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Lie Symmetries for Lattice Equations

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Abstract. Lie symmetries has been introduced by Sophus Lie to study differential equations. It has been one of the most efficient way for obtaining exact analytic solution of differential equations. Here we show how one can extend this technique to the case of differential difference and difference equations.

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1 Introduction

Sophus Lie introduced the notion of Lie groups as a unifying tool to study differential equations [1,2]. Lie groups have been used to solve differential equations, to classify them, and to establish properties of their solutions [3–8].

In particular, they provided one of the most efficient methods for obtaining exact analytic solutions for partial differential equations, i.e. symmetry reduction. This method consists of a sequence of algorithmic steps the first of which is finding the Lie group $\mathcal{G}$ of local point transformations:

$$\begin{aligned} \tilde{\mathbf{x}} &= F_g(\mathbf{x}, u) = \mathbf{x} + g\boldsymbol{\xi}(\mathbf{x}, u) + \cdots \\ \tilde{u} &= H_g(\mathbf{x}, u) = u + g\phi(\mathbf{x}, u) + \cdots \end{aligned} \tag{1}$$

where g is the set of group parameters and $\mathbf{x} \in R^p$. Given a partial differential equation of order k $E_k(\mathbf{x}, u, u_{x_i}, u_{x_{i,j}}, \ldots, u_{x_{i_1,\ldots,i_k}}) = 0$, $\mathcal{G}$ is obtained requiring that the transformation (1) leaves the set of solutions invariant, i.e. defining the infinitesimal generator of the Lie point symmetry

$$\hat{X} = \xi_i \partial_{x_i} + \phi \partial_u, \tag{2}$$

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where summation over repeated indices is understood, and requiring that

$$pr\hat{X}E_k|_{E_k=0} = 0. \tag{3}$$

where by $pr\hat{X}$ we mean the prolongation of $\hat{X}$ to all derivatives contained in $E_k = 0$. Then we look for solutions which are invariant under a subgroup $\mathcal{G}_0$ of $\mathcal{G}$. The finite group transformations, obtained by integrating the infinitesimal generators, transform solutions of $E_k = 0$ into solutions of the same equation. Other classes of exact solutions can be obtained by considering *conditional symmetries* [9].

We can consider the *evolutionary formalism* [3], an equivalent representation of the infinitesimal generator of the symmetry (2),

$$\begin{aligned} &\hat{X}_e = Q(\mathbf{x}, u, u_{x_1}, \ldots, u_{x_p})\partial_u, \\ &Q = \phi - \xi_i u_{x_i}, \quad pr\hat{X}_e E_k|_{E_k=0} = 0. \end{aligned} \tag{4}$$

Let us notice that the existence of the evolutionary symmetry (4) implies

$$u_g = Q(\mathbf{x}, u, u_{x_1}, \ldots, u_{x_p}), \tag{5}$$

i.e. that symmetries are equivalent to commuting flows. As Q is linear in the first derivatives of u eq. (5) is integrable on the characteristics being a quasilinear partial differential equation of first order and its solution gives the corresponding group transformations (1).

The notion of point symmetries can be extended [3] to the case of *generalized symmetries* by requiring that $Q = Q(\mathbf{x}, u, u_{x_{i_1}}, \ldots, u_{x_{i_1},\ldots,x_{i_s}})$. In this case eq. (5) is a partial differential equation of order s. To get the group transformation we need to solve its initial value problem starting from a generic initial data. In general this is not possible and thus we are not able to find the corresponding group transformation. However, in the case of variational symmetries, by Noether theorem [3], generalized symmetries still can provide conservation laws and their existence is usually associated to *exactly integrable equations* [10,11].

The application of Lie group theory to discrete equations, like difference equations, differential-difference equations, or q-difference equations are much more recent [12–42].

By a difference equation we mean a functional relation, linear or non-linear, between a function calculated at different points of a lattice [43–47]. These systems appear in many applications. First of all they can be written down as discretizations of a differential equation when one is trying to solve it with a computer. In such a case one reduces the differential equation to a recurrence relation:

$$\frac{du}{dx} = f(x, u) \quad \Rightarrow \quad v(n+1) = g(n, v(n)).$$

On the other hand we can consider dynamical systems defined on a lattice, i.e. systems where the real independent fields depend on a set of independent variables which vary partly on the integers and partly on the reals. For example we can consider

$$\frac{d^2u(n,t)}{dt^2} = F(t, u(n,t), u(n-1,t), \ldots, u(n-a,t), u(n+1,t), \ldots, u(n+b,t)),$$

where a and b are some integer positive numbers. These kind of equations can appear in many different setting. Among them they are associated to the evolution of many body problems, to the study of crystals, to biological and economical systems, etc. .

As an example of possible applications we consider the problem of the transmission of energy in one dimensional molecular system, problem which is of particular relevance for understanding the functioning of physical systems of biological interest [48]. This is a particularly hot topic as biological processes require the transport of energy with low dispersion along essentially one dimensional chains, such as the spines in an α helix [49]. A mechanism for the nondispersive transport of vibrational energy along hydrogenon bonded chains was proposed by Davydov and its continuous limit for small lattice spacing gave rise to a Nonlinear Schrödinger equation (NLS) which has soliton solutions [50]. If such soliton like solutions are valid also at biological temperatures is an open problem. In the case of diatomic nonlinear lattices we can describe such systems by the equations

$$\begin{aligned} M_1\ddot{x}_n - k_1(y_n - x_n) + &k_2(x_n - y_{n-1}) - \\ &\epsilon\beta_1(y_n - x_n)^2 + \epsilon\beta_2(x_n - y_{n-1})^2 = 0 \\ M_2\ddot{y}_n + k_1(y_n - x_n) - &k_2(x_{n+1} - y_n) + \\ &\epsilon\beta_1(y_n - x_n)^2 - \epsilon\beta_2(x_{n+1} - y_n)^2 = 0 \end{aligned} \tag{6}$$

where M_1 and M_2 are the different values of the two atomic masses, ϵ is a small parameter while k_1, k_2, β_1 and β_2 are four constants of order 1. When $k_2 \ll k_1$ these equation represent a molecular chain with intramolecular interaction stronger than the intermolecular one. This is the case, for example, of an hydrogen-bonded polypeptide chain.

In ref. [48] this problem has been considered from two different points of view:

(1) One has performed a molecular dynamics numerical simulation.

(2) One has performed a reductive perturbative analysis of the dynamics of the diatomic chain.

The numerical results (see Figs.1-3 of [48]) show clearly the relevance of the nonlinear terms in this non-dispersive energy transport. Moreover one can show that localized initial excitations of generic form can evolve into solitonlike solutions that travel along the chain. The numerical calculations show that such solutions are robust with respect to thermal disorder, i.e. energy can be transported with no appreciable dispersion even at room temperature. In the second case the development led to a NLS equation and to a solitonlike solution that transfers energy without dispersion.

To conclude I would like to present a result by MacKay and Aubry [51]. In a Theorem they showed that almost any Hamiltonian network of weakly coupled oscillators has a 'breather' solution while the existence of breathers for a nonlinear wave equation is rare. The results of this Theorem imply that the discrete world can be richer of interesting solutions and thus worthwhile studying by itself and symmetries are a simple and efficient way to do so.

2 Intrinsic Lie point symmetries for discrete equations

The first steps in the construction of Lie symmetries for difference equations were taken by Shiguro Maeda in 1980 [12] and later extended by many authors [13–42].

For simplicity in the following I will consider just the case of a scalar equation in two independent variables but, equivalent results can be obtained in the case of N independent and M dependent variables.

A discrete equation in R^2 is a functional relation for a field $u(P)$ at L different points P_i in R^2, i.e. $E = E(x, t, u(P_1), \ldots, u(P_L)) = 0$. A differential difference equation is obtained by considering the points P_i uniformly spaces in one direction, say t, with spacing h_t, in such a way that we are allowed to consider the continuous limit when h_t goes to zero and the number of points in this direction goes to infinity.

The points P_i in R^2 can be labelled by two discrete indexes $P_i = P_{n,m}$ which characterize the points with respect to two independent directions, say x and t. For example, in cartesian coordinates we can write:

$$P_{n,m} = (x_{n,m}, t_{n,m}) \tag{7}$$

and the function $u(P)$ reads

$$u(P_i) = u(P_{n,m}) = u(x_{n,m}, t_{n,m}) = u_{n,m}. \tag{8}$$

A difference scheme will be *a set of relations* among the values of $\{x, t, u(x, t)\}$ at a finite number, say L, of points in R^2 $\{P_1, \ldots, P_L\}$ around a reference point, say P_1. Some of these relations will define where the points are in R^2 and others how $u(P)$ transforms in R^2. The number of relations required will depend on the kind of initial - boundary value problem we are interested in. If the lattice is completely defined by those relations than we need to have five equations, four which define the two independent variables in the two independent directions in R^2, and one the dependent variable in terms of the lattice points:

$$\begin{aligned} &E_a(\{x_{n+j,m+i}, t_{n+j,m+i}, u_{n+j,m+i}\}) = 0 \\ &1 \le a \le 5; \qquad -i_1 \le i \le i_2, \quad -j_1 \le j \le j_2 \quad (i_1, i_2, j_1, j_2) \varepsilon Z^+ \\ &i_1 + i_2 = N, \qquad j_1 + j_2 = M. \end{aligned} \tag{9}$$

System (9) must be such that, starting from $\{x, t, u\}$ in L points we are able to calculate it in all points.

If a continuous limit of (9) exists, than one of the equations will go over to a partial differential equation and the others will be identically satisfied (generically $0 = 0$). We can also do partial continuous limits when only one of the independent variables become continuous while the other is still discrete. In this case only part of the lattice equations are identically satisfied and we obtain a differential difference equation for the dependent variable and an equation for the lattice variable.

To clarify the ideas let us present now some examples of difference scheme.

Let as consider at first the case of the discrete heat equation on a uniform orthogonal lattice:

$$\frac{u_{n+1,m} - u_{n,m}}{t_{n+1,m} - t_{n,m}} = \frac{u_{n,m+2} - 2u_{n,m+1} + u_{n,m}}{(x_{n,m+1} - x_{n,m})^2}, \tag{10}$$

$$\begin{aligned} x_{n,m+1} - x_{n,m} = h_x; \qquad t_{n,m+1} - t_{n,m} = 0, \\ x_{n+1,m} - x_{n,m} = 0; \qquad t_{n+1,m} - t_{n,m} = h_t, \end{aligned} \tag{11}$$

where h_x, h_t are two a priory fixed constants which define the spacing between two neighboring points in the two directions of the orthogonal lattice. The lattice equations (11) could be substituted by different ones which will provide different Lie point symmetries for the difference scheme while keeping the continuous limit. For example we can consider instead of eq. (11)

$$\begin{aligned} &x_{n,m+2} - 2x_{n,m+1} + x_{n,m} = 0; \qquad t_{n,m+1} - t_{n,m} = 0 \\ &x_{n+1,m} - x_{n,m} = 0; \qquad t_{n+2,m} - 2t_{n+1,m} + t_{n,m} = 0, \end{aligned} \tag{12}$$

or

$$\begin{aligned} &x_{n,m+2} - 2x_{n,m+1} + x_{n,m} = 0; \qquad t_{n,m+1} - t_{n,m} = 0, \\ &x_{n+1,m} - x_{n,m} = 0; \qquad t_{n+1,m} - t_{n,m} = c(x_{n,m+1} - x_{n,m})^2, \end{aligned} \tag{13}$$

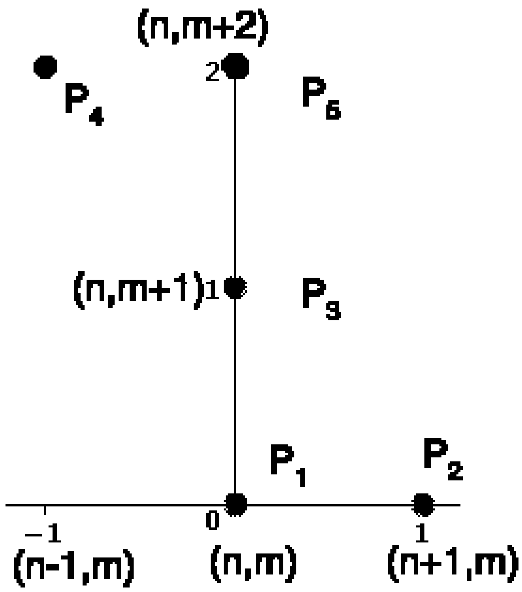

Figure 1. The set of points which are connected by the DTL (15).

or

$$x_{n,m+2} - 2x_{n,m+1} + x_{n,m} = 0; \qquad t_{n,m+1} - t_{n,m} = 0, \tag{14}$$
$$x_{n+1,m} - (1+c)x_{n,m} = 0; \qquad t_{n+1,m} - t_{n,m} = h.$$

Eq. (12) corresponds to a lattice scheme when the lattice spacing can be varied in the Lie point transformation, while the lattice (13) involves only four points as the heat equation itself and eq. (14) corresponds to an exponential lattice, i.e. a lattice that is neither equally spaced nor orthogonal. Eq. (11) or eqs. (12, 13, 14) determine completely the lattice variable $(x_{m,n}, t_{m,n})$ in terms of some initial conditions.

A second example is provided by the Discrete-time Toda Lattice equation (DTL) [52]:

$$e^{u(P_1)-u(P_3)} - e^{u(P_3)-u(P_5)} = \alpha^2 \{ e^{u(P_4)-u(P_3)} - e^{u(P_3)-u(P_{12})} \} \tag{15}$$

where α is a constant parameter. The DTL is a relation between 5 points in the two dimensional plane (see Fig. 1):

$$P_5 = (x_{n,m+2}, t_{n,m+2}); \quad P_4 = (x_{n-1,m+2}, t_{n-1,m+2}); \tag{16}$$
$$P_3 = (x_{n,m+1}, t_{n,m+1}); \quad P_2 = (x_{n+1,m}, t_{n+1,m}); \quad P_1 = (x_{n,m}, t_{n,m}).$$

i.e.

$$e^{u_{n,m}-u_{n,m+1}} - e^{u_{n,m+1}-u_{n,m+2}} = \alpha^2\{e^{u_{n-1,m+2}-u_{n,m+1}} - e^{u_{n,m+1}-u_{n+1,m}}\}. \quad (17)$$

On an orthogonal uniform unchangeable lattice such that $t_{n,m} = t = mh_t$ and $x_{n,m} = nh_x$, setting $v_n(t) = u_{n,m}$ and $\alpha = h_t^2$, when h_t goes to zero, we get;

$$\ddot{v}_n = e^{v_{n-1}-v_n} - e^{v_n - v_{n+1}}, \quad (18)$$

the usual Toda Lattice [45].

2.1 Symmetries of a difference scheme

As we are interested in Lie point symmetries, we look for transformations of the form:

$$\begin{aligned} \tilde{x} &= F_g(x,t,u) = x + g\ \xi(x,t,u) + \cdots \\ \tilde{t} &= G_g(x,t,u) = t + g\ \tau(x,t,u) + \cdots \\ \tilde{u} &= H_g(x,t,u) = u + g\ \phi(x,t,u) + \cdots \end{aligned} \quad (19)$$

where g is the group parameter. The transformation (19) is such that if $\{x,t,u\}$ satisfy the difference scheme $E_a = 0$, $\{\tilde{x},\tilde{t},\tilde{u}\}$ will be a solution of the same scheme. Such a transformation acts on the whole space of the independent and dependent variables $\{x,t,u\}$, at least in some neighborhood of P_1 including all points up to P_L. This means that the same set of functions F_g, G_g and H_g will determine the transformation in all points of the scheme. In the point P_1 we define the infinitesimal generator as:

$$\hat{X}_{P_1} = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \phi(x,t,u)\partial_u \quad (20)$$

and than we prolong it to all other $L-1$ points of the scheme. Since the transformation is given by the same set of functions $\{F_g, G_g, H_g\}$ at all points, the *prolongation* of $\hat{X}_{P_1}$ is obtained simply by evaluating $\hat{X}_{P_1}$ at all points involved in the scheme. So

$$pr\hat{X} = \sum_{i=1}^{L} \hat{X}_{P_i}, \quad (21)$$

and the invariance condition for the difference scheme is:

$$pr\hat{X}E_a|_{E_a=0} = 0. \quad (22)$$

Eq. (22) is a set of functional equations whose solution is obtained by turning them into differential equations by successive derivation with respect to the independent variables $\{x,t,u\}$ at the different points of the lattice [53].

The solution of (22) provides us with the functions $\xi(x,t,u)$, $\tau(x,t,u)$ and $\phi(x,t,u)$, the infinitesimal coefficients of the local Lie point symmetry group. The group transformation is obtained by integrating the vector field, i.e. by solving the following system of differential equations:

$$\begin{aligned} \frac{d\tilde{x}}{dg} &= \xi(\tilde{x},\tilde{t},\tilde{u}), \qquad \tilde{x}|_{g=0} = x, \\ \frac{d\tilde{t}}{dg} &= \tau(\tilde{x},\tilde{t},\tilde{u}), \qquad \tilde{t}|_{g=0} = t, \\ \frac{d\tilde{u}}{dg} &= \phi(\tilde{x},\tilde{t},\tilde{u}), \qquad \tilde{u}|_{g=0} = u. \end{aligned} \tag{23}$$

In general we expect the infinitesimal coefficients ξ and τ to be determined by the lattice equations. So according to the form of the lattice, different symmetries can appear.

In fact, in the case of eq. (11), by applying the infinitesimal generator (20) to the lattice equations we get:

$$\begin{aligned} \xi(x_{n,m+1},t_{n,m+1},u_{n,m+1}) &= \xi(x_{n,m},t_{n,m},u_{n,m}); \\ \xi(x_{n+1,m},t_{n+1,m},u_{n+1,m}) &= \xi(x_{n,m},t_{n,m},u_{n,m}). \end{aligned}$$

As $u_{n,m+1}$, $u_{n+1,m}$ and $u_{n,m}$ are independent functions we get $\xi = \xi(x,t)$. $t_{n,m+1} = t_{n,m}$ but $x_{n,m+1} \neq x_{n,m}$ and consequently $\xi = \xi(t)$. As $x_{n+1,m} = x_{n,m}$ but $t_{n+1,m} \neq t_{n,m}$ we get that the only possible value for ξ in this case is ξ=constant. In a similar fashion we derive that also τ must be a constant and that $\phi{=}u + s(x,t)$, where $s(x,t)$ is a solution of the heat equation, the linear superposition formula. Summarizing we get that the infinitesimal generators of the symmetries for the heat equation are given by

$$\hat{P}_0 = \partial_t; \quad \hat{P}_1 = \partial_x; \quad \hat{W} = u\partial_u; \quad \hat{S} = s(x,t)\partial_u. \tag{24}$$

If we associate to the heat equation (10) the lattice (12) we have an extra generator, $\hat{D} = x\partial_x + 2t\partial_t$. See [54] for other examples.

From the results presented above we see that the symmetries of a discrete equation depend crucially on the structure of the lattice. In the examples considered (11 - 14) the lattice is completely defined but we can also think of the lattice as a variable which evolves together with the system. In such a way we should need less relations to determine it. For more details on possible definitions of the lattice to generate difference equations with predefined symmetries see, for example, the results by Dorodnitzin [30–38].

3 Extensions

As one can see in the examples considered so far in the literature [13–15, 54] discrete equations will, in general have less symmetries than the corresponding continuous system. In the case of discrete equations we rarely find symmetries which involve both dependent and independent variables. The extension to a variable lattice is not always sufficient to obtain them [38].

Research for more general symmetries requires the extension of the intrinsic Lie point ansatz. We have various possibilities.

Let us consider the dependent variables $u(P_j)$ $(j = 1, 2, \cdots, L)$ as independent fields. Under such an hypothesis the difference equation is just an algebraic relation between the fields. The discrete variables are just indices of the dependent fields and the only independent variables are the continuous ones. In this case the infinitesimal generator, for one dependent field u and one independent continuous variable t, reads:

$$\hat{X} = \tau(t, \{u(P_i)\})\partial_t + \sum_{i=1}^{L} \phi_{P_i}(t, \{u(P_i)\})\partial_{u_{P_i}} \tag{25}$$

So we extend the dependence of ξ, τ and ϕ from the point P_1 to all points P_j of the lattice and to impose the invariance condition no prolongation is necessary. As, by Taylor expansion, $u_{n,m+1}$ can be expressed in term of $u_{n,m}$ and all of its derivatives, this definition of the infinitesimal generator of the symmetries is equivalent to consider a particular class of generalized symmetries. This explains the difficulties for obtaining nontrivial results [14].

On the other hand, as was shown in [26, 27] a linear difference equations admits a symmetry group which is isomorph to the one of the continuous equation by allowing the symmetry generators to depends on various points on the lattice. As an example, let us consider the discrete heat equation (10):

$$\Delta_t u = \Delta_{xx} u \tag{26}$$

where the discrete derivative Δ is defined in term of the shift operator T_z as the incremental ratio,

$$\Delta_z^+ = \frac{T_z - 1}{h_z}, \qquad T_z f(z) = f(z + h_z), \tag{27}$$

with h_z the lattice spacing in the z direction. In [26, 27] it was shown that eq. (26) admits the following symmetry group:

$$
\begin{aligned}
\hat{P}_0 &= \Delta_t^+ u \partial_u, \\
\hat{P}_1 &= \Delta_x^+ u \partial_u, \\
\hat{W} &= u \partial_u, \\
\hat{B} &= (2tT_t^{-1}\Delta_x^+ u + xT_x^{-1} u)\partial_u, \\
\hat{D} &= (2tT_t^{-1}\Delta_t^+ u + xT_x^{-1}\Delta_x^+ u + \frac{1}{2}u)\partial_u, \\
\hat{K} &= \Big\{ t^2 T_t^{-1}\Delta_t^+ u - h_t t T_t^{-2}\Delta_t^+ u + txT_t^{-1}T_x^{-1}\Delta_x^+ u + \\
&\quad + \frac{1}{4}x^2 T_x^{-2} u - \frac{1}{4}h_x x T_x^{-2} u + \frac{1}{2} t T_t^{-1} u \Big\}\partial_u.
\end{aligned}
\tag{28}
$$

In a later work [55] it has been proven that the previous result can be extended to any discrete derivative and for any linear partial differential equation by using the following prescriptions:

- we write down the vector fields in evolutionary form
- we substitute

$$
\begin{array}{rcl}
u_{,t} & \to & \Delta_t u \\
u_{,x} & \to & \Delta_x u \\
x & \to & x\beta_x \\
t & \to & t\beta_t
\end{array}
\tag{29}
$$

  where β_x, β_t are functions of the shift operators, commuting with Δ_x and Δ_t, and such that the following commutation relation is satisfied:

$$[\Delta_z, z\beta_z] = 1. \tag{30}$$

The same construction can be carried out also in the case when T_z is a dilation operator [56], i.e.

$$T_z f(z) = f(zh_z) \tag{31}$$

Equation (30) define completely the function β_z in terms of Δ_z. If by Δ_z we mean ∂_x than $\beta_x = 1$; for Δ_z given by eq. (27) than $\beta_z = T_z^{-1}$, completely in agreement with the results presented in (28). If we choose $\Delta_z^s = \frac{T_z - T_z^{-1}}{2h_z}$ than $\beta_z = 2(T_z + T_z^{-1})^{-1}$. From the correspondence (29) it follows that for any analytic solution of the linear partial differential equation we find a solution of the discrete counterpart. In the case of Δ_z^+, or $\Delta_z^- = \frac{1-T_z^{-1}}{h_z}$, the continuous case will give all solutions of the discrete one. This will not be the case for the more general Δ_z^s, when by the correspondence (29) we obtain just a subset of the possible solutions. Eqs. (29, 30) are a generalization of the *classical umbral calculus* [57, 58].

As the symmetries (28) depend on more points on the lattice, we are in a situation similar to the case of generalized symmetries and in general we are no more able to get the corresponding group transformations. However we can still use the symmetries to carry out symmetry reductions and get explicit solutions. Let us see this on the example of the heat equation (25). In the case of the symmetry generator $\hat{P}_1 - a\hat{P}_0$ eq. (5) reads:

$$\frac{du_{n,m}}{dg} = -\left(\frac{1}{h_x} - a\frac{1}{h_t}\right) u_{n,m} + \frac{1}{h_x} u_{n,m+1} - \frac{a}{h_t} u_{n+1,m}. \tag{32}$$

Eq. (32) is a linear differential difference equation and to get the group transformations we need the explicit solution of its initial problem. The symmetry reduction is obtained by setting $\frac{du_{n,m}}{dg} = 0$. In this case we get:

$$u_{n,m} = c_0 + c_1(1 + a^2 h_t)^n (1 + a h_x)^m \tag{33}$$

which is a discrete representation of the continuous solution.

What happens in the case of nonlinear difference equations? In general we are no more able to obtain a symmetry group isomorphic to the continuous one, the application of the *umbral* correspondence to a nonlinear equation is questionable and research on it is in progress [59, 60].

In all cases, both for linear and nonlinear difference equations,

- we can still compute intrinsic Lie point symmetries
- generalized symmetries, i.e. symmetries depending on a finite number of points of the lattice, are usually associated to integrable equations and their form is usually very complicate and very difficult to predict apriori without any further information on the structure of the system at study. As an example let us show few symmetries of the discrete Burgers equation [17]

  $$\Delta_t u = \frac{1 + h_x u}{1 + h_t(\Delta_x u + u T_x u)} \Delta_x(\Delta_x u + u T_x u) \tag{34}$$

  i.e.

  $$u_{,\lambda_1} = [1 + h_t(\Delta_x u + u T_x u)]\Delta_t u \tag{35}$$

  a time translation and

  $$u_{,\lambda_2} = [1 + h_x u]\Delta_x \Big\{ 2t T_t^{-1} \frac{u}{1 + h_t(\Delta_x u + u T_x u)} + \\ + (x + \frac{h_x}{2}) T_x^{-1} \frac{1}{1 + h_x u} \Big\} \tag{36}$$

a boost. In both cases they depend nonlinearly on the field u at various points of the lattice. In the case of the time translation (35) polynomially, but sometimes, as one can see in eq. (36), also rationally.

As a further exemplification of the complicate structure of symmetries for nonlinear discrete equations, let us present the simplest symmetries one obtains for the Toda Lattice equation (18) and for the dNLS (45) from their integrability properties [20]:

$$\hat{Y}_{\mu_{-1}} = \partial_{v_n} \tag{37}$$

$$\hat{Y}_{\mu_0} = t\partial_{v_n} \tag{38}$$

$$\hat{Y}_{\mu_1} = (t\dot{v}_n - 2n)\partial_{v_n} \tag{39}$$

$$\hat{X}_{\epsilon_0} = \dot{v}_n\partial_{v_n} \tag{40}$$

$$\hat{X}_{\epsilon_1} = \{\dot{v}_n^2 + e^{v_{n-1}-v_n} + e^{v_n-v_{n+1}} - 2\}\partial_{v_n} \tag{41}$$

$$\begin{aligned}\hat{X}_{\epsilon_2} = \{\dot{v}_n^3 - 2\dot{v}_n + e^{v_{n-1}-v_n}(\dot{v}_{n-1} + 2\dot{v}_n) + \\ + e^{v_n-v_{n+1}}(\dot{v}_{n+1} + 2\dot{v}_n)\}\partial_{v_n}\end{aligned} \tag{42}$$

$$\begin{aligned}\hat{Y}_\nu = [t\{\dot{v}_n^2 + e^{v_{n-1}-v_n} + e^{v_n-v_{n+1}} - 2\} - (2n-1)\dot{v}_n + w_n]\partial_{v_n} \\ w_{n-1} - w_n 2\dot{v}_n\end{aligned} \tag{43}$$

where (37 - 40) are intrinsic symmetries, (40 - 42) are associated to isospectral deformation of the discrete Schrödinger spectral problem associated to the Toda lattice while (37 - 39, 43) are associated to non isospectral deformations. Let us notice that the coefficient of t in eq. (43) is not equal to $\dot{v}_n$, as would be in the case of the Korteweg de Vries equation, but corresponds to a higher flow in the Toda Lattice hierarchy. The nonzero commutation relations between the infinitesimal generators $\hat{X}$ corresponding to the isospectral deformations and $\hat{Y}$ corresponding to the non isospectral ones are:

$$[\hat{X}_{\epsilon_i}, \hat{X}_{\epsilon_j}] = 0; \qquad [\hat{X}_{\epsilon_1}, \hat{Y}_\nu] = 4\hat{X}_{\epsilon_1} - 2\hat{X}_{\epsilon_3}. \tag{44}$$

In the case of the discrete Nonlinear Schrödinger equation [61–63]

$$i\dot{Q}_n + \frac{1}{h_x^2}[2Q_n - (1-\epsilon|Q_n|^2)(Q_{n+1} + Q_{n-1})] = 0, \tag{45}$$

where $\epsilon = \pm 1$, the symmetries read

$$\hat{X}_1 = Q_n\partial_{Q_n} - Q_n^*\partial_{Q_n^*} \tag{46}$$

$$\hat{X}_2 = (1-\epsilon|Q_n|^2)[Q_{n+1}\partial_{Q_n} - Q_{n-1}^*\partial_{Q_n^*}] \tag{47}$$

$$\hat{X}_3 = (1-\epsilon|Q_n|^2)Q_{n-1}\partial_{Q_n} - (1-\epsilon|Q_n|^2)Q_{n+1}^*\partial_{Q_n^*} \tag{48}$$

$$\hat{Y} = \Big[-\frac{2t}{h_x^2}(1-\epsilon|Q_n|^2)(Q_{n+1}-Q_{n+1}) + i(2n+1)Q_n\Big]\partial_{Q_n} + \qquad (49)$$
$$-\Big[\frac{2t}{h_x^2}(1-\epsilon|Q_n|^2)(Q^*_{n+1}-Q^*_{n+1}) + i(2n+1)Q^*_n\Big]\partial_{Q^*_n}.$$

$\hat{X}_1$ and $\hat{X} = \hat{X}_2 + \hat{X}_3$ are intrinsic Lie point symmetries, $\{\hat{X}_1,\ \hat{X}_2$ and $\hat{X}_3\}$ are isospectral symmetries while $\hat{Y}$ is a nonisospectral symmetry and they satisfy commutation relations similar to eq. (44). We can use $\frac{\hat{Y}-i\hat{X}_1}{2}$ to do a symmetry reduction for the Nonlinear Schrödinger equation (45). In the continuous limit this would correspond to a dilation symmetry which would give rise to elliptic functions or Painlevé solutions. This reduces to solving the discrete NLS (45) together with

$$-\frac{t}{h_x}(1-\epsilon|Q_n|^2)(Q_{n+1}-Q_{n+1}) + inQ_n = 0.$$

Defining $Q_n = \rho_n e^{i\theta_n}$, under the assumption that $\rho_n^2 \neq \epsilon$ we get the following nonlinear reduced equation:

$$\sqrt{\rho_n^2\rho_{n+1}^2 - h_x^4 c_0^2} + \sqrt{\rho_n^2\rho_{n-1}^2 - h_x^4 c_0^2} = \frac{nh_x^2}{t}\frac{\rho_n^2}{1-\epsilon\rho_n^2}. \qquad (50)$$

Up to now we have shown that extra symmetries can be found in the case of linear and integrable difference equations and that we can use them to get solutions. In the following we want to provide a partial answer in the generic case.

Let us consider the class of differential difference equations:

$$\dot{u}_n = F_n(u_{n+1}, u_n, u_{n-1}) \qquad (51)$$

defined on a uniform unchangeable orthogonal lattice and look for symmetries depending linearly on t, as it is the case of the symmetries $\hat{Y}$ (43, 49) for the Toda Lattice and discrete Nonlinear Schrödinger equations. To exclude the case of integrable equations we assume symmetries of the form:

$$u_{n,\lambda} = t\dot{u}_n + H_n(u_{n+1}, u_n, u_{n-1}). \qquad (52)$$

Under the assumption that the differential difference equation (51) is really nonlinear

$$\left(\frac{\partial F_n}{\partial u_{n+1}}, \frac{\partial F_n}{\partial u_{n-1}}\right) \neq (0,0) \quad \forall\, n \qquad (53)$$

we can, by a simple transformation reduce in all generality the function H_n to the form

$$H_n = H_n(u_{n+1}, u_n) \tag{54}$$

with $\frac{\partial H_n}{\partial u_{n+1}} \neq 0$, as we are not interested in intrinsic Lie point symmetries.

We can state the following theorem, proved in [24];

1 Theorem. *If a nonlinear equation (51 has a symmetry of the form (53, 54) than it is equivalent, up to a Lie point transformation*

$$\tilde{t} = \omega t, \quad \tilde{u}_n = \phi_n(u_n); \quad \omega \neq 0, \quad \phi_n^{'} \neq 0 \quad \forall \quad n \tag{55}$$

to an equation of the form

$$\begin{aligned} \dot{u}_n &= A_n + B_n \\ A_n &= a_{n+1}e^{u_{n+1}} - a_n e^{u_n} - 1 \\ B_n &= a_n e^{-u_n} - a_{n-1}e^{-u_{n-1}} - 1 \\ a_n^2 &= n^2 + \alpha n + \beta \quad \forall \quad n, \end{aligned} \tag{56}$$

where α and β are arbitrary constants. Than the symmetry is given by

$$u_{n,\lambda} = t\dot{u}_n + A_n \tag{57}$$

and the equation is linearizable.

As a consequence of Theorem 1 an equation of the form (51) can have dilation symmetries of the form (52) only if it is equivalent to a linear equation.

4 Conclusions

We have shown how one can construct in a coherent way Lie point symmetries for discrete equations.

If we want to extend the symmetries to the case when they depend on more point of the lattice than this can be done only in the case of linear, linearizable or integrable equations. This statement is very plausible but no complete proof of this statement has been given. In the case of linear, linearizable or integrable equations we can find generalized symmetries from which we are not able to construct group transformations but we can use them to get explicit solutions via symmetry reduction.

For a generic equation we can construct intrinsic Lie point and conditional symmetries [14] and we can use them, as for partial differential equations, to construct group transformations, to do symmetry reduction and obtain explicit solutions, to classify the discrete equations according to their symmetries, etc..

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