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On a functional analytic approach for transition semigroups on $L^2(\mu)$

Abdelaziz Rhandi
Department of Mathematics, University of Marrakesh,
B.P.: 2390, 40000-Marrakesh, Morocco
rhandi@ucam.ac.ma

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Abstract. By using only analytic tools we prove the positivity of the transition semigroup associated formally with the stochastic differential equation

$$dX(t) = (AX(t) + F(X(t)))dt + Q^{\frac{1}{2}}dW(t),\ X(0) = x, t \geq 0,\ x \in H$$

in the case where $F \in UCB(H, H)$. As a consequence we obtain the existence of an invariant measure of the above stochastic equation.

Introduction

The Ornstein-Uhlenbeck semigroup, acting on measurable bounded functions $\varphi\colon H \to \mathbb{R}$, can be defined by the formula

$$(R_t\varphi)(x) := \mathbb{E}[\varphi(X(t, x))], \quad x \in H,\ t \geq 0,$$

where H is a separable Hilbert space and X is the Gaussian Markov process that solves the following differential stochastic equation

$$\begin{cases} dX(t) = AX(t)dt + Q^{\frac{1}{2}}dW(t), & t \geq 0, \\ X(0) = x \in H. \end{cases} \tag{1}$$

Here $A\colon D(A) \to H$ is the generator of a C_0-semigroup $(e^{tA})_{t\geq 0}$ on H, $W(t)$, $t \geq 0$, is an H-valued cylindrical Wiener process and Q is a continuous, linear, self-adjoint and nonnegative operator in H satisfying

(H1) for each $s > 0$ the linear operator $e^{sA}Qe^{sA^*}$ is of trace-class, $\ker Q = \{0\}$ and

$$\int_0^t Tr(e^{sA}Qe^{sA^*})ds < \infty \quad \text{for all } t > 0.$$

For each $t \geq 0$, we set $Q_t := \int_0^t e^{sA}Qe^{sA^*}ds$. If (H1) holds, it is obvious that Q_t is a continuous, linear, self-adjoint and nonnegative operator on H which is of trace-class and $\ker Q_t = \{0\}$.

We denote by $B_b(H)$ the Banach space of all bounded and Borel mappings from H into $\mathbb{R}$ endowed with the norm

$$\|\varphi\|_\infty := \sup_{x\in H} |\varphi(x)|$$

and by $UCB(H)$ the closed subspace of $B_b(H)$ of all uniformly continuous and bounded functions from H into $\mathbb{R}$. It can be proved that if (H1) holds then (R_t) is given by

$$(R_t\varphi)(x) = \int_H \varphi(y)\mathcal{N}(e^{tA}x, Q_t)(dy) = \int_H \varphi(e^{tA}x + y)\mathcal{N}(0, Q_t)(dy)$$

for $\varphi \in B_b(H)$, $t \geq 0$ and $x \in H$ (see [3]). Here, $\mathcal{N}(e^{tA}x, Q_t)$ denotes the Gaussian measure with mean $e^{tA}x \in H$ and covariance Q_t. For more details concerning Gaussian measures on Banach spaces we refer to [6] and [12].

Consequently, (R_t) is *strong Feller*, i.e., $R_t\varphi \in UCB(H)$ for $\varphi \in B_b(H)$ and $t > 0$. Moreover, if A is not identically 0, the semigroup (R_t) on $UCB(H)$ is not strongly continuous (see [1] and also [9]). By the type of (e^{tA}) we understand the number $\omega(A) := \lim_{t\to\infty} \frac{1}{t} \log \|e^{tA}\|$. If $\omega(A) < 0$, we set

$$Q_\infty := \int_0^\infty e^{sA}Qe^{sA^*}\,ds.$$

Using (H1) one can see that Q_∞ is a continuous, linear, self-adjoint and non-negative operator on H of trace-class. So we can define the Gaussian measure $\mu := \mathcal{N}(0, Q_\infty)$ on H. The measure μ is the unique invariant measure for (R_t) (see [3]). This means that

$$\int_H (R_t\varphi)(x)\mu(dx) = \int_H \varphi(x)\mu(dx) \qquad \text{for all } \varphi \in UCB(H).$$

We denote by $L^2(H, \mu)$ the space of all equivalence classes of real Borel functions φ on H such that

$$\int_H |\varphi(x)|^2\mu(dx) < \infty.$$

Endowed with the inner product

$$< \varphi, \psi >_{L^2} := \int_H \varphi(x)\psi(x)\mu(dx),$$

$L^2(H, \mu)$ is a Hilbert space. Since μ is an invariant measure for (R_t), one can see that (R_t) can be uniquely extended to a C_0-semigroup of contractions in

$L^2(H,\mu)$. We denote by $\mathcal{A}$ the generator of (R_t) in $L^2(H,\mu)$. If we denote by (e_k) a complete orthonormal system of eigenvectors of Q and by $D_k\varphi$ the derivative of φ in the direction e_k, then it is well known that D_k is closable. We shall still denote by D_k its closure. We recall now the definition of Sobolev spaces. We denote by $W^{1,2}(H,\mu)$ the linear space of all functions $\varphi \in L^2(H,\mu)$ such that $D_k\varphi \in L^2(H,\mu)$ for all $k \in \mathbb{N}$ and

$$\int_H |D\varphi(x)|^2\mu(dx) = \sum_{k=1}^{\infty} \int_H |D_k\varphi(x)|^2\mu(dx) < \infty.$$

The space $W^{1,2}(H,\mu)$ endowed with the inner product

$$< \varphi,\psi >_{W^{1,2}} := \int_H \varphi(x)\psi(x)\mu(dx) + \int_H < D\varphi(x), D\psi(x) > \mu(dx),$$
$$\varphi,\psi \in W^{1,2}(H,\mu),$$

is a Hilbert space.

For $F \in UCB(H,H)$ we consider the linear operator $(B, D(B))$ on $L^2(H,\mu)$ defined by

$$D(B) = W^{1,2}(H,\mu) \text{ and } B\varphi(x) := < F(x), D\varphi(x) >$$

for $\varphi \in D(B)$ and $x \in H$.

In the sequel we will need another assumption.

(H2) For all $t > 0$ we have $e^{tA}(H) \subset Q_t^{\frac{1}{2}}(H)$ and there exists $C > 0$ and $\nu \in (0,1)$ such that $\|Q_t^{-\frac{1}{2}} e^{tA}\| \leq Ct^{-\nu}$

We note that (H2) is satisfied with $\nu = \frac{1}{2}$ if $Q = Id$ (see [3, Corollary 9.22]).

Using a Miyadera perturbation theorem (see [7], [15]), we show that $\mathcal{A} + B$ generates a compact C_0-semigroup (P_t) on $L^2(H,\mu)$ if $\omega(A) < 0$ and (H1) and (H2) are satisfied. The semigroup (P_t) is given by a Dyson–Phillips series and this permits to derive some regularity results. The positivity of (P_t) is also proved. As a consequence we obtain the existence of an invariant measure for the following stochastic differential equation

$$\begin{cases} dX(t) = (AX(t) + F(X(t)))dt + Q^{\frac{1}{2}}dW(t), \quad t \geq 0, \\ X(0) = x \in H. \end{cases} \tag{2}$$

We note here that only analytic tools will be used.

The paper is organized as follows. In Section 1 we recall the Miyadera perturbation theorem and give some well-known properties of the Ornstein–Uhlenbeck semigroup (R_t) that we will need. In Section 2 we prove that $(\mathcal{A}+B, D(\mathcal{A}))$ generates a compact C_0-semigroup (P_t) on $L^2(H,\mu)$ and give some smoothing properties of (P_t). This semigroup will be called *transition semigroup*. In Section 3 we show, by using purely analytic methods, that (P_t) is a positive semigroup on $L^2(H,\mu)$. From the positivity of (P_t) we obtain the existence of an invariant measure for (2).

1 Preliminaries

In this section we recall several results that we will use in the sequel. Let $(\mathcal{A}, D(\mathcal{A}))$ and $(B, D(B))$ be two linear operators. Recall that B is $\mathcal{A}$-bounded if $D(\mathcal{A}) \subset D(B)$ and $\|B\varphi\| \leq a\|\varphi\| + b\|\mathcal{A}\varphi\|$ for $\varphi \in D(\mathcal{A})$ and constants $a, b \geq 0$. Observe that if there exists $\lambda \in \rho(\mathcal{A})$ then B is $\mathcal{A}$-bounded if and only if $D(\mathcal{A}) \subset D(B)$ and $BR(\lambda, \mathcal{A})$ is closed (and hence bounded).

We will need the following Miyadera perturbation theorem (see [7] or [15, Theorem 1]).

Theorem 1. *Let (R_t) be a C_0-semigroup on a Banach space E with generator $(\mathcal{A}, D(\mathcal{A}))$. Consider an $\mathcal{A}$-bounded linear operator $(B, D(B))$ such that there are constants $\alpha > 0$, $\gamma \in [0,1)$ and*

$$\int_0^\alpha \|BR_t\varphi\| dt \leq \gamma \|\varphi\| \quad \text{for } \varphi \in D(\mathcal{A}) \tag{3}$$

holds. Then the following assertions hold.

(a) *The operator $G := \mathcal{A} + B$ with $D(G) = D(\mathcal{A})$ generates a C_0-semigroup (P_t) on E given by the Dyson–Phillips series*

$$P_t = \sum_{n=0}^{\infty} U_n(t), \quad t \geq 0, \tag{4}$$

where $U_0(t) := R_t$ and $U_{n+1}(t)\varphi := \int_0^t U_n(t-s)BR_s\varphi ds$ for $t \geq 0$ and $\varphi \in D(\mathcal{A})$. The series in (4) converges in the operator norm for $t \geq 0$.

(b) *For $\varphi \in D(\mathcal{A})$ and $t \geq 0$, we have*

$$P_t\varphi = R_t\varphi + \int_0^t P_{t-s}BR_s\varphi ds, \tag{5}$$

$$P_t\varphi = R_t\varphi + \int_0^t R_{t-s}BP_s\varphi ds. \tag{6}$$

Moreover, (P_t) is the only C_0-semigroup satisfying (5) for $\varphi \in D(\mathcal{A})$.

Remark 1. The last assertion in (a) is shown in [11, Proposition 2.3]. Equation (6) follows from [10, Theorem 3.1 (c)].

We denote by $UCB^k(H)$, $k \in \mathbb{N}$, the subspace of $UCB(H)$ of all functions $\varphi\colon H \to \mathbb{R}$ which are k-times Fréchet differentiable, with a bounded uniformly continuous k-derivative $D^k\varphi$.

The following regularity results of the Ornstein-Uhlenbeck semigroup (R_t) on $UCB(H)$ and $L^2(H,\mu)$ (see [4, Theorem 2.7]) are relevant.

Theorem 2. *Assume that (H1) and (H2) hold. Then for all $\varphi \in B_b(H)$ and $t > 0$, $R_t\varphi \in UCB^\infty(H)$ $(:= \cap_{k\in\mathbb{N}} UCB^k(H))$ and*

$$|D(R_t\varphi)(x)| \le Ct^{-\nu}\|\varphi\|_\infty, \quad x \in H. \tag{7}$$

Theorem 3. *If $\omega(A) < 0$ and (H1) and (H2) hold, then for any $\varphi \in L^2(H,\mu)$ and $t > 0$, we have $R_t\varphi \in W^{1,2}(H,\mu)$ and*

$$\|D(R_t\varphi)\|_{L^2} \le Ct^{-\nu}\|\varphi\|_{L^2}. \tag{8}$$

The following description of the generator $(\mathcal{A}, D(\mathcal{A}))$ of (R_t) is shown in [3].

Proposition 1. *If $\omega(A) < 0$ and (H1) are satisfied, then the subspace $\mathcal{D}_A := lin\{\varphi_h(\cdot) := e^{i<h,\cdot>},\ h \in D(A^*)\}$ of $L^2(H,\mu)$ is a core for (R_t). Moreover $\mathcal{A}$ is the closure of $\mathcal{A}_0$, where $\mathcal{A}_0$ is defined by*

$$\mathcal{A}_0\varphi(x) := \frac{1}{2}Tr[QD^2\varphi(x)] + < Ax, D\varphi(x) > \qquad \text{for } \varphi \in \mathcal{D}_A.$$

2 A Miyadera perturbation of the Ornstein-Uhlenbeck semigroup on L^2

In this and the next section we suppose that $\omega(A) < 0$ and that (H1) and (H2) hold. By $(\mathcal{A}, D(\mathcal{A}))$ we denote the generator of the Ornstein-Uhlenbeck semigroup (R_t) on $L^2(H,\mu)$ and $(B, D(B))$ the operator defined by

$$D(B) := W^{1,2}(H,\mu) \text{ and } B\varphi(x) := < F(x), D\varphi(x) >,\ x \in H,$$

where $F \in UCB(H,H)$.

First of all we establish the following auxiliary result.

Lemma 1. *For any $\lambda > 0$ and $\varphi \in L^2(H,\mu)$ we have $R(\lambda,\mathcal{A})\varphi \in W^{1,2}(H,\mu)$ and $BR(\lambda,\mathcal{A}) \in \mathcal{L}(L^2(H,\mu))$. In particular, $D(\mathcal{A}) \subset W^{1,2}(H,\mu)$ holds.*

Proof. From Theorem 3 we have for any $\varphi \in L^2(H,\mu)$ and $t > 0$, $R_t\varphi \in W^{1,2}(H,\mu)$ and

$$\begin{aligned} \|D(R_t\varphi) - D(R_s\varphi)\|_{L^2} &= \|DR_s(R_{t-s}\varphi - \varphi)\|_{L^2} \\ &\le Cs^{-\nu}\|R_{t-s}\varphi - \varphi\|_{L^2} \end{aligned}$$

for $t > s > 0$. This implies that the function

$$0 < t \mapsto DR_t \text{ is strongly continuous.}$$

Consequently, it follows from (8) that

$$\int_0^\infty e^{-\lambda t} \|D(R_t\varphi)\|_{L^2} dt < \infty \text{ for all } \varphi \in L^2(H,\mu) \text{ and } \lambda > 0.$$

Therefore, for each $\varphi \in L^2(H,\mu)$ and $\lambda > 0$, we have

$$R(\lambda,\mathcal{A})\varphi \in W^{1,2}(H,\mu) \text{ and } D(R(\lambda,\mathcal{A})\varphi) = \int_0^\infty e^{-\lambda t} D(R_t\varphi)dt.$$

Since, $F \in UCB(H,H)$, it is now easy to see that $BR(\lambda,\mathcal{A}) \in \mathcal{L}(L^2(H,\mu))$ for $\lambda > 0$. QED

We state now the main result of this section.

Theorem 4. *Assume that $\omega(A) < 0$ and that (H1) and (H2) hold. Let $(\mathcal{A}, D(\mathcal{A}))$ and $(B, D(B))$ be defined as above. Then the operator $G := \mathcal{A} + B$ with $D(G) := D(\mathcal{A})$ generates a compact C_0-semigroup (P_t) on $L^2(H,\mu)$ satisfying the following integral equation*

$$P_t\varphi = R_t\varphi + \int_0^t P_{t-s}BR_s\varphi ds \tag{9}$$

for all $t \geq 0$ and $\varphi \in L^2(H,\mu)$. Moreover for each $T > 0$ there exists $C_T > 0$ such that

$$P_t\varphi \in W^{1,2}(H,\mu) \text{ and } \|D(P_t\varphi)\|_{L^2} \leq C_T t^{-\nu} \|\varphi\|_{L^2} \tag{10}$$

for $t \in (0,T]$ and $\varphi \in L^2(H,\mu)$. Further, (P_t) satisfies

$$P_t\varphi = R_t\varphi + \int_0^t R_{t-s}BP_s\varphi ds \tag{11}$$

for all $t \geq 0$ and $\varphi \in L^2(H,\mu)$. Finally, $\mathcal{D}_A$ is a core for (P_t) and G is the closure of G_0, where

$$G_0\varphi(x) := \frac{1}{2}Tr[QD^2\varphi(x)] + < Ax, D\varphi(x) > + < F(x), D\varphi(x) >$$

for $x \in H$ and $\varphi \in \mathcal{D}_A$.

Proof.

1. In order to apply Theorem 1 and by Lemma 1 it suffices to prove (3) for B and (R_t). From the proof of Lemma 1 one can see that the function $0 < t \mapsto BR_t\varphi \in L^2(H,\mu)$ is continuous and by (8) we have

$$\begin{aligned}\int_0^\alpha \|BR_t\varphi\|_{L^2}dt &\leq C\|F\|_\infty\|\varphi\|_{L^2}(\int_0^\alpha t^{-\nu}dt) \\ &= \left(\frac{C\|F\|_\infty}{1-\nu}\alpha^{1-\nu}\right)\|\varphi\|_{L^2}\end{aligned}$$

for all $\alpha > 0$ and $\varphi \in L^2(H,\mu)$. One can choose α sufficiently small such that $\gamma := \frac{C\|F\|_\infty}{1-\nu}\alpha^{1-\nu} \in (0,1)$ and thus (3) is satisfied for all $\varphi \in L^2(H,\mu)$. Therefore, $G := \mathcal{A} + B$ with $D(G) := D(\mathcal{A})$ generates a C_0-semigroup (P_t) on $L^2(H,\mu)$ and (9), (11) hold for all $\varphi \in D(\mathcal{A})$. Since $D(\mathcal{A})$ is dense in $L^2(H,\mu)$, it follows from (8) and the dominated convergence theorem that (9) holds for all $\varphi \in L^2(H,\mu)$. From Proposition 1 and Lemma 1 follow that $\mathcal{D}_A$ is a core for (P_t) and G is the closure of G_0. On the other hand, since the embedding $W^{1,2}(H,\mu) \hookrightarrow L^2(H,\mu)$ is compact (see [2]), if we show that $P_t\varphi \in W^{1,2}(H,\mu)$ for $t > 0$ and $\varphi \in L^2(H,\mu)$, then (P_t) is compact.

2. We prove now (10) and (11) for all $\varphi \in L^2(H,\mu)$. By the same argument as above it follows from Theorem 1 and 3 that (P_t) is given by

$$P_t\varphi = \sum_{n=0}^{\infty} U_n(t)\varphi \quad \text{for } t \geq 0 \text{ and } \varphi \in L^2(H,\mu),$$

where $U_0(t)\varphi := R_t\varphi$ and $U_{n+1}(t)\varphi := \int_0^t U_n(t-s)BR_s\varphi ds$ for all $t \geq 0$ and $\varphi \in L^2(H,\mu)$.

First we have, from Theorem 3, that $R_t\varphi \in W^{1,2}(H,\mu)$ and

$$\|D(R_t\varphi)\|_{L^2} \leq Ct^{-\nu}\|\varphi\|_{L^2}$$

for all $t > 0$ and $\varphi \in L^2(H,\mu)$. For $U_1(\cdot)$ we also have $U_1(t)\varphi \in W^{1,2}(H,\mu)$

and

$$\begin{aligned}\|D(U_1(t)\varphi)\|_{L^2} &= \left\|D\int_0^t R_{(t-s)}BR_s\varphi ds\right\|_{L^2} \\ &\le \int_0^t \|D(R_{(t-s)}BR_s\varphi)\|_{L^2}ds \\ &\le C\int_0^t (t-s)^{-\nu}\|BR_s\varphi\|_{L^2}ds \\ &\le C^2\|F\|_\infty t^{-\nu}\left[t^{1-\nu}\int_0^1(1-s)^{-\nu}s^{-\nu}ds\right]\|\varphi\|_{L^2} \\ &\le (C^2\|F\|_\infty T^{1-\nu}K)t^{-\nu}\|\varphi\|_{L^2},\end{aligned}$$

for $\varphi \in L^2(H,\mu)$ and $t \in (0,T]$, where $K := \int_0^1 (1-s)^{-\nu}s^{-\nu}ds$.
By induction one can see that for each $\varphi \in L^2(H,\mu)$ and $t \in (0,T]$

$$U_n(t)\varphi \in W^{1,2}(H,\mu)$$

and

$$\|D(U_n(t)\varphi)\|_{L^2} \le C(C\|F\|_\infty T^{1-\nu}K)^n t^{-\nu}\|\varphi\|_{L^2},\ n \in \mathbb{N}.$$

If we choose T sufficiently small, then $P_t\varphi \in W^{1,2}(H,\mu)$ and

$$\begin{aligned}\|D(P_t\varphi)\|_{L^2} &\le \sum_{n=0}^{\infty}\|D(U_n(t)\varphi)\|_{L^2} \\ &\le C_T t^{-\nu}\|\varphi\|_{L^2},\end{aligned}$$

for $\varphi \in L^2(H,\mu)$ and $t \in (0,T]$. The semigroup property yields

$$P_t\varphi \in W^{1,2}(H,\mu) \text{ and } \|D(P_t\varphi)\|_{L^2} \le C_T t^{-\nu}\|\varphi\|_{L^2},$$

for all $\varphi \in L^2(H,\mu)$ and $t \in (0,T]$, where C_T is a constant depending on T. Now from the last inequality, the density of $D(\mathcal{A})$ in $L^2(H,\mu)$ and (6) it follows that (10) is satisfied for all $\varphi \in L^2(H,\mu)$ and the proof is finished.

QED

Remark 2. Let $\mathbf{1}$ be the constant function equal to 1. Since $R_t\mathbf{1} = \mathbf{1}$ for all $t \ge 0$, it follows from (9) that $P_t\mathbf{1} = \mathbf{1}$ for all $t \ge 0$. On the other hand, since the operator P_t, $t > 0$, is compact in $L^2(H,\mu)$, the same is true for its adjoint P_t^*, $t > 0$. Therefore, 1 is also an eigenvalue for P_t^* and $\mathrm{Ker}(Id - P_t^*)$ is a finite dimensional non trivial subspace of $L^2(H,\mu)$.

3 Positivity of the transition semigroup on $L^2(H,\mu)$

We denote by $Lip_b(H,H)$ the space of all bounded Lipschitz functions from H into H. It is proved in [14] and [13] that $Lip_b(H,H)$ is dense in $UCB(H,H)$. Using this result, we prove the positivity of the transition semigroup (P_t) for $F \in UCB(H,H)$.

For the main result of this section we will use the following consequence of the Trotter-Kato theorem due to Voigt [16].

Theorem 5. *Let (R_t) be a C_0-semigroup on a Banach space E, with generator $(\mathcal{A}, D(\mathcal{A}))$. Let B_n, B be $\mathcal{A}$-bounded operators, and suppose that there exist $\alpha \in (0,\infty]$ and $\gamma \in [0,1)$ such that*

$$\int_0^\alpha \|B_n R_t \varphi\| dt \leq \gamma \|\varphi\| \quad \text{for all } \varphi \in D(\mathcal{A}) \text{ and } n \in \mathbb{N}.$$

Further assume

$$\int_0^\alpha \|(B_n - B) R_t \varphi\| dt \to 0 \quad (n \to \infty),$$

for all $\varphi \in D(\mathcal{A})$. Then

$$P_t \varphi = \lim_{n\to\infty} P_t^{(n)} \varphi \quad \text{for all } \varphi \in E$$

uniformly for t in bounded subsets of $\mathbb{R}_+$, where (P_t) (resp. $(P_t^{(n)})$) is the semigroup generated by $\mathcal{A} + B$ (resp. $\mathcal{A} + B_n$).

We can now prove the main result of this section.

Theorem 6. *Assume that $\omega(A) < 0$ and that (H1) and (H2) hold. Let $(\mathcal{A}, D(\mathcal{A})$ and $(B, D(B))$ be defined as in Section 1 and 2. Then the transition semigroup (P_t) is positive. Therefore there exists an invariant measure σ for (P_t) which is absolutely continuous with respect to μ. Moreover,*

$$\frac{d\sigma}{d\mu}(x) \in L^2(H,\mu).$$

Proof. The proof is carried out in two steps.

Step 1. We first suppose that $F \in Lip_b(H,H)$.

By standard arguments one sees that there is $T > 0$ such that the nonlinear equation

$$\begin{cases} \frac{\partial}{\partial t}\eta(t,x) = F(\eta(t,x)), & t \in [0,T], x \in H \\ \eta(0,x) = x \in H \end{cases}$$

has a unique solution $\eta(\cdot,\cdot)$ satisfying

$$\eta(t,x) = x + \int_0^t F(\eta(s,x))\,ds \quad \text{for } t \in [0,T] \text{ and } x \in H.$$

Since F is bounded and so by the uniqueness it follows that

$$\text{the function } [0,T] \ni t \mapsto \eta(t,x) \text{ is continuous uniformly in } x \in H \tag{12}$$

and

$$\eta(s,\eta(t,x)) = \eta(t+s,x) \tag{13}$$

for $x \in H$ and $t,s \in [0,T]$ such that $t+s \in [0,T]$. We consider now the family of bounded operators $(S_t)_{t\in[0,T]}$ on $UCB(H)$ defined by

$$S_t\varphi(x) := \varphi(\eta(t,x))$$

for $t \in [0,T], x \in H$ and $\varphi \in UCB(H)$. By (13) we obtain $S_{t+s} = S_tS_s$ for $t,s \in [0,T]$ such that $t+s \in [0,T]$. The strong continuity of (S_t) on $[0,T]$ follows from (12). For $t \geq 0$ there is $n \in \mathbb{N}$ such that $\frac{t}{n} \leq T$. With this n we define $S_t := (S_{\frac{t}{n}})^n$. One can see that this definition is unambiguous. Hence $(S_t)_{t\geq 0}$ is a positive C_0-semigroup of contractions on $UCB(H)$. If we denote by $(\mathcal{B}, D(\mathcal{B}))$ its generator, then

$$\begin{aligned}&UCB^1(H) \subset D(\mathcal{B}) \text{ and}\\ &(\mathcal{B}\varphi)(x) = < F(x), D\varphi(x) > = (B\varphi)(x) \quad \text{for } \varphi \in UCB^1(H)\end{aligned}$$

(cf. [8, B-II, Example 3.15]). Hence,

$$\lim_{m\to\infty} \mathcal{B}_m\varphi = B\varphi \text{ in } UCB(H) \quad \text{for all } \varphi \in UCB^1(H),$$

where $\mathcal{B}_m := m\mathcal{B}(m-\mathcal{B})^{-1}$ is the Hille–Yosida approximation of $\mathcal{B}$.

On the other hand, if we put

$$R(\lambda)\varphi(x) := \int_0^\infty e^{-\lambda t}(R_t\varphi)(x)\,dt$$

for $\lambda > 0$, $\varphi \in UCB(H)$ and $x \in H$, then by [1, Proposition 6.2 and 3.1], $R(\lambda) \in \mathcal{L}(UCB(H))$ and by a simple computation one can see that

$$R(\lambda)\varphi = R(\lambda,\mathcal{A})\varphi \quad \text{for } \varphi \in UCB(H) \text{ and } \lambda > 0.$$

Hence from Theorem 2 it follows that $R(\lambda,\mathcal{A})\varphi \in UCB^\infty(H)$ and there is $\lambda_0 > 0$ such that

$$\|BR(\lambda,\mathcal{A})\varphi\|_\infty \leq \frac{1}{2}\|\varphi\|_\infty \tag{14}$$

for all $\varphi \in UCB(H)$ and $\lambda > \lambda_0$. This implies that

$$Id - BR(\lambda, \mathcal{A})\colon UCB(H) \to UCB(H)$$

is invertible and $(Id - BR(\lambda, \mathcal{A}))^{-1} = \sum_{n=0}^{\infty}[BR(\lambda, \mathcal{A})]^n$ for $\lambda > \lambda_0$. Hence,

$$R(\lambda, \mathcal{A} + B)\varphi = R(\lambda, \mathcal{A}) \sum_{n=0}^{\infty}[BR(\lambda, \mathcal{A})]^n \varphi \in UCB^{\infty}(H) \tag{15}$$

for all $\varphi \in UCB(H)$ and $\lambda > \lambda_0$. Since $R(\lambda, \mathcal{A})\mathbf{1} = \frac{1}{\lambda}$ and $R(\lambda, \mathcal{A}) \geq 0$ on $UCB(H)$, it follows that

$$\|R(\lambda, \mathcal{A})\|_{\infty} \leq \frac{1}{\lambda} \quad \text{for } \lambda > 0.$$

On the other hand, the estimate in (14) implies that

$$\|\mathcal{B}_m R(\lambda, \mathcal{A})\|_{\infty} = \|m(m - \mathcal{B})^{-1} BR(\lambda, \mathcal{A})\|_{\infty} \leq \frac{1}{2}$$

for $\lambda > \lambda_0$. So from the dissipativity of $\mathcal{B}_m$ on $UCB(H)$, and since $\|R(\lambda, \mathcal{A})\|_{\infty} \leq \frac{1}{\lambda}$ for $\lambda > 0$, follows that

$$(\lambda_0, \infty) \subset \rho(\mathcal{A} + \mathcal{B}_m) \text{ and } \|R(\lambda, \mathcal{B}_m + \mathcal{A})\|_{\infty} \leq \frac{1}{\lambda} \tag{16}$$

for $\lambda > \lambda_0$ and $m \in \mathbb{N}$. So by (16) we obtain

$$\begin{aligned} \|R(\lambda, \mathcal{B}_m + \mathcal{A})\varphi - R(\lambda, \mathcal{A} + B)\varphi\|_{\infty} &= \\ &= \|R(\lambda, \mathcal{B}_m + \mathcal{A})(B - \mathcal{B}_m)R(\lambda, \mathcal{A} + B)\varphi\|_{\infty} \\ &\leq \frac{1}{\lambda}\|(B - \mathcal{B}_m)R(\lambda, \mathcal{A} + B)\varphi\|_{\infty} \\ &\qquad \downarrow (m \to \infty) \\ &\qquad 0 \end{aligned}$$

for all $\varphi \in UCB(H)$ and $\lambda > \lambda_0$. It remains to show that

$$R(\lambda, \mathcal{B}_m + \mathcal{A})\varphi \geq 0 \quad \text{for all } \varphi \in UCB(H)_+, m \in \mathbb{N} \text{ and } \lambda > \lambda_0.$$

The positivity of $e^{t\mathcal{B}_m}$ follows from that of S_t. Moreover, from [8, Theorem C-II.1.11] we have $T_m := \mathcal{B}_m + \|\mathcal{B}_m\| Id \geq 0$ for $m \in \mathbb{N}$. Hence,

$$\begin{aligned} R(\lambda, \mathcal{B}_m + \mathcal{A}) &= R(\lambda + \|\mathcal{B}_m\|, T_m + \mathcal{A}) \\ &= R(\lambda + \|\mathcal{B}_m\|, \mathcal{A}) \sum_{n=0}^{\infty}[T_m R(\lambda + \|\mathcal{B}_m\|, \mathcal{A})]^n \geq 0 \end{aligned}$$

for all $\lambda > \|\mathcal{B}_m\|$. We fix now $m \in \mathbb{N}$ and consider the set

$$M := \{\lambda > \lambda_0 \mid R(\lambda, \mathcal{B}_m + \mathcal{A}) \geq 0\}.$$

Then M is a closed and open subset of (λ_0, ∞). In fact, let $\lambda \in M$. Then for small $\varepsilon > 0$ one has $R(\lambda - \varepsilon, \mathcal{B}_m + \mathcal{A}) = \sum_{n=0}^{\infty} \varepsilon^n R(\lambda, \mathcal{B}_m + \mathcal{A})^{n+1} \geq 0$. On the other hand, since $R(\lambda, \mathcal{B}_m + \mathcal{A}) = R(\lambda + \|\mathcal{B}_m\|, T_m + \mathcal{A}) \geq 0$, it follows from [17, Theorem 1.1] that $r(T_m R(\lambda + \|\mathcal{B}_m\|, \mathcal{A})) < 1$. Furthermore, we have

$$0 \leq T_m R(\lambda + \varepsilon + \|\mathcal{B}_m\|, \mathcal{A}) \leq T_m R(\lambda + \|\mathcal{B}_m\|, \mathcal{A}).$$

Therefore, $r(T_m R(\lambda + \varepsilon + \|\mathcal{B}_m\|, \mathcal{A})) < 1$ and hence,

$$0 \leq R(\lambda + \varepsilon + \|\mathcal{B}_m\|, T_m + \mathcal{A}) = R(\lambda + \varepsilon, \mathcal{B}_m + \mathcal{A}).$$

The claim "M is a closed subset of (λ_0, ∞)" follows from the resolvent equation and (16). Thus,

$$R(\lambda, \mathcal{B}_m + \mathcal{A}) \geq 0$$

on $UCB(H)$ and by density on $L^2(H, \mu)$ for all $\lambda > \lambda_0$. This proves the positivity of (P_t) on $L^2(H, \mu)$.

Step 2. For $F \in UCB(H, H)$ there is $F_n \in Lip_b(H, H)$ such that

$$\lim_{n \to \infty} \|F_n - F\|_\infty = 0.$$

We associated with F_n the operator defined by

$$D(B_n) = D(B) = W^{1,2}(H, \mu)$$

and $B_n \varphi(x) := < F_n(x), D\varphi(x) >$, $\varphi \in W^{1,2}(H, \mu), x \in H$ and $n \in \mathbb{N}$. So by Theorem 3 and Lemma 1 we obtain that B and B_n satisfy the assumptions of Theorem 5. Hence,

$$P_t \varphi = \lim_{n \to \infty} P_t^{(n)} \varphi \quad \text{ for all } \varphi \in L^2(H, \mu) \text{ and } t \geq 0.$$

From Step 1 we have the positivity of (P_t) on $L^2(H, \mu)$.

We prove now the last statement of the theorem.

From Remark 2 and the spectral mapping theorem for the point spectrum (cf. [5, IV-3.6]) it follows that there is $\psi \in D(G^*)$, $\psi \not\equiv 0$ such that $G^* \psi = 0$, where $(G^*, D(G^*))$ denotes the generator of (P_t^*). Hence,

$$P_t^* \psi - \psi = \int_0^t P_s^* (G^* \psi) \, ds = 0 \quad \text{ for all } t \geq 0.$$

Since (P_t) is positive it follows that $|\psi| = |P_t^*\psi| \leq P_t^*|\psi|$ and from

$$< P_t^*|\psi|, \mathbf{1} >=< |\psi|, P_t\mathbf{1} >=< |\psi|, \mathbf{1} >=< |P_t^*\psi|, \mathbf{1} >$$

we obtain

$$|P_t^*\psi| = P_t^*|\psi| = |\psi| \quad \text{for all } t \geq 0.$$

If we put $\psi_0 := \frac{1}{\|\psi\|_{L^2}}|\psi|$ then the measure $\sigma(dx) := \psi_0(x)\mu(dx)$ has the asserted properties.

QED

Remark 3. The above result generalizes the one given in [4, Theorem 3.1].

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