] N. L. Johnson: *Derivable nets can be embedded in nonderivable planes*, Trends in Mathematics, (1998), Birkhäuser Verlag Basel/Switzerland, 123–144.

[3] Norman L. Johnson., Subplane Covered Nets, Marcel Dekker, Pure and Applied Mathematics, Vol. 222 (2000).