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Homogeneous Manifolds in Codimension Two Revisited

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Abstract. In this paper we study Riemannian homogeneous submanifolds of Euclidean spaces in codimension two. If the index of relative nullity of the second fundamental form is relatively low, we prove that the submanifold is a product $M_1^m \times \mathbf{R}^k$ where M_1^m is either isometric to a sphere or to a compact isoparametric hypersurface of the sphere or covered by $S^{m-1} \times \mathbf{R}$. For homogeneous Einstein manifolds we obtain a complete classification which improves the result in [1].

Keywords: relative nullity, rigid immersions, isometry, einstein manifolds.

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Introduction

The purpose of this paper is to extend the results obtained by the authors in [1] and [5] on codimension two homogeneous submanifolds of Euclidean spaces. In the study of isometric immersions of Riemannian homogeneous manifolds, the first step is to investigate the equivariance of the immersion which in turn implies that its image is an extrinsically homogeneous manifold, that is, it is the orbit of an isometric action in the ambient space. Such a property can be established by studying the rigidity of such immersions. To this end, in our previous articles we used a result of do Carmo-Dajczer in [4] for rigidity of isometric immersions in higher dimensions and proved the following result.

(Theorem $[CN_1]$**).** *Let* $f\colon M^n \to \mathbf{R}^{n+2}$ *be an isometric immersion of a Riemannian homogeneous manifold such that the minimum index of relative nullity* $\bar{\nu} = \min_{x\in M} \nu_f(x) \le n-5$*. Then either* f *is rigid or for every point* p

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in M there exist orthonormal vectors $\xi, \eta \in T_pM^{\perp}$ such that $\operatorname{rank} A_\eta \leq 2$ *and if $g \in Iso\,(M)$, ξ can be oriented so that $g_\star \circ A_\xi = A_\xi \circ g_\star$.*

It is easy to see from the Gauss equation and homogeneity of M that if $\operatorname{rank} A_\eta \leq 2$ then either $\operatorname{rank} A_\eta \leq 1$ for all points of M or $\operatorname{rank} A_\eta \equiv 2$. The study of the case $\operatorname{rank} A_\eta \leq 1$ showed that such immersions (as expected) are not equivariant, since they are essentially compositions of hypersurfaces with isometric immersions of $\mathbf{R}^{n+1}$ into $\mathbf{R}^{n+2}$. In [5] we showed that if $\operatorname{rank} A_\eta \equiv 2$ then the immersion is still equivariant and we classified it in the case that M is compact. It is worth pointing out that the equivariance came as consequence of the classification of homogeneous manifolds which admit immersions with this type of second fundamental form (see [5, Theorem 4.1]).

However, codimension two homogeneous manifolds of minimum index of relative nullity $\bar{\nu} \leq n-5$ were not completely classified in [5], since we have not found in the mathematical literature a characterization of non-compact extrinsically homogeneous manifolds. It is well known that the compact ones are isoparametric hypersurfaces of the sphere.

The first result of this paper describes the non-compact homogeneous submanifolds in codimension two.

Theorem 1. *Let $f\colon M^n \to \mathbf{R}^{n+2}$, $n \geq 2$, be an isometric immersion of a non-compact Riemannian homogeneous manifold. If f is equivariant then $f(M) = M_1 \times \mathbf{R}^k$, where M_1 is a compact isoparametric hypersurface of the sphere.*

Now, if the nullity of the second fundamental form is relatively low, Theorem 1 and the results in [5] give the following complete classification.

(Theorem $[CN_2]$**).** *Let $f\colon M^n \to \mathbf{R}^{n+2}$ be an isometric immersion of a Riemannian homogeneous manifold such that $\bar{\nu} = k \leq n-5$. Then $M = M_1^m \times \mathbf{R}^k$ and $f = f_1 \times i$, where $i\colon \mathbf{R}^k \to \mathbf{R}^k$ is the identity map and $\bar{\nu}_{f_1} = \min_{x \in M_1} \nu_{f_1}(x) = 0$. Moreover, one of the following occurs for M_1 and f_1:*

(a) *M_1 is isometric to a sphere S^m and f_1 is the composition of a standard embedding into a hyperplane with an isometric immersion of this hyperplane into $\mathbf{R}^{n+2}$.*

(b) *M_1 is covered by $S^{m-1} \times \mathbf{R}$ and $f_1 \circ p$ is the composition of $h \circ (g \times i)$ where p is the covering map, g is a standard embedding of S^{m-1} into a hyperplane, i the identity map and h an isometric immersion of this hyperplane into $\mathbf{R}^{n+2}$.*

(c) *$f_1(M_1)$ is a compact isoparametric hypersurface of the sphere.*

It also follows from Theorem 1 that the remaining cases for a complete understanding of codimension two homogeneous submanifolds are non-rigid immersions for which $\bar{\nu} = n-2, n-3, n-4$. The methods used in [5] do not apply to these cases due to the rigidity problem for codimensions greater than 1.

However, for homogeneous Einstein manifolds we obtain a complete classification. We recall that the class of homogeneous Einstein manifolds includes the *isotropy-irreducible* homogeneous spaces by a result of J.A. Wolf (see [3, p. 187]). Since homogeneous surfaces have constant curvature and 3-dimensional Einstein manifolds have constant sectional curvature, we study the case $n \geq 4$. Notice that either M is Ricci flat and hence flat by a result of D.V. Alekseevskii and B.N. Kimelfeld (see [3, p. 191]) or $\nu_f \equiv 0$. Therefore for $n \geq 5$ we use the classification above. For $n = 4$ we use a result of Jensen [7] which states that an Einstein homogeneous manifold of dimension 4 is a symmetric space.

The second result of this paper is the following theorem which improves the result in [1], since it does not require compactness for M.

Theorem 2. *Let $f\colon M^n \to \mathbf{R}^{n+2}$, $n \geq 4$, be an isometric immersion of a Riemannian homogeneous Einstein manifold. Then one of the following holds:*

(a) *M is flat and hence the product $T^k \times \mathbf{R}^{n-k}$ where T^k is a torus of dimension $k \leq 2$*

(b) *M is either a sphere or a product of two spheres, each of which is of dimension greater than 1.*

1 Proof of Theorem 1

Definition 1. A submanifold M of $\mathbf{R}^{n+p}$ is said to be (extrinsically) reducible if M splits in a Riemannian product $M_1 \times M_2$ and $\alpha(X, Y) = 0$ for all $X \in TM_1$ and all $Y \in TM_2$, where α denotes the second fundamental form.

Proposition 1. *Let $M^n = G(v)$, $n \geq 2$, be a homogeneous (extrinsically) irreducible full submanifold of $\mathbf{R}^{n+p}$, where G is a Lie subgroup of the isometry group of $\mathbf{R}^{n+p}$. If $p = 2$ then M is a compact isoparametric hypersurface of the sphere.*

Proof. It follows from a result of Olmos, in the appendix of [10], that M lies in a cylinder $S^m \times \mathbf{R}^k$. Let η be a unit normal vector field of the inclusion $i\colon S^m \times R^k \to \mathbf{R}^{n+2}$ and $\tilde{A}_\eta$ its corresponding Weingarten operator. Then $\tilde{A}_\eta$ has two distinct eigenvalues, 0 and c. Let E_c denote the eigenspace corresponding to the eigenvalue c.

For x in M^n, we consider the spaces $V(x) = T_xM \cap E_c(x)$ and $U(x) = T_xM \cap \operatorname{Ker} \tilde{A}_{\eta(x)}$. Let $y \in M$ and $g \in G$ such that $g(x) = y$. Then its differential $g_\star$ maps $\eta(x)$ to $\eta(y)$, $E_c(x)$ to $E_c(y)$ and $\operatorname{Ker} \tilde{A}_{\eta(x)}$ to $\operatorname{Ker} \tilde{A}_{\eta(y)}$. Moreover, $g_\star(T_xM) = T_yM$ and hence V and U are distributions defined on M that are invariant by isometries in G. Notice that

$$\dim\ V = \dim T_xM + \dim E_c(x) - \dim(T_xM\ +\ E_c(x)) \geq n+m-(n+1) = m-1$$

$$\begin{aligned}\dim\ U &= \dim T_xM + \dim \operatorname{Ker} \tilde{A}_{\eta(x)} - \dim(T_xM\ +\ \operatorname{Ker} \tilde{A}_{\eta(x)}) \\ &\geq n+k-(n+1) = k-1\end{aligned}$$

and then $W = V + U$ is at least $n-1$-dimensional, since $m + k = n + 1$.

Now we consider η as a normal vector field of the immersion $f\colon M \to \mathbf{R}^{n+2}$. Let ξ be another unit normal vector field which is orthogonal to η and $\nabla^\perp$ denote the normal connection of f. Since for X in W, $\tilde{A}_\eta(X)$ is orthogonal to ξ we have

$$\langle \nabla^\perp_X \eta, \xi\rangle = 0, \quad \forall X \in W.$$

Therefore, if $\dim W = n$, we conclude that ξ and η are parallel normal sections of the normal bundle which in turn implies $\operatorname{rank}(M) = 2$. A result of Olmos (see [9] and [10]) implies that M is contained in a sphere and since $\operatorname{rank}(M) = 2$ we conclude that M is an orbit of an s-representation. Further, the normal bundle is flat and hence M is an isoparametric submanifold of $\mathbf{R}^{n+2}$. Since M is irreducible, M must be compact, for complete non-compact isoparametric submanifolds are Riemannian products of compact ones with Euclidean spaces (see [12]). In particular, M is a compact isoparametric hypersurface of the sphere.

Suppose that $\dim W = n-1$, that is, $\dim V = m-1$ and $\dim U = k-1$. Then there exists $Z_1 \in E_c(x)$ and $Z_2 \in \operatorname{Ker} \tilde{A}_{\eta(x)}$ such that $Z_1 \perp V$ and $Z_2 \perp U$. Let A_η denote the Weingarten operator corresponding to η as normal vector of the immersion f. Then for $X \in TM$, $A_\eta(X)$ is given by the orthogonal projection of $\tilde{A}_\eta(X)$ onto TM. Let Z be the unit vector field orthogonal to the distribution W. It follows that $\beta = \{Z, X_1, \ldots, X_{m-1}, Y_1, \ldots, Y_{k-1}\}$ is an orthonormal basis of eigenvectors of A_η, where $X_i \in V$ and $Y_i \in U$. We will show that either $\langle \nabla^\perp_Z \eta, \xi\rangle = 0$ or β also diagonalizes A_ξ. In the former case we will have again two parallel sections; in the latter case we obtain from the Ricci equation that normal bundle of the immersion f is flat. In both cases we conclude again that M is a compact isoparametric submanifold of $\mathbf{R}^{n+2}$.

For that, let $\overline{\nabla}$ denote the Riemannian connection of the cylinder $S^m \times \mathbf{R}^k$. Since ξ is tangent to the cylinder, we write

$$\xi = aZ_1 + bZ_2 \qquad Z = bZ_1 - aZ_2,$$

where Z_1 and Z_2 are as above. Observe $\mathit{span}\{\xi\}, TM, E_c$ and $\operatorname{Ker} \tilde{A}_\eta$ are distributions invariant by $g_\star, \forall g \in G$ and then so is the distribution $\mathit{span}\{Z\}$. These facts imply that a and b are constant on M. Further, the Riemannian product structure of $S^m \times R^k$ implies that E_c and $\operatorname{Ker} \tilde{A}_\eta$ are parallel distributions with respect to the connection $\overline{\nabla}$. Then, for all $X \in TM$, we have $\overline{\nabla}_X Z_1 \in V$, since it is orthogonal to Z_1 and to $\operatorname{Ker} \tilde{A}_\eta$. Similarly, $\overline{\nabla}_X Z_2 \in U$. Therefore

$$\overline{\nabla}_X \xi = a \overline{\nabla}_X Z_1 + b \overline{\nabla}_X Z_2 \in W,$$

and in particular, $A_\xi(X) \in W$ for all $X \in W$. This immediately implies that Z is an eigenvector of A_ξ. Moreover, the above for the vector field Z gives

$$\overline{\nabla}_Z \xi = a \overline{\nabla}_Z Z_1 + b \overline{\nabla}_Z Z_2 \in W,$$

and then the eigenvalue corresponding to Z is zero. If either $k = 1$ or $m = 1$ then it is clear that $A_\eta \circ A_\xi = A_\xi \circ A_\eta$.

For the case that $m > 1$ and $k > 1$, notice first that if $b = 0$ then $Z \in \operatorname{Ker} \tilde{A}_\eta$ and hence $\langle \nabla^\perp_Z \eta, \xi \rangle = 0$. Let us then suppose that $\langle \nabla^\perp_Z \eta, \xi \rangle \neq 0$. We will show that in this case the distributions V and U are involutive. In fact, let Y_1 and Y_2 be vector fields in U. Since $\nabla^\perp_{Y_i} \eta = 0$, from the Codazzi equation

$$\nabla_{Y_1} A_\eta Y_2 - A_\eta \nabla_{Y_1} Y_2 = \nabla_{Y_2} A_\eta Y_1 - A_\eta \nabla_{Y_2} Y_1$$

we obtain

$$A_\eta [Y_1, Y_2] = 0.$$

Observe that

$$\tilde{A}_\eta Z = b \tilde{A}_\eta Z_1 = bc Z_1 = bc\,(a\xi + bZ) = abc\xi + b^2 c Z.$$

Then $A_\eta Z = b^2 c Z$ and since we are supposing $b \neq 0$, we conclude that the eigenvalue of A_η corresponding to Z is non-null. Therefore U is integrable and its leaves N are homogeneous submanifolds (see [5, Lemma 4.4]) of M immersed in $\mathbf{R}^{n+p}$. We also have

$$\langle \overline{\nabla}_{Y_1} Y_2, X \rangle = 0 \quad \forall X \in V \quad \text{and} \quad Y_1, Y_2 \in U$$

$$\langle \overline{\nabla}_{Y_1} Y_2, Z_1 \rangle = 0 \quad \text{and} \quad \langle A_\eta(Y_1), Y_2 \rangle = 0$$

implying that the first normal space of the immersion from $g = f_{|N} : N \to \mathbf{R}^{n+2}$ is spanned by Z_2. Moreover,

$$\langle \overline{\nabla}_Y Z_2, X \rangle = 0 \quad \forall X \in V, Y \in U \quad \text{and} \quad \langle \overline{\nabla}_Y Z_2, Z_1 \rangle = 0$$

and then the first normal space is parallel. Therefore each leaf N is a homogeneous hypersurface of Euclidean space $\mathbf{R}^k$. Likewise, the same Codazzi equation for $X_1, X_2 \in V$ gives

$$A_\eta([X_1, X_2]) = c[X_1, X_2],$$

which in turn implies that V is involutive, for if Z is an eigenvector of A_η corresponding to the eigenvalue c then $b = 1$ and hence $a = 0$; thus $Z = Z_1$ contradicting that $\dim W = n-1$. Therefore V is integrable and let S denote denote a maximal leaf through a point. The immersion $h = f_{|S} \colon S \to \mathbf{R}^{n+2}$ has first normal space spanned by Z_1 and η which is parallel, since

$$\langle \overline{\nabla}_X Z_1, Y\rangle = 0, \quad \langle \bar{\nabla}_X Z_1, Z_2\rangle = 0 \quad \text{and} \quad \langle A_\eta(X), Y\rangle = 0, \quad \forall X \in V, Y \in U.$$

It follows that each leaf S is a homogeneous hypersurface of the sphere S^m.

Let us consider now the product of immersions $S \times N \to S^m \times \mathbf{R}^k$. Then the manifold $\overline{M} = S \times N$ is extrinsically reducible in the sense of Definition 1. Therefore the second fundamental form

$$(\overline{\nabla}_X Y)^\perp = 0, \ \forall X \in V, Y \in U.$$

It follows then that $\langle \nabla_X Y, Z\rangle = 0$ and $\langle [X,Y], Z\rangle = 0$. This fact implies that the normal curvature (of the immersion f) $\langle R^\perp(X,Y)\xi, \eta\rangle = 0$, and from the Ricci equation we get that $A_\eta \circ A_\xi = A_\xi \circ A_\eta$. QED

Theorem 3. *Let $f \colon M^n \to \mathbf{R}^{n+2}$, $n \geq 2$, be an isometric immersion of a non-compact Riemannian homogeneous manifold. If f is equivariant then $f(M) = M_1 \times \mathbf{R}^k$, where M_1 is a compact isoparametric hypersurface of the sphere.*

PROOF. Since f is equivariant we have $f(M) = G(v)$. If $f(M)$ is non-compact, the proof of the theorem in the Appendix of [10] implies that $f(M)$ splits in a Riemannian product $M_1 \times \mathbf{R}^m$ where M_1 is irreducible. Now the previous proposition implies that M_1 is a compact isoparametric hypersurface of the sphere. QED

2 Codimension two homogeneous Einstein submanifolds

In this section we prove Theorem 2, but first we recall some results concerning 4-dimensional manifolds.

Let us consider M an oriented Riemannian manifold of dimension 4. Let Λ^2 denote the bundle of exterior 2-forms and $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ the eigenspace splitting for the Hodge $\star$-operator. The Weyl tensor W leaves $\Lambda^2_\pm$ invariant and we denote by $W^\pm$ its restriction to $\Lambda^2_\pm$. An oriented 4- manifold is called *self-dual* if $W^- = 0$.

Proposition 2. *Let $f \colon M^4 \to \mathbf{R}^6$, be an isometric immersion of an orientable, locally irreducible, locally symmetric Riemannian manifold. Then M has constant sectional curvature.*

Proof. It follows from Corollary 4 of [6] that M is self-dual for some orientation. Moreover, if M does not have constant sectional curvature, then either M or a double cover of M is a self-dual Kähler manifold. We will show that if M is isometrically immersed in $\mathbf{R}^6$, the latter case implies that M is flat contradicting the local irreducibility of M.

Suppose first that $R^{\perp}=0$. Then from the Ricci equation we get that the curvature tensor of M is pure and hence $W^+=W^-=0$ (see [3, Lemma 16.20, p.439]). Then M is conformally flat and since a locally irreducible, locally symmetric Riemannian manifold is Einstein, M has constant sectional curvature. Otherwise, let ξ,η be vectors of $T_{f(p)}M^{\perp}$. If $\{Y_1,\dots,Y_4\}$ is an orthonormal basis of T_pM and X and Y two arbitrary vectors, the Gauss equation implies

$$Ric(X,Y)=\sum_{i=1}^{4}\langle R(X,Y_i)Y_i,Y\rangle =$$

$$=t_1\langle A_\xi(Y),X\rangle \;+\; t_2\langle A_\eta(Y),X\rangle \;-\; \langle A_\xi(X),A_\xi(Y)\rangle \;-\; \langle A_\eta(X),A_\eta(Y)\rangle$$

where $t_1=traceA_\xi$ and $t_2=traceA_\eta$.

Now let $\{X_1,\dots,X_4\}$ and $\{Z_1,\dots,Z_4\}$ be orthonormal bases of eigenvectors of A_ξ and A_η respectively. Let us denote the eigenvalues of A_ξ by λ_i and of A_η by μ_i. We then suppose that neither of these two bases diagonalize simultaneously A_ξ and A_η. Since X_1 is not an eigenvector of A_η, there exist, say Z_1,Z_2 such that $\langle X_1,Z_i\rangle\neq 0$, for $i=1,2$. Since $Ric(X_1,X_j)=0$ for all $j\neq 1$ and X_1 is an eigenvector of A_ξ, we get from the above that

$$A_\eta(t_2X_1-A_\eta(X_1))=\gamma X_1.$$

Writing $X_1=\sum_i a_iZ_i$, the equation above implies that $a_i\sum_{j\neq i}\mu_j\mu_i=\gamma a_i$, for all i and, without loss of generality, we can assume that $a_1\neq 0, a_2\neq 0$. We obtain

$$\sum_{j\neq 1}\mu_j\mu_1=\gamma \qquad \sum_{j\neq 2}\mu_j\mu_2=\gamma.$$

Therefore, μ_1 and μ_2 are distinct roots of the quadratic equation $x^2-t_2x+\gamma=0$ and hence $t_2=\mu_1+\mu_2$ and $\mu_1\mu_2=\gamma$. This implies $\sum_{i\neq 1,2}\mu_i=0$. This also shows that if $\langle X_1,Z_i\rangle\neq 0$, for $i\geq 3$, the corresponding eigenvalue of Z_i is either μ_1 or μ_2. Therefore we can suppose that X_1 lies in the plane spanned by Z_1,Z_2. Now we repeat the procedure for A_ξ. There exists, say X_2, such that $\langle X_2,Z_1\rangle\neq 0$, and we have $t_1=\lambda_1+\lambda_2$, $\sum_{i\neq 1,2}\lambda_i=0$ and $\lambda_1\lambda_2=\delta$, where

$$A_\xi(t_1Z_1-A_\xi(Z_1))=\delta Z_1.$$

Again X_2 can be chosen such that $Z_1 \in span\{X_1, X_2\}$. Thus $span\{X_1, X_2\} = span\{Z_1, Z_2\}$ and the Ricci curvatures are given by the sectional curvature $K(X_1, X_2) = \lambda_1\lambda_2 + \mu_1\mu_2$. Moreover, the plane $span\{X_1, X_2\}$ is invariant by A_ξ and A_η. This implies that the plane $span\{X_3, X_4\}$ is also invariant by A_ξ and A_η and hence the 2-forms $X_1 \wedge X_2$ and $X_3 \wedge X_4$ are eigenvectors of the curvature operator $\mathcal{R}$ with same eigenvalue, because $K(X_1, X_2) = K(X_3, X_4)$. This implies that M has constant sectional curvature, otherwise, since $W^- = 0$, we conclude that W^+ has a null eigenvalue, denoted by $W_1^+ = 0$. Therefore, the other two eigenvalues satisfy $W_2^+ + W_3^+ = 0$. If either M or a double cover is a self-dual Kähler manifold, Proposition 2 of [6] shows that on Kähler manifold (with natural orientation) $\sharp spec W^+ \leq 2$. Therefore $W_2^+ = 0$ and $W_3^+ = 0$ and being Kähler and locally symmetric is flat. QED

Proof of Theorem 2. Since M is Einstein, the Ricci curvatures are given by S/n, where S is the scalar curvature. First we remark that if $S = 0$, being homogeneous, M is flat and hence the product $T^k \times \mathbf{R}^{n-k}$ where T^k is a torus (see [3, Vol.I, p. 191]). But since M is isometrically immersed in $\mathbf{R}^{n+2}$, a classical result of Tompkins, [13] implies that $k \leq 2$. Now, if $S \neq 0$, we conclude that $\nu(p) = 0$, for every p in M. Let us suppose first that f is rigid. Since $\nu(p) = 0$, Theorem 1 implies that $f(M)$ is compact and lies in a sphere S^{n+1}. Therefore, it is an Einstein hypersurface of the sphere. A result of Ryan in [11, p. 376] implies then that $f(M)$ and hence M itself, is isometric either to a sphere or to a product of two spheres each of which is of dimension greater than 1.

If f is non-rigid we consider first the case $n \geq 5$. Since $\nu(p) = 0$, for every point in M, there exist orthonormal sections ξ, η of the normal bundle such that $\operatorname{rank} A_\eta \leq 2$ and A_ξ is constant. From Theorem $[CN_2]$ we conclude that M is isometric either to a sphere S^n or $f(M)$ is compact Einstein hypersurface of the sphere S^{n+1} and hence M is the round sphere or the Riemannian product $S^2 \times S^{n-2}$. If $n = 4$, a result of Jensen, [7], states that a homogeneous Einstein 4-manifold is a symmetric space. If M is locally irreducible, we get from Proposition 2 that M has constant sectional curvature. Since a 4-dimensional hyperbolic space cannot be immersed in $\mathbf{R}^6$, M has positive constant curvature and hence is isometric to a sphere S^4 (recall that positively curved codimension 2 submanifolds are simply connected, [8]). If M is reducible, being Einstein, it is covered by the product of surfaces with the same constant curvature. By the result in [2] we conclude that the immersion is a product of two hypersurface immersions and since the hyperbolic plane cannot be isometrically immersed in $\mathbf{R}^3$, M a product of two 2-spheres with the same constant curvature. QED

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