Riemannian manifolds structured by a $T$-parallel exterior recurrent connection

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**Abstract.** Geometrical and structural properties are proved for Riemannian manifolds which are equipped with a $T$-parallel exterior recurrent connection.

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**Introduction**

Riemannian manifolds structured by a $T$-parallel connection have been defined in [9] and have also been studied in [6]. Let $M$ be a $2m$-dimensional $C^\infty$-manifold and $\nabla$ be the Levi-Civita connection. We recall that if $M$ carries a globally defined vector field $T(T^A)$ and the connection forms satisfy

$$\theta^A_B = \langle T, e_B \wedge e_A \rangle,$$

where $\wedge$ denotes the wedge product of vector fields, then one says that $M$ is structured by a $T$-parallel connection. In the present paper we assume in addition that $\theta^A_B$ are exterior recurrent forms [2], which means that

$$d\theta^A_B = 2\alpha \wedge \theta^A_B,$$

where $\alpha = T^\flat$, (2)

having $T^\flat$ as recurrence form. This implies that the curvature forms $\Theta^A_B$ are also exterior recurrent. In consequence of this fact, we adopt the terminology that $M$ is structured by a $T$-parallel exterior recurrent connection.

For the above mentioned structures, we prove the following properties:

(i) $T$ is a concurrent vector field and defines an infinitesimal conformal transformation of $\theta^A_B$ and $\Theta^A_B$ and the differential system $\nabla_{e_A}$ corresponding to the vector basis $\mathcal{O} = \{e_A\}$ admits an infinitesimal transformation with generator $T$;
(ii) $\|T\|^2$ is an isoparametric function [13], and an eigenfunction of $\Delta$ having $4(2m + \|T\|^2)$ as eigenvalue;

(iii) if $V$ is any parallel vector field, one has by the Weitzenböck formula that

$$(\Delta T) V = -4m\|T\|^2 g(T, V);$$

(iv) if

$$\Theta^{(p)}_{u^1, \ldots, u^{2p}} = \Theta^{u_2}_{u_1} \wedge \Theta^{u_3}_{u_2} \wedge \cdots \wedge \Theta^{2p}_{2p-1}$$

denotes the Bianchi forms (in the sense of Tachibana [12]), these forms are exterior recurrent with $3(2m - 1)\alpha$ as recurrence form;

(v) any vector field $X$ such that

$$\nabla X = X \wedge T$$

is a skew symmetric Killing vector field [11] and $X$ defines an infinitesimal transformation of the conformal symplectic form $\Omega$, i.e.

$$\mathcal{L}_X \Omega = -2g(X, T)\Omega.$$

In Section 4 we consider some properties of the tangent bundle manifold $TM$ having the manifold $M$, studied in Section 3, as basis. On $TM$ the canonical vector field $V(V^A) (A = 1, \cdots 2m)$ is called the Liouville vector field [3], and the complete lift [14] $\Omega^C$ of the structure 2-form of rank $4m$ on $TM$ is given by

$$\Omega^C = \sum dV^a \wedge \omega^a + \omega^a \wedge dV^{a^*}, \quad a = 1, \cdots m; a^* = a + m. \quad (3)$$

In Section 3, the following relation will be derived (see formula (24)):

$$d\omega^A = \alpha \wedge \omega^A.$$

By exterior differentiation of (3), and taking into account the above formula, one gets

$$d\Omega^C = \alpha \wedge \Omega^C; \quad (4)$$

and

$$\mathcal{L}_V \Omega^C = \Omega^C. \quad (5)$$

The above equations express that the 2-form $\Omega^C$ is a homogeneous 2-form of class 1 [4] on $TM$. Next, the Liouville form $\mu$ (i.e. $\mu = V^\flat$) is expressed by

$$\mu = \sum V^A \omega^A \quad A = 1, \cdots 2m \quad (6)$$
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and one finds by exterior differentiation that

$$d\mu = \alpha \wedge \mu + \psi,$$

where we have set

$$\psi = \sum dV^A \wedge \omega^A.$$  \hspace{1cm} (8)

One also derives that

$$\mathcal{L}_V \psi = \psi,$$  \hspace{1cm} (9)

and this shows that, like $\Omega^C$, the form $\psi$ is a homogeneous 2-form of class 1. Moreover, making use of the vertical operator $i_v$ of Godbillon [3], one calculates that

$$i_v \psi = 0,$$  \hspace{1cm} (10)

which together with (9) proves that $\psi$ is a Finslerian form. In addition, if $\mathcal{T}^V$ denotes the vertical lift of $\mathcal{T}$, one also finds that

$$\mathcal{L}_{\mathcal{T}^V} \psi = 0,$$

which shows that $\mathcal{T}^V$ defines an infinitesimal automorphism of $\psi$. Some other properties regarding the principal almost symplectic form $\Pi = \|T\|^2 \psi$ are also discussed.

1 Preliminaries

Let $(M, g)$ be a Riemannian $C^\infty$-manifold and let $\nabla$ be the covariant differential operator with respect to the metric tensor $g$. We assume that $M$ is oriented and $\nabla$ is the Levi-Civita connection of $g$. Let $\Gamma TM = \Xi(M)$ be the set of sections of the tangent bundle, and

$$\flat: TM \xrightarrow{\flat} T^*M \quad \text{and} \quad \sharp: TM \xleftarrow{\sharp} T^*M$$  \hspace{1cm} (11)

the classical isomorphisms defined by $g$ (i.e. $\flat$ is the index lowering operator, and $\sharp$ is the index raising operator).

Following [8], we denote by

$$A^q(M, TM) = \Gamma \text{Hom}(\Lambda^q TM, TM),$$  \hspace{1cm} (12)

the set of vector valued $q$-forms ($q < \dim M$), and we write for the covariant derivative operator with respect to $\nabla$

$$d^\nabla: A^q(M, TM) \to A^{q+1}(M, TM).$$  \hspace{1cm} (13)
It should be noticed that in general \( d^{\nabla^2} = d^{\nabla} \circ d^{\nabla} \neq 0 \), unlike \( d^2 = d \circ d = 0 \). We denote by \( I \in A^1(M, TM) \) the canonical vector valued 1-form of \( M \), which is also called the soldering form of \( M \) [2]. Since \( \nabla \) is symmetric one has that \( d^{\nabla} (I) = 0 \).

A vector field \( Z \in \Xi(M) \) which satisfies
\[
d^{\nabla}(\nabla Z) = \nabla^2 Z = \pi \wedge I \in A^2(M, TM); \quad \pi \in \Lambda^1 M
\]
is defined to be an exterior concurrent vector field [9] (see also [6]). The 1-form \( \pi \) in (14) is called the concurrence form and is defined by
\[
\pi = \lambda Z^\flat, \quad \lambda \in \Lambda^0 M.
\]

Let \( \mathcal{O} = \{e_A \mid A = 1, \cdots, 2m\} \) be a local field of orthonormal frames over \( M \) and let \( \mathcal{O}^* = \text{covect}\{\omega^A\} \) be its associated coframe. Then E. Cartan’s structure equations can be written in indexless manner as
\[
\begin{align*}
\nabla e &= \theta \otimes e, \quad (16) \\
d\omega &= -\theta \wedge \omega, \quad (17) \\
d\theta &= -\theta \wedge \theta + \Theta. \quad (18)
\end{align*}
\]
In the above equations \( \theta \) (respectively \( \Theta \)) are the local connection forms in the tangent bundle \( TM \) (respectively the curvature 2-forms on \( M \)).

2 Manifolds with \( T \)-parallel exterior recurrent connection

Let \( M(\Omega, T, g) \) be a \( 2m \)-dimensional manifold with almost symplectic 2-form \( \Omega \) and with structure vector field \( T(T^A) \) \((A = 1, \cdots, 2m)\). Now, by reference to [9] (see also [6]), we assume that \((M, g)\) is structured by a \( T \)-parallel connection, which means that the connection forms satisfy
\[
\theta^A_B = \langle T, e_B \wedge e_A \rangle, \quad (19)
\]
where \( \wedge \) stands for the wedge product of vector fields. In addition, we also assume that the connection forms \( \theta^A_B \) are exterior recurrent [2] with \( 2T^\flat \) as recurrence forms, which means that
\[
d\theta^A_B = 2T^\flat \wedge \theta^A_B. \quad (20)
\]
Since
\[
\theta^A_B = T^B \omega^A - T^A \omega^B,
\]
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it follows that

$$dT^A = T^A \alpha,$$  \hspace{1cm} (21)

where we have set $\alpha := T^b$. Now, in view of the structure equations (17) and invoking the curvature forms $\Theta^A_B$, one derives

$$\Theta^A_B = \|T\|^2 \omega^B \wedge \omega^A + \alpha \wedge \theta^A_B.$$  \hspace{1cm} (22)

Since one has

$$d\|T\|^2 = 2\|T\|^2 \alpha,$$  \hspace{1cm} (23)

then by (21) one gets

$$d\omega^A = \alpha \wedge \omega^A.$$  \hspace{1cm} (24)

By exterior differentiation of (22), one derives that

$$d\Theta^A_B = 3\alpha \wedge \Theta^A_B.$$  \hspace{1cm} (25)

The above equation expresses the fact that the connection forms being exterior recurrent implies the same property for the curvature forms $\Theta^A_B$ also. Taking moreover the Lie derivatives of $\theta^A_B$ and $\Theta^A_B$ with respect to the structure vector field $T$, and using (23), one finds

$$\mathcal{L}_T \theta^A_B = 2\|T\|^2 \theta^A_B,$$

$$\mathcal{L}_T \Theta^A_B = 3\|T\|^2 \Theta^A_B.$$  \hspace{1cm} (26)

Hence, $T$ defines an infinitesimal conformal transformation of both the connection forms and the curvature forms.

On the other hand, by (19) one finds that

$$\nabla e_A = T^A I - \omega^A \otimes T,$$  \hspace{1cm} (27)

and in this way one gets by (21) also that

$$\nabla T = \|T\|^2 I.$$  \hspace{1cm} (28)

This shows that $T$ is a concurrent vector field (it is well known [1] that concurrency is of conformal nature). From (27) and (28) it follows that

$$[T, e_A] = -\|T\|^2 e_A,$$  \hspace{1cm} (29)

and this proves that the differential system $\{e_A\}$ corresponding to the vector basis admits an infinitesimal transformation with generator $T$. We also notice that operating on (28) with $\nabla$ (the operator $\nabla$ acts inductively) one gets

$$\nabla(\nabla T) = \nabla^2 T = \|T\|^4 \alpha \wedge I.$$  \hspace{1cm} (30)
This shows that $T$ is an exterior concurrent vector field [10] (see also [7]). In consequence of (30) one may now also write
\[
\mathcal{R}(T, Z) = -(2m - 1)\|T\|^4 g(T, Z), \quad Z \in \Xi(M),
\] (31)
where $\mathcal{R}$ means the Ricci tensor field of $\nabla$. In the same way one can also calculate that
\[
\nabla^3 e_A = \|T\|^4 (\alpha \wedge \omega^A) \wedge I,
\] (32)
and consequently one can conclude that the elements of the vector basis $\{e_A\}$ are exterior concurrent vector fields; in the sequel we will use the terminology of a 2-exterior vector basis for this case.

We recall that a function $f: \mathbb{R}^{2m} \to \mathbb{R}$ is called isoparametric [13] if both $\|\text{grad} f\|^2$ and $\text{div}(\text{grad} f)$ are functions of $f$. In the case under discussion, one has first of all that
\[
\text{grad} \|T\|^2 = \|T\|^2 T,
\] (33)
from which there follows that
\[
\|\text{grad} \|T\|^2\|^2 = \|T\|^4.
\] (34)
Next, one also derives that
\[
\text{div} \text{grad} \|T\|^2 = 4(2m + \|T\|^2) \|T\|^2,
\] (35)
from which one may conclude that $\|T\|^2$ is an isoparametric function. Next, by the general formula
\[
\Delta \mu = -\text{div} \nabla \mu, \quad \mu \in \Lambda^0 M,
\]
where $\Delta$ denotes the Laplacian, and in virtue of (33), we see that $\|T\|^2$ is an eigenfunction of $\Delta$, having $4(2m + \|T\|^2)$ as eigenvalue of $\Delta$. Recall now that if $Z$ is any vector field, one has
\[
\text{tr} \nabla^2 Z = \sum \nabla e_A (\nabla e_A Z).
\]
Then, by (30) one derives
\[
\text{tr} \nabla^2 T = 2\|T\|^2 T.
\] (36)
With $\mathcal{R}$ denoting the Ricci tensor field, one now has
\[
\mathcal{R}(T, V) = -2(2m - 1)\|T\|^2 g(T, V), \quad V \in \Xi(M).
\] (37)
Then, by reference to [8], if $V$ is a parallel vector field, one has the Weitzenbock formula:

$$(\Delta^T) V = \mathcal{R}(V, T) - <\text{tr}\nabla^2 T, V > = -4m\|T\|^2 g(T, V). \quad (38)$$

On the other hand, regarding the almost symplectic form $\Omega$, one writes with standard notation

$$\Omega = \sum \omega^a \wedge \omega^{a*}, \quad a = 1, \ldots, m, a^* = a + m. \quad (39)$$

Taking the exterior derivative of $\Omega$, and in view of (24), one finds that

$$d\Omega = 2\alpha \wedge \Omega, \quad \alpha = T^b. \quad (40)$$

This affirms the fact that $\Omega$ defines a locally conformal symplectic structure on $M$ having $\alpha$ as covector of Lee. Then, as is known from [5], calling the mapping $Z \rightarrow -i_Z \Omega = \flat \ Z$ the symplectic isomorphism, one has

$$-\flat \ T = i_T \Omega = \sum (T_a \omega^{a*} - T^{a*}_a \omega^a), \quad (41)$$

and by (21) and (24) one finds that

$$\mathcal{L}_T \Omega = 2\|T\|^2 \Omega. \quad (42)$$

Hence, following a known definition [5], the above equation means that $T$ defines a infinitesimal conformal transformation of $\Omega$. On the other hand, regarding the curvature forms, we recall that the Bianchi forms in the sense of Tachibana [12] are defined by

$$\Theta^{(p)}_{u_1, \ldots, u_{2p}} = \Theta^{a_2}_{u_1} \wedge \Theta^{u_3}_{a_2} \wedge \cdots \wedge \Theta^{2p}_{2p-1}. \quad (43)$$

Then, by exterior differentiation one gets from (43)

$$d \left( \Theta^{(p)}_{u_1, \ldots, u_{2p}} \right) = 3(2m - 1)\alpha \wedge \Theta^{(p)}_{u_1, \ldots, u_{2p}}, \quad (44)$$

and we may consequently observe that the Bianchi forms $\Theta^{(p)}_{u_1, \ldots, u_{2p}}$ are exterior recurrent, with $3(2m - 1)\alpha$ as recurrence form.

In another perspective, let $X$ be any vector field on $M$; if the covariant differential of $X$ is the wedge product of $X$ with the structure vector field $T$, this means that $X$ is a skew symmetric Killing vector field (in the sense of [11]), i.e.

$$\nabla X = X \wedge T = \alpha \otimes X - X^b \otimes T. \quad (45)$$
One may also remark that the above relation is indeed in correspondence with Rosca’s lemma [11] concerning skew-symmetric Killing and conformal skew-symmetric Killing vector fields.

\[ dX^b = 2X \wedge X^b. \]

In this case, the differentials of the components of \( X \), i.e. \( dX^A \) satisfy

\[ dX^A = -g(X, T)\omega^A + X^A\alpha. \tag{46} \]

In view of the mentioned facts, and taking the Lie derivative of \( \Omega \) with respect to \( X \), one calculates that

\[ \mathcal{L}_X \Omega = -2g(X, T)\Omega. \tag{47} \]

This proves the property that any skew symmetric Killing vector field \( X \), having the structure vector field \( T \) as generative, defines an infinitesimal conformal transformation of the conformal symplectic form \( \Omega \).

Summing up, we state the following

**Theorem 1.** Let \( M(\Omega, T, \alpha) \) be a \( 2m \)-dimensional Riemannian manifold structured by a \( T \)-parallel exterior recurrent connection. In this case, the structure vector field \( T \) is concurrent and defines an infinitesimal conformal transformation of the connection forms \( \theta^A_B \), of the curvature forms \( \Theta^A_B \) and of the conformal symplectic form \( \Omega \). In addition, one has the following properties:

(i) \( \|T\|^2 \) is an isoparametric function;

(ii) the differential system \( \{e_A\} \) admits an infinitesimal transformation with generator \( T \), i.e.

\[ [T, e_A] = \|T\|^2 e_A; \]

(iii) all the basis vector fields \( e_A \) are 2-exterior concurrent vector fields, i.e.

\[ \nabla^3 e_A = 2\|T\|^2 (\alpha \wedge \omega^A) \wedge I, \quad \alpha = T^b. \]

(iv) \( \|T\|^2 \) is an eigenfunction of \( \Delta \) having \( 4(2m + \|T\|^2) \) as eigenvalue of \( \Delta \);

(v) if \( V \) denotes any parallel vector field, then one has the Weitzenböck formula

\[ \Delta \alpha(V) = \mathcal{R}(V, T) - \langle tr\nabla^2 T, V \rangle = -4m\|T\|^2 g(T, V); \]

(vi) if \( \Theta^{(p)}_{u_1, \ldots, u_{2p}} = \Theta^{u_1}_{v_1} \wedge \Theta^{u_2}_{v_2} \wedge \cdots \wedge \Theta^{2p}_{2p-1} \) means the Bianchi form of type \((2p, 2p)\), in the sense of Tachibana, then \( \Theta^{(p)}_{u_1, \ldots, u_{2p}} \) is exterior recurrent with \( 3(2m - 1)\alpha \) as recurrence form;
(vii) any skew symmetric Killing vector field $X$, having $\mathcal{T}$ as generative, defines an infinitesimal conformal transformation of $\Omega$, i.e.

$$\mathcal{L}_X \Omega = -2g(X, \mathcal{T})\Omega.$$ 

### 3 Geometry of the tangent bundle

In this section we will discuss some properties of the tangent bundle manifold $TM$ having as basis manifold $M$ studied in Section 3. Denote by $V(V^A)$ ($A = 1, \cdots 2m$) the Liouville vector field (or the canonical vector field on $TM$ [4]). Accordingly, one may consider the set

$$B^* = \{ \omega^A, dV^A \mid A = 1, \cdots 2m \}$$

as an adapted cobasis in $TM$ (see also [6]). Let $T^r_s$ be the set of all tensor fields of type $(r, s)$ on $M$. It is well known [14] that the vertical and complete lifts are linear mappings of $T^r_s M$ into $T^r_s(TM)$, and one has

$$(T_1 \otimes T_2)^C = T_1^V \otimes T_2^C + T_1^C \otimes T_2^V. \quad (48)$$

Hence, in the case under discussion we may define the complete lift $\Omega^C$ of the structure conformal 2-form $\Omega$ of $M$ to be the 2-form of rank $4m$ on $TM$ given by

$$\Omega^C = \sum (dV^a \wedge \omega^{a*} + \omega^a \wedge dV^{a*}), \quad a = 1, \cdots m; a^* = a + m. \quad (49)$$

On the other hand, the Liouville vector field $V$ is expressed by

$$V = \sum V^A \frac{\partial}{\partial V^A}; \quad (50)$$

it is also known that the associated basic 1-form

$$\mu = \sum V^A \omega^A \quad (51)$$

is called the Liouville form. (Alternatively, one can also write that $\mu = V^\flat$.)

Next, taking the Lie differential of $\Omega^C$ with respect to the Liouville vector field $V$ and taking into account (24), one finds that

$$\mathcal{L}_V \Omega^C = \Omega^C. \quad (52)$$

Hence, with reference to [4], the above equation proves that $\Omega^C$ is a homogeneous 2-form of class 1 on $TM$. 
Taking moreover the Lie differential of $\Omega^C$ with respect to the structure vector field $T$, one also derives that

$$\mathcal{L}_T \Omega^C = \|T\|^2 \Omega^C. \quad (53)$$

The above equation shows that $T$ defines also for $\Omega^C$ an infinitesimal conformal transformation.

By exterior derivation of the Liouville form $\mu$ defined by (51), and taking into account (24), one gets that

$$d\mu = \alpha \wedge \mu + dV^A \wedge \omega^A. \quad (54)$$

Introducing the notation

$$\psi = \sum dV^a \wedge \omega^a, \quad (55)$$

and by reference to (24), it follows that

$$d\psi = \alpha \wedge \psi, \quad (56)$$

which shows that $\psi$ is an exterior recurrent form with $\alpha$ as recurrence form.

Then, since one first calculates that

$$i_V \psi = \mu, \quad \alpha(V) = 0, \quad (57)$$

one finally gets that

$$\mathcal{L}_V \psi = \psi, \quad (58)$$

which shows that, as $\Omega^C$, the form $\psi$ is also a homogeneous 2-form of class 1.

We remind that the vertical operator $i_v$ in the sense of [3] possesses by definition the following properties:

$$i_v \lambda = 0, \quad i_v \omega^A = 0, \quad i_v dV^A = \omega^A, \quad (59)$$

from which one calculates that

$$i_v \psi = 0. \quad (60)$$

On behalf of (58) and (60) we conclude from this that $\psi$ is a Finslerian form [3].

In another order of ideas, we recall that the vertical lift $Z^V$ [14] of any vector field $Z$ on $M$ with components $Z^A$ is expressed by

$$Z^V = \begin{pmatrix} 0 \\ Z^A \end{pmatrix} = Z^A \frac{\partial}{\partial v^A}, \quad (A = 1, \cdots 2m).$$
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Therefore, in the case under consideration, the vertical lift $\mathcal{T}^V$ of $\mathcal{T}$ is given by

$$\mathcal{T}^V = \sum \mathcal{T}^A \frac{\partial}{\partial V^A}, \quad A \in \{1, \cdots, 2m\},$$

(61)

and by (55) one finds respectively that

$$i_{\mathcal{T}^V} \psi = \alpha, \quad \text{and} \quad \mathcal{L}_{\mathcal{T}^V} \psi = 0.$$  

(62)

On behalf of the above, one may conclude that $\mathcal{T}^V$ defines an infinitesimal automorphism of the 2-form $\psi$.

Finally, consider the 2-form

$$II = f\psi;$$

(63)

following [4], $f$ is called the energy scalar. Now, in view of (23), one has

$$dII = f \left( \frac{df}{f} + \frac{d\|T\|^2}{2\|T\|^2} \right) \wedge II.$$  

(64)

By reference to [4] and in case that

$$\frac{df}{f} + \frac{d\|T\|^2}{2\|T\|^2} = 0,$$

this shows that $II$ can then be seen as the canonical symplectic form of the $4m$-dimensional manifold $TM$. Finally, we set

$$r = fv,$$

where $v = \frac{1}{2} \sum (V^A)^2$ denotes the Liouville function; then, by reference to [4], the pair $(r, II)$ defines a regular mechanical system (in the sense of Klein) having $r$ as kinetic energy.

**Theorem 2.** Let $TM$ be the tangent bundle manifold having as basis the conformal symplectic manifold $M(\Omega, \mathcal{T}, \alpha)$ structured by a $\mathcal{T}$-parallel connection and having $\alpha = \mathcal{T}^0$ as covector of Lee. Let $V$, $\mu$, and $v$ be the Liouville vector field, the Liouville form, and the Liouville function of $TM$ respectively. One has the following properties:

(i) the complete lift $\Omega^C$ on $TM$ of the conformally symplectic form $\Omega$ of $M$, is a homogeneous 2-form of class 1, i.e.

$$\mathcal{L}_V \Omega^C = \Omega^C;$$
(ii) the vertical lift $T^V$ of $T$ defines an infinitesimal automorphism of the 2-form $\psi = \sum dV^A \wedge \omega^A, (A = 1, \cdots, 2m)$;

(iii) if $f$ stands for the energy function of $M$, then the 2-form $II = f\psi$ is the canonical symplectic form on $TM \left(\frac{df}{f} + \frac{d\|T\|^2}{2\|T\|^2} = 0\right)$, and the pair $(r, II)$, consisting of the scalar $r = f\nu$ and the 2-form $f\psi$, defines a regular mechanical system (in the sense of Klein) on $TM$.

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