Note di Matematica **23,** n. 2, 2004/2005, 113–137.

Generalized time-dependent oscillators: results from a group-theoretical approach and their application to cosmology

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Received: 30/5/2004; accepted: 30/5/2004.

Abstract. Some results following from the analysis of generalized time-dependent oscillators in the framework of the Lie group theory are reviewed. Their role in treating aspects concerning the loss of coherence in cosmological models is discussed.

Keywords: generalized time-dependent oscillators, Nöther invariants, alternative Lagrangians, quantum states in cosmology.

MSC 2000 classification: 58D19 (primary), 83F05.

1 Introduction

Symmetry methods have been widely applied for more than one century to examine physical systems. At the classical level, their usefulness is well recognized in providing a powerful way of searching exact solutions to the differential equations ruling dynamical systems. At the quantum level, they play a role in the characterization of dynamical groups, which are non-invariance groups whose generators not all commute with the Hamiltonian of a dynamical system. These groups are of fundamental importance in several branches of physics, such as nuclear physics, particle physics, condensed matter physics and quantum optics (see *e.g.* [1]). They yield, in fact, the energy spectrum and the degeneracy of levels and can be employed to build up the transition probabilities between states [2]. Nevertheless, the concept of a dynamical group does not appear to be uniquely defined in literature. A possible way to overcome the ambiguity was

proposed by Dothan in [3], where the definition of the dynamical group of a given system is based on the symmetry group of the corresponding quantum-mechanical equation of motion for the system. In [3] a clean and deep discussion regarding the concepts of *quantum-generating algebra* and *symmetry algebra* is presented. Dothan's definition arises naturally from the quantum version of the time-dependent symmetry transformation. The key point is that, as remarked by Malkin and Man'ko in [4], if $\psi(\vec{x},t)$ solves the time-dependent Schrödinger equation,

$$i\hbar\partial_t\psi = \hat{H}\psi, \tag{1}$$

then $\hat{K}(\vec{x},\vec{p},t)\,\psi(\vec{x},t)$ is a solution as well provided that $\hat{K}(\vec{x},\vec{p},t)$ is a generally time-dependent conserved quantity, that is

$$i\hbar\partial_t\hat{K} + \left[\hat{K},\hat{H}\right] = 0. \tag{2}$$

Dothan thus suggested to define the dynamical group as the group whose generators are provided by Eq. (2). Finding explicitly the invariant operators $\hat{K}$ for the system under study is a basic issue in the attempt of solving complicated Schrödinger equations by reducing them to much simplified problems [5,6]. In [1] the search for invariant operators was handled by means of a technique described by D'Hoker and Vinet [7] in the context of spectrum-generating superalgebras. A more direct procedure which works out algorithmically was followed in [8] where time-dependent constants of motion ("charges") associated with generalized oscillators[1] have been determined which can be interpreted as the dynamical group generators. The group approach (see *e.g.* [9]) underlies the strategy there exploited.

In this Communication we present a brief overview of some basic aspects concerning the Lie group-theoretical framework and its role to dwell upon topics of the physical interest, such as the mechanism of particle creation in cosmological models. The outline is as follows. In Section 2 symmetries and (classical) invariants of generalized time-dependent oscillators (GTDO's) shall be discussed. In Section 3 we shall argue on the way non-Nöther invariants can be used to construct alternative classical Lagrangians for GTDO's. Section 4 concerns the GTDO dynamical group. The quantum theory of GTDO's is discussed in Section 5 while in Section 6 we shall be faced with the application of specific results to problems arising in theoretical cosmology. Last Section is for conclusions. Basic definitions and few properties of the so-called *coherent states* and *squeezed states* are finally summarized in the Appendix.

[1] We shall make use of this very long-time traditional terminology although the locution is actually not appropriate. To be precise, it should be better to refer to these systems as linear time-dependent systems. A "generalized oscillator" also indicates, in fact, a system whose generalized time-dependent "frequency" is not periodic.

2 Symmetries of the GTDO equation

Though we know that they may even appear in others contributions to this Volume, for the sake of completeness we shall briefly recall some notions and definitions concerning symmetries which would make the Communication self-contained.

Let G be a Lie group of local transformations, depending on one parameter ϵ and with nonzero Jacobian, acting on variables (t, q) according to

$$t' = R(t, q; \epsilon), \qquad q' = S(t, q; \epsilon), \tag{3}$$

where the functions R and S are t-differentiable and the value $\epsilon = 0$ corresponds to the identity transformation $t = R(t, q; 0)$, $q = S(t, q; 0)$. The group G of transformations is called a *symmetry group* of the second order ordinary differential equation

$$\ddot{q} = f(t, q, \dot{q}), \tag{4}$$

where dot means time derivative and f is a known function, iff $q'(t') = g \circ q(t')$ is a solution of Eq. (4) for $g \in G$ so that $g \circ q$ is defined whenever $q(t)$ satisfies Eq. (4) [9]. The symmetry group G, which transforms solutions of Eq. (4) to other solutions, can be obtained via an algorithmic procedure. This allows us to write down the Lie algebra of vector fields underlying the Lie group G as follows. The transformations (3) are generated by the infinitesimal operator (vector field)

$$V = \xi(t, q)\partial_t + \varphi(t, q)\partial_q, \tag{5}$$

where

$$\xi(t, q) = \partial_\epsilon R(t, q; \epsilon) \mid_{\epsilon=0}, \quad \varphi(t, q) = \partial_\epsilon S(t, q; \epsilon) \mid_{\epsilon=0}. \tag{6}$$

If ϵ is regarded as a pertubative parameter, then Eqs. (3) give rise to the infinitesimal transformations

$$t' = t + \epsilon\xi(t, q), \quad q' = q + \epsilon\varphi(t, q) \tag{7}$$

at the first order in ϵ. The variation of $\dot{q}$ under (7) is given by

$$\dot{q}' = \dot{q} + \epsilon\varphi^t, \qquad \varphi^t = \varphi_t + [\varphi_q - (\xi_t + \xi_q\dot{q})], \tag{8}$$

where $\dot{q}' = \frac{dq'}{dt'}$, and subscripts denote partial derivatives. Introduce the first and the second prolongation of the vector field V [9]

$$\mathrm{pr}^1 V = V + \varphi^t\partial_{\dot{q}}, \qquad \mathrm{pr}^2 V = V + \varphi^t\partial_{\dot{q}} + \varphi^{tt}\partial_{\ddot{q}},$$

where φ^t is given by (8) and

$$\varphi^{tt} = \frac{d^2}{dt^2}(\varphi - \xi\dot{q}) + \xi\dddot{q}. \tag{9}$$

The group G can be extended to comprise the transformation of $\dot{q}$. The finite transformations take the form

$$t' = [\exp(\epsilon V)]t, \quad q' = [\exp(\epsilon V)]q, \quad \dot{q}' = [\exp(\epsilon \ \mathrm{pr}^1 V)]\dot{q}. \tag{10}$$

From the relation

$$\mathrm{pr}^2 V[\ddot{q} - f(t, q, \dot{q})] = 0, \tag{11}$$

whenever $\ddot{q} - f(t, q, \dot{q}) = 0$, for every infinitesimal generator V of G we can find the coefficients ξ and φ appearing in (5). Equation (11) is the starting point to derive all the Lie point symmetries for a differential equation of the form (4). It is useful to distinguish between *Nöther symmetries* (divergence symmetries) and *additional symmetries* (see *e.g.* [9]). Both are Lie-point symmetries in the sense that the functions ξ and φ appearing in the vector field (5) do not depend on derivatives of q. The divergence symmetries, which lead to the constants of motion of the Nöther type, can be singled out in the following way. Let (4) be the Euler-Lagrange equation for the Lagrangian $L(t, q, \dot{q})$. When eq. (4) possesses a *Nöther symmetry group* G_N, then the conservation equation

$$\frac{dI}{dt} = 0, \quad I = (\xi\dot{q} - \varphi)\partial_{\dot{q}}L - \xi L + B, \tag{12}$$

holds, if and only if the action integral $A = \int L(t, q, \dot{q})\ dt$ is invariant with respect to G_N. B in (12) stands for a proper function of t and q. This is an extended version of the original Nöther theorem. G_N is a subgroup of the (complete) symmetry group G. The conserved quantities (12) are the *Nöther invariants*. The Nöther symmetries have the property of generating Nöther invariants, while additional symmetries do not enjoy this feature. Additional generators constitute a subalgebra of the complete symmetry algebra whose subgroup does not preserve the action integral. They lead to alternative Lagrangians which may be t-dependent and give rise to the same Euler-Lagrange equation as the conventional Lagrangian (see later).

So far, we have been referring to a general case. Now, we shall handle the problem of symmetries of equations of the generalized time-dependent oscillator (GTDO) type in some detail. They can be obtained as the equations of motion for the Hamiltonians

$$H(t) = \frac{p^2}{2m} + \frac{1}{2}m\,\omega^2 q^2, \qquad m = m(t),\ \omega = \omega(t) \tag{13}$$

and read

$$\ddot{q} + M\,\dot{q} + \omega^2 q = 0, \quad M = M(t) = \frac{\dot{m}}{m}. \tag{14}$$

Before to continue, is useful to remark that for practical purposes Eq. (14) can be cast into the form

$$\ddot{y} + \Omega^2(t)\, y = 0, \tag{15}$$

where

$$y = q e^{\frac{1}{2}\int_{t_0}^{t} M(t')\, dt'}, \quad \Omega^2(t) = \frac{1}{4}\left(4\omega^2 - 2\dot{M} - M^2\right). \tag{16}$$

The Ermakov-Pinney-Milne (EMP) equation

$$\ddot{\sigma} + \Omega^2(t)\, \sigma = \frac{K}{\sigma^3}, \tag{17}$$

($K = \text{const}$) can be related to Eq. (15): if y_1 and y_2 are two independent solutions of Eq. (15) then the general solution of the auxiliary EMP equation (17) can be written as

$$\sigma = (Ay_1^2 + By_2^2 + 2Cy_1y_2)^{\frac{1}{2}}, \tag{18}$$

A, B, C being constants such that

$$AB - C^2 = \frac{K}{W_0^2}, \qquad W_0 = y_1\dot{y}_2 - \dot{y}_1 y_2 = const. \tag{19}$$

The complete Lie point symmetry algebra of Eq. (14) is constituted by eight (independent) generators, say $V_1, \ldots, V_8$. To see this, let us substitute the quantity

$$f = -\omega^2(t)q - M(t)\dot{q}$$

into Eq. (14). Equating coefficients of powers of $\dot{q}$ to zero we obtain

$$\xi = a_1 q + a_2, \quad \varphi = (\dot{a}_1 - Ma_1)q^2 + b_1 q + b_2, \tag{20}$$

where a_1, a_2, b_1, b_2 are time-dependent functions of integration satisfying the constraints

$$\ddot{a}_1 - M\dot{a}_1 + (\omega^2 - \dot{M})a_1 = 0, \tag{21}$$

$$2\dot{b}_1 - \ddot{a}_2 + \dot{M}a_2 + M\dot{a}_2 = 0, \tag{22}$$

$$\ddot{b}_1 + M\dot{b}_1 + 2\omega^2\dot{a}_2 + 2\omega\dot{\omega}a_2 = 0, \tag{23}$$

$$\ddot{b}_2 + Mb_2 + \omega^2 b_2 = 0. \tag{24}$$

with c_1, c_2 arbitrary constants. Equations (21)-(24) provide

$$a_1 = m^{1/2}\,\eta\ (c_1\cos\gamma + c_2\sin\gamma), \tag{25}$$

$$a_2 = \sigma^2\,(c_5\cos\theta + c_6\sin\theta + c_7), \tag{26}$$

$$b_1 = c_5\left[\left(\sigma\dot\sigma - \frac{M}{2}\sigma^2\right)\cos\theta - \frac{1}{2}\sin\theta\right] + \tag{27}$$

$$+\, c_6\left[\left(\sigma\dot\sigma - \frac{M}{2}\sigma^2\right)\sin\theta + \frac{1}{2}\cos\theta\right] + c_7\left(\sigma\dot\sigma - \frac{M}{2}\sigma^2\right) + c_8, \tag{28}$$

$$b_2 = m^{-1/2}\,\eta\ (c_3\cos\gamma + c_4\sin\gamma), \tag{29}$$

where $c_3, \ldots, c_8$ are arbitrary constants, the phases γ, θ are defined via $\gamma(t) = \int_{t_0}^{t}\eta^{-2}(s)\,ds$, $\theta(t) = \int_{t_0}^{t}\sigma^{-2}(s)\,ds$ and the functions $\eta = \eta(t)$ and $\sigma = \sigma(t)$ arise as solutions of the auxiliary EMP equation (17) with $K = 1$ and $K = 1/4$ respectively. Notice that the phase θ can be given in the form

$$\theta = -i\ln\frac{[\sqrt{A}\,e^{i\alpha}\,y_1(\eta) - \sqrt{B}\,e^{i\beta}\,y_2(\eta)]}{[\sqrt{A}\,e^{i\alpha}\,y_1(\eta) - \sqrt{B}\,e^{i\beta}\,y_2(\eta)]^*} \quad , \quad AB\sin^2(\alpha-\beta) = \frac{1}{4W_0^2}, \tag{30}$$

with α, β real constants [14]. Then, Eqs. (20) become

$$\xi = m^{1/2}\,\eta\ (c_1\cos\gamma + c_2\sin\gamma) + \sigma^2(c_5 cos\theta + c_6\sin\theta + c_7), \tag{31}$$

$$\varphi = m^{1/2}\,q^2\left\{c_1\left[\left(\dot\eta - \frac{M}{2}\eta\right)\cos\gamma - \frac{\sin\gamma}{\eta}\right] + c_2\left[\left(\dot\eta - \frac{M}{2}\eta\right)\sin\gamma + \frac{\cos\gamma}{\eta}\right]\right\} +$$
$$+\ q\left\{c_5\left[\left(\sigma\dot\sigma - \frac{M}{2}\sigma^2\right)\cos\theta - \frac{\sin\theta}{2}\right] + c_6\left[\left(\sigma\dot\sigma - \frac{M}{2}\right)\sigma^2\sin\theta + \frac{\cos\theta}{2}\right] +\right.$$
$$\left. +\ c_7\left(\sigma\dot\sigma - \frac{M}{2}\sigma^2\right) + c_8\right\} + m^{-\frac{1}{2}}\,\eta\ (c_3\cos\gamma + c_4\sin\gamma). \tag{32}$$

The functions η, σ and γ, θ are mutually dependent, $\gamma = \theta/2$, $\eta = \sqrt{2}\sigma$. This entails that, once the quantities (31)-(32) are introduced into Eq. (5), the eight generators of the complete Lie point symmetry algebra of the GTDO (14) can be explicitly written [8]. Each operator $V_1, \ldots, V_8$ generates a one-parameter subgroup of Lie point symmetry for Eq. (14). By focusing on the generalized oscillator, Nöther invariants can be determined on the basis of the procedure outlined below. Let us deal with Eq. (12). By using the expression

$$L = \frac{1}{2}m(t)[\dot q^2 - \omega^2(t)q^2] \tag{33}$$

for the Lagrangian, from Eq. (12) we have

$$(\frac{1}{2}\dot{m}\dot{q}^2 - \frac{1}{2}\dot{m}\omega^2 q^2 - m\omega\dot{\omega}q^2)\xi + \{\varphi_t + [\varphi_q - (\xi_t + \xi_q\dot{q})\}m\dot{q} + \\ - m\omega^2 q\varphi + (\frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2)(\xi_t + \xi_q\dot{q}) = B_t + B_q\dot{q}. \quad (34)$$

Equating the coefficient of powers of $\dot{q}$ to zero, Eq. (34) produces

$$\varphi = \frac{1}{2}(\dot{\xi} - M\xi)q + \chi, \quad B = \frac{m}{4}(\ddot{\xi} - \dot{M}\xi - M\dot{\xi})q^2 + m\chi q, \quad (35)$$

where the functions $\xi = \xi(t)$ and $\chi = \chi(t)$ obey the constraints

$$\dddot{\xi} + 4\Omega^2\dot{\xi} + 4\Omega\dot{\Omega}\xi = 0, \quad \ddot{\chi} + M\dot{\chi} + \omega^2\chi = 0, \quad (36)$$

Equations (36) can be solved as

$$\xi = \sigma^2\left(\kappa_1\cos\theta + \kappa_2\sin\theta + \kappa_3\right), \quad \chi = \sqrt{\frac{2}{m}}\sigma\left(\kappa_4\cos\frac{\theta}{2} + \kappa_5\sin\frac{\theta}{2}\right) \quad (37)$$

where $\kappa_1, \ldots, \kappa_5$ are arbitrary constants, $\theta = \theta(t) = \int \sigma^{-2}dt$, and $\sigma = \sigma(t)$ satisfying the EMP equation with $K = 1/4$. The Nöther invariants arise from Eqs. (12), (35), (37) after setting $\kappa_1, \kappa_j = 0$ $(j \neq 1)$, $\kappa_2 = 1, \kappa_j = 0$ $(j \neq 2)$, and so on. They read

$$I_1 = \frac{m}{2}\left\{\left[\sigma\dot{q} - \left(\dot{\sigma} - \frac{M}{2}\sigma\right)q\right]^2 - \frac{q^2}{4\sigma^4}\right\}\cos\theta + \\ + \frac{m}{2}\left[q\dot{q} - \frac{1}{\sigma}\left(\dot{\sigma} - \frac{M}{2}\sigma\right)q^2\right]\sin\theta, \quad (38)$$

$$I_2 = \frac{m}{2}\left\{\left[\sigma\dot{q} - \left(\dot{\sigma} - \frac{M}{2}\sigma\right)q\right]^2 - \frac{q^2}{4\sigma^4}\right\}\sin\theta + \\ - \frac{m}{2}\left[q\dot{q} - \frac{1}{\sigma}\left(\dot{\sigma} - \frac{M}{2}\sigma\right)q^2\right]\cos\theta, \quad (39)$$

$$I_3 = \frac{m}{2}\left\{\left[\sigma\dot{q} - \left(\dot{\sigma} - \frac{M}{2}\sigma\right)q\right]^2 + \frac{q^2}{4\sigma^4}\right\}, \quad (40)$$

$$I_4 = \sqrt{2m}\left\{\left[-\sigma\dot{q} + \left(\dot{\sigma} - \frac{M}{2}\sigma\right)q\right]\sin\frac{\theta}{2} + \frac{q}{2\sigma}\cos\frac{\theta}{2}\right\}, \quad (41)$$

$$I_5 = \sqrt{2m}\left\{\left[-\sigma\dot{q} + \left(\dot{\sigma} - \frac{M}{2}\sigma\right)q\right]\cos\frac{\theta}{2} - \frac{q}{2\sigma}\sin\frac{\theta}{2}\right\}. \quad (42)$$

Of the Nöther invariants (38)-(42) only two are functionally independent, in the sense that

$$I_1 = \frac{1}{2}(I_4^2 - I_5^2), \quad I_2 = I_4 I_5, \quad I_3 = \frac{1}{2}(I_4^2 + I_5^2). \tag{43}$$

We observe that Eqs. (41)-(42) give the general solution to Eq. (14),

$$q = \sqrt{\frac{2}{m}}\sigma \left(I_5 \cos\frac{\theta}{2} - I_4 \sin\frac{\theta}{2} \right). \tag{44}$$

Furthermore, the conjugate momentum $p = m\dot{q}$ takes the form

$$p = \sqrt{\frac{m}{2}}\frac{1}{\sigma}\left\{ \left[2\sigma\left(\dot{\sigma} - \frac{M}{2}\sigma \right) I_5 - I_4 \right] \cos\frac{\theta}{2} - \left[2\sigma\left(\dot{\sigma} - \frac{M}{2}\sigma \right) I_4 + I_5 \right] \sin\frac{\theta}{2} \right\}. \tag{45}$$

3 Alternative Lagrangians and Hamiltonians via non-Nöther operators

In [10] Edwards showed that a particular class of alternative (inequivalent) classical Hamiltonians can be written for a damped harmonic oscillator which make the quantization of the system ambiguous. The result motivates this Section. We would like to stress indeed that although the vector fields V_6, V_7, V_8 do not lead to Nöther invariants, notwithstanding they are crucial in the construction of alternative Lagrangians for Eq. (14). All these Lagrangians produce the same classical equation of motion. However, at the quantum level, the problem of the interpretation of these Lagrangians is not yet well settled up, and many questions remain to be clarified (see [11]). In this Section, we would like to expound some considerations on the additional generators $\tilde{V}_6$, $\tilde{V}_7$, $\tilde{V}_8$ (not of the Nöther type) and on the invariance properties of the Lagrangian for the standard harmonic oscillator (14) with $M = 0$ and $\omega = const.$ To this purpose, let us write down the additional generators associated with this case. They are (see also [12])

$$\tilde{V}_6 = (q\cos\omega t)\ \partial_t - (q^2\omega\sin\omega t)\ \partial_q, \tag{46}$$

$$\tilde{V}_7 = (q\sin\omega t)\ \partial_t + (q^2\omega\cos\omega t)\ \partial_q, \tag{47}$$

$$\tilde{V}_8 = q\,\partial_q. \tag{48}$$

Let us deal first with the operator $\tilde{V}_8$ for which $\xi = 0$ and $\varphi = q$ (see (5)). The first prolongation of $\tilde{V}_8$ thus takes the form

$$\mathrm{pr}^1\,\tilde{V}_8 = \tilde{V}_8 + (D_t)\,\partial_{\dot{q}} = q\,\partial_q + \dot{q}\,\partial_{\dot{q}}. \tag{49}$$

Applying (49) to the conventional Lagrangian $L = \frac{1}{2}\left(\dot{q}^2 - \omega^2 q^2\right)$, we therefore get

$$\left(\mathrm{pr}^1\,\tilde{V}_8\right)\,L = \left(q\,\partial_q + \dot{q}\,\partial_{\dot{q}}\right)\left[\frac{1}{2}\left(\dot{q}^2 - \omega^2 q^2\right)\right] = \left(\dot{q}^2 - \omega^2 q^2\right) = 2L. \tag{50}$$

We are now interested in determining how the action integral

$$A = \int_{t_1}^{t_2} dt\; L(t, q, \dot{q}) \tag{51}$$

behaves under the transformation $(t, q, \dot{q}) \to (t', q', q'_{t'})$. We have

$$L(t, q, \dot{q}) = L\left(e^{-\varepsilon V}\,t', e^{-\varepsilon V}\,q', e^{-\varepsilon\;\mathrm{pr}^1\,V}\,q'_{t'}\right) = e^{-\varepsilon\;\mathrm{pr}^1\,V'}L\left(t', q', q'_{t'}\right), \tag{52}$$

where $V' \equiv V\left(t', q', q'_{t'}\right)$ and $\mathrm{pr}^1\,V'$ depends on the coordinates $t', q', q'_{t'}$. Since t can be considered as a function of the variables t' and q', then

$$A = \int_{t'_1}^{t'_2} e^{-\varepsilon\;\mathrm{pr}^1\;V'}L\left(t', q', q'_{t'}\right)\;\left(\frac{\partial t}{\partial t'} + q'_{t'}\,\frac{\partial t}{\partial q'}\right)\,dt'. \tag{53}$$

So we deduce the new Lagrangian

$$L'_\varepsilon = L'_\varepsilon\left(t', q', q'_{t'}; \varepsilon\right) = \left[e^{-\varepsilon\;\mathrm{pr}^1\,V\left(t', q', q'_{t'}\right)}L\left(t', q', q'_{t'}\right)\right]\;\left(\frac{\partial t}{\partial t'} + q'_{t'}\,\frac{\partial t}{\partial q'}\right). \tag{54}$$

In case $L'_\varepsilon\left(t', q', q'_{t'}; \varepsilon\right) = L\left(t', q', q'_{t'}\right)$ the action integral (53) is called *invariant*. Then, the symmetry group maps solutions of the Euler-Lagrange equation into solutions of the same equation. We point out that such a property of the Euler-Lagrange equation is valid even if L'_ε is of the form:

$$L'_\varepsilon\left(t', q', q'_{t'}; \varepsilon\right) = L\left(t', q', q'_{t'}\right) + D_{t'}\phi\left(t', q', q'_{t'}\right), \tag{55}$$

ϕ being an arbitrary (differentiable) function. Thus, we would assume, as a definition of *invariance* of the action integral, formula (53) where the expression (55) holds.

The Lagrangian L'_ε can be written in terms of L. To this aim, let us start from (54) to obtain

$$L'_\varepsilon = \left(\frac{\partial}{\partial t'} + q'_{t'}\frac{\partial}{\partial q'}\right)\;e^{-\varepsilon V'}\,t'\,e^{-\varepsilon\;\mathrm{pr}^1\,V'}\,L\left(t', q', q'_{t'}\right), \tag{56}$$

which in turn provides, after performing a series expansion in the parameter ε and stopping the calculation at the first order, the expression

$$L'_\varepsilon = L - \varepsilon\;\left[\left(\mathrm{pr}^1\,V'\right)\;L + L\,D_{t'}\xi\right]. \tag{57}$$

If we introduce the quantity

$$L_1 = \left(\mathrm{pr}^1\, V'\right)\, L + L\, D_{t'}\xi, \tag{58}$$

(57) can be written as

$$L'_\varepsilon = L - \varepsilon L_1. \tag{59}$$

Now let us pay attention on the additional generator $\tilde{V}_8 = q\partial_q$. Since in this case $\xi = 0$, from (58) we have

$$L_1 = \left(\mathrm{pr}^1\, \tilde{V}_8\right)\, L = 2\, L. \tag{60}$$

Hence, (59) becomes

$$L'_\varepsilon = (1 - 2\varepsilon)\, L, \tag{61}$$

which tells us that, at the first order in ε, the transformed Lagrangian L'_ε is proportional to the original Lagrangian L. Below, we prove that the property of $L'_\varepsilon \propto L$ is fulfilled at any order in ε. In doing so, let us write down the finite transformations identifying V by $\tilde{V}_8$,

$$t' = e^{\varepsilon \tilde{V}_8}\, t = e^{\varepsilon\, q\, \partial_q}\, t = t, \tag{62}$$

$$q' = e^{\varepsilon \tilde{V}_8}\, q = e^{\varepsilon\, q\, \partial_q}\, q = e^{\varepsilon\, q}, \tag{63}$$

$$q'_{t'} = e^{\varepsilon\ \mathrm{pr}^1\, \tilde{V}_8}\ \dot{q} = e^{\varepsilon\left(q\partial_q + \dot{q}\partial_{\dot{q}}\right)}\, \dot{q} = e^{\varepsilon}\, \dot{q}. \tag{64}$$

On the other hand, Eq. (54) can be elaborated to give

$$L'_\varepsilon = e^{-\varepsilon\left(t'\partial_{t'} + q'_{t'}\partial_{q'_{t'}}\right)}\, L\, \left(\partial_{t'} + q'_{t'}\partial_{q'}\right)\, t = e^{-\varepsilon\left(q'\partial_{q'} + q'_{t'}\partial_{q'_{t'}}\right)} L = e^{-2\varepsilon} L\,, \tag{65}$$

which ensures the assertion. An important consequence of Eq. (65) is that it breaks the condition (55) of invariance of the action integral with respect to the symmetry subgroup generated by $\tilde{V}_8 = q\partial_q$. We can thus conclude that a Nöther invariant (constant of the motion) associated with $\tilde{V}_8$ does not exist. However, since the new Lagrangian L'_ε can be derived from the old Lagrangian L multiplying the last by a constant factor, the equation of motion (*i.e* (14) with $M = 0$, ω =const) is again invariant under the subgroup generated by $\tilde{V}_8$. Therefore, notwithstanding the lack of a Nöther invariant related to $\tilde{V}_8$, the subgroup generated by $\tilde{V}_8$ still has the property to transform solutions of the equation of motion into solutions of the same equation.

The action of the other additional generators (46)-(47) can be analyzed by adopting the same procedure carried out for $\tilde{V}_8$. The operators $\tilde{V}_6$ and $\tilde{V}_7$ have a similar structure, so that here we shall develop in some details the computations

relative to only one of these operators, say $\tilde{V}_7$. We note that the expression (47) for $\tilde{V}_7$ is given by choosing $\xi = q\sin\omega t$ and $\phi = q^2\cos\omega t$. Furthermore, we have

$$\mathrm{pr}^1\,\tilde{V}_7 = q\sin\omega t\,\partial_t + q^2\omega\cos\omega t\,\partial_q + \left[-\left(\dot{q}^2+\omega^2q^2\right)\sin\omega t + q\dot{q}\omega\cos\omega t\right]\,\partial_{\dot{q}} \quad (66)$$

so that

$$\left(\mathrm{pr}^1\,\tilde{V}_7\right)\,L = 2\omega q\cos\omega t\;L - \dot{q}\sin\omega t\;\left(\dot{q}^2+\omega^2q^2\right), \quad (67)$$

and

$$L_1 = 3\omega q\cos\omega t\;L - \frac{1}{2}\dot{q}\;\left(\dot{q}^2+3\omega^2q^2\right)\;\sin\omega t, \quad (68)$$

where Eq. (58) has been used. The Lagrangian L'_ε, which comes from the transformation of $L = \frac{1}{2}\left(\dot{q}^2-\omega^2q^2\right)$ with respect to the symmetry subgroup generated by $\tilde{V}_7$, reads $L'_\varepsilon = L - \varepsilon L_1$ (see (59)) with L_1 given by (68). It is a simple matter to see that

$$\frac{d}{dt}\frac{\partial L_1}{\partial\dot{q}} - \frac{\partial L_1}{\partial q} = 3\left(\ddot{q}+\omega^2q\right)\;I_5, \quad (69)$$

where $I_5 = (\omega q\cos\omega t - \dot{q}\sin\omega t)$ is a Nöther invariant. If the variational principle is applied to the Lagrangian L_1, the condition (69) implies two cases: *a)* $\ddot{q}+\omega^2q = 0$; *b)* $I_5 = 0$. Case *a)* means that q is (again) solution of the equation of motion of the harmonic oscillator. In order to understand case *b)*, we resort to the following considerations. Let us first remind the generator of the Nöther invariant I_5, *i.e.* $V_5 = \sin\omega t\,\partial_q$. We can now ask what is the behaviour of the general solution $q = k_1\cos\omega t + k_2\sin\omega t$ of the standard harmonic oscillator under the action of the symmetry subgroup generated by V_5. To answer the question, let us take

$$e^{\varepsilon V_5}\,(q - k_1\cos\omega t - k_2\sin\omega t) = q - k_1\cos\omega t + (\varepsilon - k_2)\sin\omega t \quad (70)$$

at the first order in ε. The symmetry subgroup generated by $V_5 = \sin\omega t\,\partial_q$ thus transforms a given solution of the equation of motion ((14) with $M = 0$, ω =const) characterized by the arbitrary constants (k_1, k_2) into another solution characterized by the arbitrary constants $(k_1, \varepsilon - k_2)$. In both cases we get $I_5 = k_1\omega$. This fact leads to the statement that *the symmetry subgroup generated by* V_5 *preserves the property of constant of the motion possessed by* I_5. In the light of this result, case *b)* is consistent with the choice $k_1 = 0$; both $q = k_2\sin\omega t$ and $\tilde{q} = (\varepsilon - k_2)\sin\omega t$ are (particular) solutions of the standard harmonic oscillator, Eq. (14) with $M = 0$ and and ω =const. Similar features are shared by all the Nöther generators.

4 The GTDO's dynamical group

Invariants $I_1, \ldots, I_5$ (see Eqs. (38)-(42)) are such that

$$\frac{dI_j}{dt} = \{I_j, H\} + \frac{\partial I_j}{\partial t} \equiv 0, \quad (71)$$

where H is the Hamiltonian (13) of the GTDO (14), and the symbol $\{,\}$ stands for the Poisson bracket with respect to the canonical variables q, p, *i.e.* $\{A, B\} = \partial_q A\, \partial_p B - \partial_p A\, \partial_q B$. The Poisson brackets involving the Nöther invariants $I_1, \ldots, I_5$ are

$$\{I_1, I_2\} = 2I_3, \quad \{I_2, I_3\} = -2I_1, \quad \{I_3, I_1\} = -2I_2, \tag{72}$$

$$\{I_4, I_5\} = 1, \tag{73}$$

$$\{I_1, I_4\} = I_5, \quad \{I_1, I_5\} = I_4, \quad \{I_2, I_4\} = -I_4, \tag{74}$$

$$\{I_2, I_5\} = I_5, \quad \{I_3, I_4\} = -I_5, \quad \{I_3, I_5\} = I_4. \tag{75}$$

The main result coming from the analysis of Eqs. (72)-(75) is that the Nöther invariants I_1, I_2, I_3 form a Lie algebra under the Poisson bracket operation (see in particular relations (72)). The algebra is of the $su(1,1)$ type, underlying the noncompact group $SU(1,1)$. This is a relevant fact because it allows us to define unambiguously the *dynamical group* associated with the GTDO (14). From the purpose to dwell upon the dynamical group of Eq. (14), the following considerations are in order. In the quantum theory of the GTDO the commutation rules between the operators $\hat{I}_1, \ldots, \hat{I}_5$ corresponding to the classical constants of the motion (38)-(41) are derived from (72)-(75) by resorting to the substitution

$$\{I_j, I_k\} \to \frac{1}{i\hbar}[\hat{I}_j, \hat{I}_k]. \tag{76}$$

These commutation rules define a noncompact Lie algebra $su(1,1)$ under the commutation bracket operation. The noncompact group $SU(1,1)$, associated with the $su(1,1)$ algebra, can be identified as the *dynamical group* of Eq. (14). The existence of the dynamical group $SU(1,1)$ is dictated by the Nöther symmetry properties of Eq. (14), and the invariant operators of the Nöther type $\hat{I}_1, \hat{I}_2$ and $\hat{I}_3$ represent a natural realization of the Lie algebra of $SU(1,1)$.

5 Quantum theory of the GTDO

The quantum theory of the GTDO can be developed starting from the Hamiltonian operator

$$\hat{H} = \frac{\hat{P}^2}{2m} + \frac{1}{2} m\omega^2 \hat{Q}^2, \tag{77}$$

where $m = m(t)$, $\omega = \omega(t)$ and and $[\hat{Q}, \hat{P}] = i\hbar$. A set of invariant operators of the Nöther type, corresponding to the classical ones, can be constructed from Eqs. (38)-(42) by adopting the prescription $q \to \hat{Q}$, $\dot{q} \to \frac{\hat{P}}{m}$ and taking properly

care of the operator ordering. In doing so, we obtain

$$\hat{I}_1 = m\left\{\left[\frac{\sigma}{m}\hat{P} - \left(\dot{\sigma} - \frac{M}{2}\sigma\right)\hat{Q}\right]^2 - \frac{\hat{Q}^2}{4\sigma^4}\right\}\cos\theta + \\ + m\left[\hat{Q}\hat{P} - \frac{1}{\sigma}\left(\dot{\sigma} - \frac{M}{2}\sigma\right)\hat{Q}^2 - \frac{i\hbar}{2m}\right]\sin\theta, \tag{78}$$

etc., where σ obeys the EMP auxiliary equation (17) with $K = 1/4$. The invariant quantities $\hat{I}_1, \ldots, \hat{I}_5$ satisfy an equation of the type (2), *i.e.*

$$\frac{d\hat{I}_j}{dt} = \frac{1}{i\hbar}[\hat{I}_j, \hat{H}] + \frac{\partial \hat{I}_j}{\partial t} = 0, \tag{79}$$

with $\hat{H}$ given by (77). At this stage, the time-dependent lowering and raising operators

$$\hat{a} = \sqrt{\frac{m}{2}}\left\{\frac{\hat{Q}}{2\sigma} + i\left[\frac{\sigma}{m}\hat{P} - \left(\dot{\sigma} - \frac{M}{2}\sigma\right)\hat{Q}\right]\right\}, \tag{80}$$

$$\hat{a}^\dagger = \sqrt{\frac{m}{2}}\left\{\frac{\hat{Q}}{2\sigma} - i\left[\frac{\sigma}{m}\hat{P} - \left(\dot{\sigma} - \frac{M}{2}\sigma\right)\hat{Q}\right]\right\}, \tag{81}$$

which fulfill the commutation relation $[\hat{a}, a^\dagger] = \hat{1}$ can be introduced [8]. In terms of $\hat{a}, \hat{a}^\dagger$, the canonically conjugate operators $\hat{Q}$ and $\hat{P}$ therefore take the form

$$\hat{Q} = \sqrt{\frac{\hbar}{m}}\sigma(\hat{a} + \hat{a}^\dagger), \qquad \hat{P} = \sqrt{\hbar m}\left(\zeta\hat{a} + \zeta^*\hat{a}^\dagger\right) \tag{82}$$

where

$$\zeta = -\left[\frac{i}{2\sigma} + \left(\frac{M}{2}\sigma - \dot{\sigma}\right)\right]. \tag{83}$$

This would yield the Hermitian operators

$$\hat{I}_1 = -\frac{\hbar}{2}(e^{i\theta}\hat{a}^2 + e^{-i\theta}\hat{a}^{\dagger\,2}), \qquad \hat{I}_2 = i\frac{\hbar}{2}(e^{i\theta}\hat{a}^2 - e^{-i\theta}\hat{a}^{\dagger}\,2), \tag{84}$$

$$\hat{I}_3 = \hbar(\hat{a}^\dagger\hat{a} + \frac{1}{2}), \qquad \hat{I}_4 = i\sqrt{\frac{\hbar}{2}}(e^{i\frac{\theta}{2}}\hat{a} - e^{-i\frac{\theta}{2}}\hat{a}^\dagger), \tag{85}$$

$$\hat{I}_5 = \sqrt{\frac{\hbar}{2}}(e^{i\frac{\theta}{2}}\hat{a} + e^{-i\frac{\theta}{2}}\hat{a}^\dagger). \tag{86}$$

We have

$$\hat{I}_1 = \frac{1}{2}(\hat{I}_4^2 - \hat{I}_5^2), \qquad \hat{I}_2 = \frac{1}{2}(\hat{I}_4\hat{I}_5 + \hat{I}_5\hat{I}_4), \qquad \hat{I}_3 = \frac{1}{2}(\hat{I}_4^2 + \hat{I}_5^2). \tag{87}$$

These properties are consistent with what happens for the classical case, where the invariants I_1, I_2, I_3 can be expressed in terms of I_4 and I_5. The commutation rules involved by the invariant operators $I_1, \ldots, I_5$ are of the type (72)-(75) with the Poisson brackets properly substituted by commutators, $\{\,,\} \to \frac{1}{i\hbar}[\,,]$.

The Hamiltonian (77) can be written in terms of the $SU(1,1)$ Nöther invariant operators $\hat{J}_0, \hat{J}_+, \hat{J}_-$ according to

$$\hat{H} = \gamma_1 \hat{J}_0 + \gamma_2 \hat{J}_+ + \gamma_2^* \hat{J}_-, \tag{88}$$

where γ_1 and γ_2 are the time-dependent functions

$$\gamma_1 = 2\hbar \left\{ \left[\frac{1}{4\sigma^2} + \left(\dot{\sigma} - \frac{M}{2}\sigma \right)^2 \right] + \omega^2 \sigma^2 \right\}, \tag{89}$$

$$\gamma_2 = \hbar \left\{ \left[\frac{i}{2\sigma} + \left(\dot{\sigma} - \frac{M}{2}\sigma \right) \right]^2 + \omega^2 \sigma^2 \right\} e^{i\theta}, \tag{90}$$

and

$$\hat{J}_+ = -\frac{(\hat{I}_1 + i\hat{I}_2)}{2\hbar} = \frac{1}{2} e^{-i\theta} a^{\dagger 2}, \qquad \hat{J}_- = -\frac{(\hat{I}_1 - i\hat{I}_2)}{2\hbar} = \frac{1}{2} e^{i\theta} a^2, \tag{91}$$

$$J_0 = \frac{\hat{I}_3}{2\hbar} = \frac{1}{2}(a^\dagger a + \frac{1}{2}), \tag{92}$$

so that

$$[\hat{J}_+, \hat{J}_-] = -2\hat{J}_0, \quad [\hat{J}_0, \hat{J}_+] = \hat{J}_+, \quad [\hat{J}_0, \hat{J}_-] = -\hat{J}_-. \tag{93}$$

The form (88) of the Hamiltonian (77), which is Hermitian, is the one usually employed to solve the energy spectrum problem. We point out that (88) belongs to the class of the most general Hamiltonian preserving an arbitrary initial $SU(1,1)$ coherent state under time evolution, that is $H = \gamma_1(t)\, J_0 + \gamma_2(t)\, J_+ + \gamma_2^*(t)\, J_- + \gamma_3(t)$ where $\gamma_1(t)$, $\gamma_3(t)$ are real functions and $\gamma_2(t)$ is a complex function (see *e.g.* [13]).

5.1 Bogolubov coefficients and the uncertainty relation for the GTDO

Variances between generalized coherent states $|\alpha\rangle$ (see the Appendix) are:

$$(\Delta_\alpha \hat{Q})^2 = \langle\, \hat{Q}^2 \,\rangle - \langle\, \hat{Q} \,\rangle^2, \quad (\Delta_\alpha \hat{P})^2 = \langle\, \hat{P}^2 \,\rangle - \langle\, \hat{P} \,\rangle^2 \tag{94}$$

where the expectation value is given by $\langle \ldots \rangle = \langle \alpha | \ldots | \alpha \rangle$. Thus, we have $(\Delta_\alpha \hat{Q})^2 = \hbar\sigma^2/m$, $(\Delta_\alpha \hat{P})^2 = \hbar\, m\ |\zeta|^2$, which entail the uncertainty formula

$$(\Delta_\alpha \hat{Q})\ (\Delta_\alpha \hat{P}) = \frac{\hbar}{2} \left[1 + 4\sigma^2 \left(\dot{\sigma} - \frac{M}{2}\sigma \right)^2 \right]^{\frac{1}{2}}. \tag{95}$$

Introduction of the Schrödinger picture fixed photon annihilator and creation operators at the time t_0, $\hat{a}_0 = \hat{a}_0(t_0)$ and $\hat{a}_0^\dagger = \hat{a}_0^\dagger(t_0)$, provides

$$\hat{a}(t) = \mu(t)\,\hat{a}_0 + \nu(t)\,\hat{a}_0^\dagger, \tag{96}$$

where the Bogolubov coefficients $\mu(t)$ and $\nu(t)$ take the form

$$\mu(t) = \sqrt{\frac{m}{2m_0\omega_0}}\left[\frac{m_0\omega_0}{m}\sigma - i\zeta^*\right], \qquad \nu(t) = -\sqrt{\frac{m}{2m_0\omega_0}}\left[i\zeta^* + \frac{m_0\omega_0}{m}\sigma\right] \tag{97}$$

($|\mu(t)|^2 - |\nu(t)|^2 = 1$). The quantities $\mu(t)$, $\nu(t)$, can be even involved in the uncertainty formula, *i.e.* $(\Delta_\alpha\hat{Q})\,(\Delta_\alpha\hat{P}) = \frac{\hbar}{2}\,|\mu(t)+\nu(t)|\;|\mu(t)-\nu(t)|$. Since the description of minimum uncertainty states for time-dependent oscillators is an essential step in many branches of physics, a *criterium* of minimum uncertainty for the generalized oscillator (14) deserves attention to be paid. In doing so, notice that the product (95) reaches its minimum value whenever $M\sigma = 2\dot{\sigma}$, which implies

$$\sigma = c\,m^{1/2}, \quad m(t)\,\omega(t) = \frac{1}{2c^2}. \tag{98}$$

For the standard time-independent harmonic oscillator ($\omega = const$ and $m = (2\omega c^2)^{-1}$), the unique exact solution of the auxiliary EMP equation minimizing the uncertainty product is $\sigma = (2\omega)^{-1/2}$. Other solutions can minimize the uncertainty formula only approximately. Notice that in the minimum uncertainty case the phase would take the form $\theta(t) = 2\int_{t_0}^{t}\omega(t')dt'$. Further, the Hamiltonian operator would read $\hat{H}(t) = \frac{m_0}{m(t)}\,\hat{H}(0)$ so that $\langle 0|\;\hat{H}(t)\;|0\rangle = \frac{\hbar\omega(t)}{2}$.

The way the operators of the physical interest transform under the displacement and the squeeze operators can be straightforwardly obtained. For instance, expressions

$$\hat{D}^\dagger(\alpha)\,\hat{a}_0\,\hat{D}(\alpha) = \hat{a}_0 + \alpha\;, \qquad \hat{D}^\dagger(\alpha)\,\hat{a}_0^\dagger\,\hat{D}(\alpha) = \hat{a}_0^\dagger + \alpha^*\;,$$

lead to the expectation value

$$\left\langle \alpha, z \,\middle|\, \hat{N}_0 \,\middle|\, \alpha, z \right\rangle = |\alpha|^2 + \sinh^2 r\;, \tag{99}$$

$\hat{N}_0 = \hat{a}_0^\dagger\hat{a}_0$ being the number operator in the Schrödinger representation. We would like to recall that

$$\left\langle 0 \,\middle|\, \hat{H}(t) \,\middle|\, 0 \right\rangle = \left[\frac{\hbar}{8\sigma^2} + \frac{\hbar}{2}\left(\frac{M}{2}\sigma - \dot{\sigma}\right)^2\right] + \frac{\hbar}{2}\omega^2(t)\,\sigma^2. \tag{100}$$

The term in the square bracket corresponds to the vacuum expectation value of the kinetic energy of the system, while the last term is related to the vacuum expectation value of the potential energy. The quantity

$$E_{NM} \doteqdot \frac{\hbar}{2}\left(\frac{M}{2}\sigma - \dot{\sigma}\right)^2 \tag{101}$$

(NM=non-minimum) is interpreted as the energy associated with the squeezed states which do not satisfy the criterium of minimum uncertainty. When the criterium is verified, $E_{NM} = 0$ in fact. In such a case the vacuum expectation values of the kinetic energy and the potential energy turn out to be proportional to σ^2 and σ^{-2}, respectively. In the case of minimum uncertainty we also have

$$\left\langle 0 \left| \hat{S}\,\hat{H}(t)\,\hat{S}^\dagger \right| 0 \right\rangle = \hbar\left(\frac{e^{-2r}}{8\sigma^2} + \frac{e^{2r}}{2}\omega^2\sigma^2\right). \tag{102}$$

A useful property is $\hat{H}(t) = \frac{\omega(t)}{\omega_0}\hat{S}^\dagger(r)\,\hat{D}^\dagger(\alpha)\,\hat{H}(0)\,\hat{D}(\alpha)\,\hat{S}(r)$, from which the matrix element

$$\left\langle 0 \left| \hat{H}(t) \right| 0 \right\rangle = \hbar\omega(t)\left(|\alpha|^2 + \sinh^2 r + \frac{1}{2}\right) \tag{103}$$

can be determined. This provides a simple way to obtain the eigenstates of $\hat{H}(t)$. To this aim, let $\{|n\rangle\}$ denote the eigenstates of $\hat{H}(0)$, $\hat{H}(0)\,|n\rangle = \hbar\omega(0)\,\left(n+\frac{1}{2}\right)\,|n\rangle$. Eigenstates $\{|n\rangle_t\}$ of the time-dependent Hamiltonian $\hat{H}(t)$ are expressed by $|n\rangle_t = \hat{S}^\dagger(r)\,\hat{D}^\dagger(\alpha)\;|n\rangle$ and it results

$$\begin{aligned}\hat{H}(t)\;|n\rangle_t &= \frac{\omega(t)}{\omega_0}\hat{S}^\dagger(r)\,\hat{D}^\dagger(\alpha)\,\hat{H}(0)\;|n\rangle = \\ &= \hbar\,\omega(t)\;\left(n+\frac{1}{2}\right)\hat{S}^\dagger(r)\,\hat{D}^\dagger(\alpha)\;|n\rangle\; = \hbar\,\omega(t)\;\left(n+\frac{1}{2}\right)\,|n\rangle_t\,,\end{aligned}$$

where $\hat{S}\hat{S}^\dagger = \hat{S}^\dagger\hat{S} = \hat{1}$ and $\hat{D}\hat{D}^\dagger = \hat{D}^\dagger\hat{D} = \hat{1}$ have been used.

6 Applications to cosmology

Modern cosmology investigates, among the others, the idea that sudden changes in the spacetime metric very far in the past could give rise to observable effects in present days. Field modes can be created out from the vacuum during the cosmological expansion and squeezed quantum states finally show up. In the presence of a change of regime under the cosmological evolution, the occupation number of a given initial state would get indeed amplified. As long as

the change can be considered as adiabatic, the amplification factor approaches to one. But in case the change is sudden the amplification mechanism cannot be neglected and even the vacuum state transforms into a multiparticle state in the Fock space appropriate to the new regime. The amplification mechanism may provide the possibility to detect quantum effects (*e.g.* relic gravitons) at scales considerably above the Planck one. By virtue of all this, we would like to discuss on the case study of a free massless scalar field ϕ in a Robertson-Walker inflationary background, $ds^2 = a^2(\eta)\ (d\eta^2 - \delta_{ij}\, dx^i dx^j)$. As it is widely known, the field can be expanded into mode functions, say $\phi(\eta, \overrightarrow{x}) = \sum_{\mathbf{n}} q_{\mathbf{n}}(\eta)\, u_{\mathbf{n}}(\overrightarrow{x})$ where $\mathbf{n}$ denotes the comoving wavevector, whose spatial parts satisfy the Laplace equation and whose temporal parts evolve like time-dependent oscillators (GTDO's) with constant frequencies, $\omega = n = |\mathbf{n}|$, and time-dependent mass, $m = a^2(\eta)$. Since one idea underlying recent attempts to shed light on the possible role of trans-Planckian effects in inflationary cosmology is to mimic them by means of time-dependent effective mode dispersion relations n_{eff} which deviate from the linear behaviour for large momenta (see [15] and Refs. therein), afterwards we shall generally refer to GTDO's of the type (the index $\mathbf{n}$ is omitted)

$$q_{\eta\eta} + 2\frac{a_\eta}{a}\, q_\eta + n_{eff}^2\, q = 0, \tag{104}$$

where the subscript η denotes conformal time derivative. So, to investigate the dynamics of a massless scalar field in a Robertson-Walker spacetime one has to choose the initial state and let it evolve under the GTDO dynamics. Choosing the initial state is a key point which clearly results into a system for coefficients A, B, C entering in the definition of the function σ (see Eq. (18)). Due to the semiclassical approximation underlying these studies, the natural arena for the choice of the initial state turns out to be that of coherent states (*e.g.* the vacuum defined via $\hat{a}_0|0>_0 = 0$). They display features which are in the borderline between classical and quantum mechanics, in fact. The evaluation of the GTDO Hamiltonian expectation values shows that the departure from the minimum uncertainty enhances the squeezing mechanism through terms depending on the decoherence energy E_{NM} [14]. A suitable choice for the initial state could be therefore that of a coherent state minimizing the squeezing action of the gravitational field at the initial time η_0. This can be achieved by looking for coefficients A, B, C which make possible the vanishing of the decoherence energy E_{NM} at η_0. The request that at the initial time the time-dependent annihilation and creation operators go into the standard Dirac form is accomplished for $\sigma(\eta_0) = [2n_{eff}(\eta_0)]^{-\frac{1}{2}}$ (see (96)). Since taking account of the condition (19) is

always due, then A, B, C can be determined by means of the system

$$AB - C^2 = \frac{1}{4{W_0}^2} , \qquad \sigma(\eta_0) = [2n_{eff}(\eta_0)]^{-1/2} , \qquad \sigma_\eta(\eta_0) = \frac{a_\eta(\eta_0)}{a(\eta_0)}\sigma(\eta_0) . \tag{105}$$

Let us focus on a period under which the Universe undergoes an accelerating adiabatic expansion. The prototype of the models of inflationary cosmology is based on the de Sitter spacetime for which $a(\eta) = (-H_0\eta)^{-1}$, where $\eta < 0$ and H_0 denotes the Hubble constant. While it is customary to resort to the spatially flat inflationary model based on the de Sitter metric, a more general and realistic description of the inflation is provided by a quasi-de Sitter spacetime. In this case, the Hubble rate is not exactly constant but, rather, it weakly conformally changes with time according to $\tilde{H}_\eta = -\epsilon a^2 \tilde{H}^2$, where ϵ is a constant parameter. When ϵ vanishes one gets just the ordinary de Sitter spacetime. For small values of ϵ, a quasi-de Sitter spacetime is associated with the scale factor $a(\eta) = -\tilde{H}(1-\epsilon)\eta$, with $\eta < 0$. In the quasi-de Sitter spatially flat scenario $\Omega^2 = \frac{(2+3\epsilon)}{(1-\epsilon)^2\eta^2}$ and Eq. (15) can be solved in terms of Bessel functions. Precisely, one has the two independent solutions

$$y_1 = \sqrt{-n\eta} J_\nu(-n\eta) , \qquad y_2 = \sqrt{-n\eta}\, Y_\nu(-n\eta) \tag{106}$$

where $\nu = \sqrt{\frac{1}{4} + \frac{(2+3\epsilon)}{(\epsilon-1)^2}}$. The procedure outlined in previous Sections can be applied and we are led to the introduction of the basic function

$$\sigma = \sqrt{-n\eta}\left[A J_\nu^2(-n\eta) + B Y_\nu^2(-n\eta) + 2C J_\nu(-n\eta) Y_\nu(-n\eta)\right]^{\frac{1}{2}} \tag{107}$$

where A, B, C are determined via the conditions (105). Once we are interested in a situation in which the system started both very far in the past and in a vacuum state, the Bessel function expansions for ν fixed and $n\eta \to -\infty$ assists us in finding suitable constants A, B, C. Whenever the limit of arbitrary large negative initial times is concerned a natural choice is therefore given by $A = B = \frac{\pi}{4n}$, $C = 0$. So we obtain

$$\sigma(\eta) = \sqrt{-\frac{\pi}{4}\eta}\, \left|H_\nu^1(-n\eta)\right| . \tag{108}$$

The decoherence energy E_{NM} at the time η of gravitational waves in a quasi-de Sitter model of inflation can be evaluated by inserting (108) into formula (101).

Moreover, since

$$\mu(\eta) = \sqrt{\frac{a^2(\eta)}{2n\,a^2(\eta_i)}}\left\{\left[\frac{1}{2\sigma} + \frac{n\,a^2(\eta_i)}{a^2(\eta)}\sigma(\eta)\right] - i\left[\sigma_\eta - \frac{a_\eta}{a}\sigma\right]\right\}, \tag{109}$$

$$\nu(\eta) = \sqrt{\frac{a^2(\eta)}{2n\,a^2(\eta_i)}}\left\{\left[\frac{1}{2\sigma} - \frac{n\,a^2(\eta_i)}{a^2(\eta)}\sigma(\eta)\right] - i\left[\sigma_\eta - \frac{a_\eta}{a}\sigma\right]\right\} \tag{110}$$

at arbitrary times η the Bogolubov coefficients are given by setting (108) into

$$\mu(\eta) = \sqrt{\frac{1}{2n}\left(\frac{\eta_i}{\eta}\right)^{\frac{2}{1-\epsilon}}}\left\{\left[\frac{1}{2\sigma} + \left(\frac{\eta}{\eta_i}\right)^{\frac{2}{1-\epsilon}} n\sigma\right] - i\left[\sigma_\eta + \frac{\sigma}{(1-\epsilon)\,\eta}\right]\right\}, \tag{111}$$

$$\nu(\eta) = \sqrt{\frac{1}{2n}\left(\frac{\eta_i}{\eta}\right)^{\frac{2}{1-\epsilon}}}\left\{\left[\frac{1}{2\sigma} - \left(\frac{\eta}{\eta_i}\right)^{\frac{2}{1-\epsilon}} n\sigma\right] - i\left[\sigma_\eta + \frac{\sigma}{(1-\epsilon)\,\eta}\right]\right\}. \tag{112}$$

The phase θ can be evaluated to furnish $\theta(\eta) = 2\,\theta^1_\nu|^\eta_{\eta_i}$, where θ^1_ν denotes the phase of the Hankel function H^1_ν. In the case of a standard Sitter inflation $\epsilon = 0$, $\nu = \frac{3}{2}$ and the standard exact and normalized solution for the quantum fluctuations of a generic massless scalar field turns out to be associated with $\sigma = [\left(1 + n^{-2}\eta^{-2}\right)/2n]^{1/2}$. Moreover, in the standard de Sitter phase we would have [14]

$$|\nu(\eta)|^2 = \frac{1}{4}\left(\frac{\eta_i}{\eta} - \frac{\eta}{\eta_i}\right)^2. \tag{113}$$

In tackling more refined studies of cosmological effects in expanding Universe, it could be even useful to introduce a cosmological model which allows one to take into account different evolutionary phases of the Universe. Once the model has been specified and Eq. (15) solved, one can get an insight into physical effects related to different cosmological stages. For instance, one can consider a simple cosmological model which includes the so-called inflationary, radiation-dominated and matter-dominated epochs, respectively. Results and comments concerning this case can be found by the interested reader in [14]. In going further, another step can be formulated by putting $n^2_{eff} = a^2(\eta)\,F^2(k) = a^2(\eta)\,F^2[n/a(\eta)]$, where the function F is required to behave linearly for physical wavenumbers smaller than a characteristic scale k_C (see *e.g.* [15] and Refs. therein). Regardless the actual form of the function F, three different prototypical wavelength Regions can be distinguished as follows: *a)* Region I, where the wavelength of a given mode is much smaller than the characteristic length, $\lambda = \frac{2\pi}{n}a \ll l_C\ (\equiv k_C^{-1})$ and the nonlinearity of F is crucial;

b) Region II, with the mode's wavelength larger than the characteristic length but still smaller than the Hubble radius, $l_C \ll \lambda \ll l_H$, and the dispersion relation is linear; *c)* Region III, where the mode is outside the Hubble radius and the dispersion relation can be neglected. Hence, we have to evaluate the function σ in each Region according to

$$\sigma_j = \left(A_j y_{1,j}^2 + B_j\, y_{2,j}^2 + 2C_j\, y_{1,j}\, y_{2,j}\right)^{\frac{1}{2}}, \quad j = I, II, III, \tag{114}$$

where $y_{1,j}$, $y_{2,j}$ denote the solutions to (15) in the three Regions (henceforth, lowerscripts $j = I, II, III$ will refer to Regions I, II and III respectively). As for Region I, the basic system is (105). In order to deal with the other Regions, the system for the matching data is needed to be considered:

$$A_j B_j - C_j^2 = \frac{1}{4{W_{0,j}}^2}\ , \qquad \sigma_j(\eta_l) = \sigma_{j-1}(\eta_l)\ , \qquad \sigma_{j,\eta}\left(\eta_l\right) = \sigma_{j-1,\eta}(\eta_l)\ , \tag{115}$$

where $l = 1, 2, \quad j = II, III$, and η_1 and η_2 are the beginning of the second and third Region, respectively. Simple calculations show that solutions to (115) can be written in the form

$$A_j = Z_j\left[y_{2,j}, y_{2,j}, \eta_l\right], \quad B_j = Z_j\left[y_{1,j}, y_{1,j}, \eta_l\right], \quad C_j = -Z_j\left[y_{1,j}, y_{2,j}, \eta_l\right],$$

$$Z_j\left[f, g, \eta_l\right] = \left\{ \frac{fg}{4W_{0,j-1}^2\, \sigma_{j-1}^2} \left[1 + 4\,\sigma_{j-1}^2\, fg\, \left(\frac{\sigma_{j-1}}{f}\right)_\eta \left(\frac{\sigma_{j-1}}{g}\right)_\eta\right]\right\}\Bigg|_{\eta_l}. \tag{116}$$

After above systems have been solved, quantities of the physical interest (*e.g.* the Bogolubov coefficients) can be computed starting from the results so far expounded.

As an explicit example of nonlinear dispersion cosmologies, the generalized Corley-Jacobson relation of dispersion

$$n_{eff}^2(\eta) = n^2 + n^2 \sum_{q=1}^{m} \frac{b_q}{(2\pi)^{2q}} \left[\frac{n l_C}{a(\eta)}\right]^{2q}, \qquad b_m > 0 \tag{117}$$

can be adopted. We can consider the general scale factor of the form $a(\eta) = l_0(-\eta)^{1+\beta}$ with $\beta \leq -2$ (the standard de Sitter case is recovered for $\beta = -2$) so that $\eta_1 = -\left(n\epsilon\,/\,2\pi\right)^{\frac{1}{1+\beta}} |b_m|^{\frac{1}{2m(1+\beta)}}$ and $\eta_2 = \frac{2\pi}{n}(1+\beta)$. By inserting the dispersion relation (117) into (15) and by keeping the leading terms, the mode

solutions in the three Regions can be straightforwardly found. We therefore get

$$\sigma_I = \sqrt{-\eta}\left\{A_I\, J^2_{\frac{1}{2b}}(z) + B_I Y^2_{\frac{1}{2b}}(z) + 2C_I\, J_{\frac{1}{2b}}(z)\, Y_{\frac{1}{2b}}(z)\right\}^{\frac{1}{2}} \quad , \tag{118}$$

$$\sigma_{II} = \left(A_{II}\cos^2 n\eta + B_{II}\,\sin^2 n\eta + 2C_{II}\,\sin n\eta\,\cos n\eta\right)^{\frac{1}{2}}, \tag{119}$$

$$\sigma_{III} = \left(A_{III}\,\eta^{2(1+\beta)} + B_{III}\,\eta^{-2\beta} - 2C_{III}\,\eta\right)^{\frac{1}{2}}, \tag{120}$$

where $z = \gamma(-\eta)^b$, $\gamma = \frac{\sqrt{|b_m|}}{b(2\pi)^m}\epsilon^m n^{m+1}$, $b = 1 - m\,(1+\beta)$ and $\epsilon = l_C\,/\,l_0$. The initial condition and the matching data between Regions determine the constants A_j, B_j, C_j. We choose as the initial state any coherent state minimizing the uncertainty relation. We shall restrict ourselves on Region I [15], for which $W_0 = -2b/\pi$. If we proceed as customary by thinking about an adiabatic limit in the infinite past, we can neglect the term a_η/a in the system of the type (105) for A_I, B_I, C_I, and resort to the asymptotic expansion for Bessel functions (see *e.g.* [16]). Under these assumptions, it turns out that we can set $A_I = B_I = \frac{\pi}{4b}$, $C_I = 0$. That is, we would obtain

$$\sigma_I = \sqrt{-\frac{\pi\eta}{4b}}\left|H^1_{\frac{1}{2b}}(z)\right| . \tag{121}$$

A gravitational phase $\theta_I(\eta)$ twice with respect to the phase of the Hankel function $H^1_{\frac{1}{2b}}$ hence arises. An insight into the decoherence energy can be given via $E_{NM,I} = \frac{\hbar}{2}(\sigma_{I,\eta} - \frac{a_\eta}{a}\sigma_I)^2$ while the time dependence of the amount of modes which are created out of the vacuum can be straightforwardly evaluated by inserting (121) into

$$|\nu_I(\eta)|^2 = \frac{1}{2\gamma b}\left(\frac{\eta_0^{2+m}}{\eta^2}\right)^{(1+\beta)}\left\{\frac{1}{4\sigma_I^2(\eta)}\left[1 - \left(\frac{\eta_0}{\eta}\right)^{2(1+\beta)}\frac{\sigma_I^2(\eta)}{\sigma_I^2(\eta_0)}\right]^2 + \right.$$
$$\left. + \left[\sigma_{I,\eta}(\eta) - \frac{a_\eta(\eta)}{a(\eta)}\sigma_I(\eta)\right]^2\right\}. \tag{122}$$

At this point, comments are in order. First, it is definitively worth to recall that in the special case $m = 1$ the minimum uncertainty criterium $a^2(\eta)\, n_{eff}(\eta) = const$ approximatively holds [14] and the dynamics of the field ϕ turns out to be very weakly affected by quantum decoherence effects. Secondly, in evaluating the constants A_I, B_I and C_I we basically imposed the vanishing of a_η/a for asymptotically negative times in the last of conditions of the type (105) for A_I, B_I and C_I. This is a good approximation, but one might wonder about

setting all initial conditions at the particular scale η_0. This would avoid that in Bogolubov coefficients there will be a mixing of quantities referring to an adiabatic limit in the infinite past and quantities which actually do not. So, let us keep the initial time finite also in the last of equations (105). On general ground, one expects explicit leading corrections of the order $O(1/z_0)$, where $z_0 = \gamma(-\eta_0)^b$, for instance. Precisely, it results that A_I and B_I are no longer equal and C_I does not vanish [15]. The net consequence is that oscillations on the $1/z_0$ scale originated by the term

$$\frac{\varrho z^{1/b}}{\gamma^{1/b} z_0} \left\{ \sin 2\chi \ \left[Y^2_{\frac{1}{2b}}(z) - \ J^2_{\frac{1}{2b}}(z) \right] + 2(1 - 2\sin^2\chi)\, Y_{\frac{1}{2b}}(z) J_{\frac{1}{2b}}(z) \right\}, \qquad (123)$$

with $\varrho = [\pi(\beta + 4b + 4)\,/\,4b^2]$ and $\chi = z_0 - \pi(b+1)\,/\,4b$, would appear in σ. They imply the finite time corrections to the quantities previously expressed in terms of the zeroth-order mode amplitude (121). A comparison with the case in which one has a de Sitter spacetime and a dispersion relation of the Unruh type can be straightforwardly performed [15]. In this case, in fact, in Region I one has $n^2_{eff}(\eta) \approx (2\pi/\,\epsilon\,\eta)^2$ so that a nonvanishing initial friction term a_η/a would not introduce fluctuations whose amplitude scale is fixed by the initial time η_0. Rather, corrections with respect to the parameter ϵ enter in the matter [15].

7 Conclusions

The "generalized time-dependent oscillator" (GTDO) is a paradigmatic linear system whose "mass" and "frequency" can vary with time. A huge literature therefore does exist on the topic and very many aspects have been pointed out in the years which are interesting both from a mathematical and a physical point of view. In this Communication, we described some significant features pertinent to GTDO's by selecting just a few of the several many notable results achieved up to now. That is, we focused on the problem of characterizing the GTDO dynamical group and on the aspects related to the loss of coherence in cosmological models. Precisely, we have remarked that in the case of GTDO's the Nöther invariants form an algebra of the $su(1,1)$ type under the Poisson bracket operation. Namely, the dynamical group of a time-dependent oscillator is $SU(1,1)$. Another important achievement is represented by the alternative Lagrangians, which arise from the so-called additional infinitesimal operators emerging beyond those generating the Nöther invariants. When moving to the quantum theory, a set of invariant operators of the Nöther type $\hat{I}_i$ can be constructed from the classical Nöther invariants I_i by adopting a suitable procedure. Once the substitution $\{\,,\} \to \frac{1}{i\hbar}[\,,]$ from Poisson bracket to commutator is performed, the commutation rules thus define a noncompact Lie algebra $su(1,1)$

under the commutation bracket operation [8]. To conclude, once the framework of the theory of invariants is concerned, effective tools are provided to study the dynamics of a massless scalar field with arbitrary dispersion relation in a Robertson-Walker spacetime. A new perspective is gained in that the way gravitational decoherence generally works within the semiclassical approximation is made extremely manifest. The $SU(1,1)$ symmetry underlying the GTDO dynamics enables to express quantities of the physical interest in a rather compact form without losing in generality. The role of the dispersion relation, of the gravitational scale factor and of the mode time-dependent amplitude modulation, is explicitly clarified. Furthermore, formula (101) for the decoherence energy supplies a new tool for tackling the problem of imposing initial conditions on the scalar field modes without worrying about the adiabaticity of their successive evolution.

Appendix: Coherent and Squeezed States

The concept of *coherent state* was introduced by Schrödinger who investigated quantum states for the harmonic oscillator in such a way that the expectation value of the position and the momentum operators were the same as the classical solutions [17]. The basic properties of these states for the harmonic oscillator can be briefly synthesized as follows: *i)* they are eigenstates of the annihilation operator; *ii)* they are created from the ground states by a unitary operator; *iii)* they minimize the uncertainty relation; *iv)* they are overcomplete, normalized but not orthogonal. Coherent states for the GTDO in the context of Lewis-Riesenfeld theory [5] were constructed by Hartley and Ray in 1982 [18]. These states share all the features of the coherent states of the conventional (time-independent) oscillator except that the uncertainty formula, *i.e.* the product of position and momentum is *not* minimum. A few years later, Pedrosa [19] showed that the coherent states devised by Hartley and Ray for the GTDO are actually *squeezed* states. Generally speaking, squeezed states enter in the matter whenever the quadratic operators $\hat{a}^2$ and $\hat{a}^{\dagger\,2}$ are involved in the Hamiltonian. Many references exist where both coherent and squeezed states are discussed (see *e.g.* [20]). Squeezed states $|\alpha, z\rangle$ are defined as $|\alpha, z\rangle = \hat{D}(\alpha)\,\hat{S}(z)\,|0\rangle$ where $\hat{D}(\alpha)$ and $\hat{S}(\alpha)$ are the so-called *displacement operator* and *squeeze operator*. Quantities α, z are complex parameters. For any fixed α, the choice $z = 0$ yields the coherent state $|\alpha\rangle = |\alpha, 0\rangle = \hat{D}(\alpha)\ |0\rangle$. Coherent states $|\alpha\rangle$ are eigenstates of the (non-Hermitian) annihilation operator a with eigenvalue α, *i.e.* $a\,|\alpha\rangle = \alpha\,|\alpha\rangle$. It is noteworthy that the squeeze operator $\hat{S}(z)$ induces a canonical transformation of the annihilation and creation operators. Indeed, by defining the operators $\hat{b} \doteqdot \hat{S}^\dagger \hat{a}_0 \hat{S}$, $\hat{b}^\dagger \doteqdot \hat{S}^\dagger \hat{a}_0^\dagger \hat{S}$, the relation

$\left[\hat{b}, \hat{b}^\dagger\right] = \hat{1}$ holds. The Bogolubov transformation can be naturally embedded into above relations where the operators $\hat{b}$, $\hat{b}^\dagger$ can be identified with $\hat{a}(t)$ and $\hat{a}^\dagger(t)$.

References

[1] O. Castaños, A. Frank, R. Lopez-Peña: *Noether theorem and dynamic groups in quantum mechanics*, J. Phys. A: Math. Gen. **23**, 5141 (1990).

[2] G.B. Wybourne: Classical Groups for Physicists, Wiley, New York, 1974.

[3] Y. Dothan: *Finite-dimensional spectrum generating algebras*, Phys. Rev. **D2**, 2944 (1970).

[4] I.A. Malkin, V.I. Man'ko: *Symmetry of the hydrogen atom*, Pis'ma Zh. Eksp. Teor. Fit. **2**, 230 (1965) [JETP Lett **2**, 146 (1965)].

[5] H.R. Lewis, W.B. Riesenfeld: *An exact quantum theory of the time dependent harmonic oscillator and of a charged particle time dependent electromagnetic field*, J. Math. Phys. **10**, 1458 (1969).

[6] X.C. Gao, J.B. Xu, T.Z. Quian: *Geometric phase and the generalized invariant formulation*, Phys. Rev. **A44**, 7016 (1991).

[7] E. D'Hoker, L. Vinet: *Dynamical supersymmetry of the magnetic monopole and the $1/r^2$-potential*, Comm. Math. Phys. **97**, 391 (1985).

[8] G. Profilo, G. Soliani: *Group-theoretical approach to the classical and quantum oscillator with time-dependent mass and frequency*, Phys. Rev. **A44**, 2057 (1991).

[9] P.J. Olver: Applications of Lie Groups to Differential Equations, Springer, New York, 1986.

[10] I.K. Edwards: *Quantization of inequivalent classical Hamiltonians*, Am. J. Phys. **47**, 153 (1979).

[11] G. Morandi, C. Ferrario, G. Lo Vecchio, G. Marmo, C. Rubano: *The inverse problem in the calculus of variations and the geometry of the tangent bundle*, Phys. Rep. **3-4** , 188 (1990).

[12] M. Lutzky: *Symmetry groups and conserved quantities for the harmonic oscillator*, J. Phys. A: Math. Gen. **11**, 249 (1978).

[13] C.C. Gerry: *Application of $SU(1,1)$ coherent states to the interaction of squeezed light in an anharmonic oscillator*, Phys. Rev. **A35**, 2146 (1987).

[14] A. Geralico, G. Landolfi, G. Ruggeri, G. Soliani: *Novel approach to the study of quantum effects in the early Universe*, Phys. Rev. **D69**, 043504 (2004).

[15] G. Landolfi, G. Soliani: *Trans-Planckian effects in nonlinear dispersion cosmologies*, Phys. Lett. **B588**, 1 (2004).

[16] M. Abramowitz, I.A. Stegun: Handbook of Mathematical Tables, Dover Publ., New York, 1970.

[17] E. Schrödinger: *Der stetige Übergang von der Mikro-zur Makromechanik*, Naturwissenschaften **14**, 664 (1926).

[18] J.G. Hartley, J.R. Ray: *Coherent states for the time-dependent harmonic oscillator*, Phys. Rev. **D25**, 382 (1982).

[19] I.A. Pedrosa: *Comment on "Coherent states for the time-dependent harmonic oscillator"*, Phys. Rev. **D36**, 1279 (1987).

[20] W.P. Schleich: Quantum Optics in Phase Space, Wiley-VCH, Berlin, 2001.