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Geometric methods of solving boundary-value problems

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Abstract. In this paper, we list several geometrical methods of solving the boundary problems for equations of the mathematical physics. Constructing solutions invariant w.r.t. symmetries of the problem is discussed. A method based on the representation of the equation at hand in the evolutionary form is pointed out. The methods based on the deformation of the boundary problem along discrete or continuous parameters are described. Among these methods, we note the direct iterations, the boundary conditions homotopy, the relaxation method, and the deformation of the initial equation. Then, comparative analysis of the results of computer experiments in applying these methods is carried out.

Keywords: Boundary-value problems, deformations, geometry of PDE

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Introduction

In this note, we review several methods for solving boundary-value problems (mainly, the Dirichlet problem) for nonlinear equations of mathematical physics. Then we compare and analyse the results of a computer experiment in applying the described algorithms. The methods which we study are based on the geometry of the jet spaces [2, 7, 12] and treating differential equations $\mathcal{E}$ (and their prolongations $\mathcal{E}^\infty$ as well) as submanifolds $\mathcal{E} \subset J^k(\pi)$ of the jet space of order k for a certain fibre bundle π (respectively, we have $\mathcal{E}^\infty \subset J^\infty(\pi)$). By using this approach, we pay attention to the analysis of the following objects: they are

- differential equations and sections s such that the jet $j_k(s) \subset \mathcal{E}$ defines a solution of $\mathcal{E}$,
- boundary-value problems $\mathcal{P} = (\mathcal{E}, \mathcal{D}, f = s|_{\partial\mathcal{D}})$ in a domain $\mathcal{D}$,

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- and their deformations $\dot{\mathcal{P}}$.

We emphasize that the geometrically motivated technology of solving the boundary-value problems is discussed. We do not stop on particular properties of some solutions in practical situations. Basic definitions and concepts follow [2,7,11,12], see also [5]. We use the elliptic Liouville equation

$$\mathcal{E}_{\text{Liou}} = \{u_{z\bar{z}} = \exp(2u)\}$$

as a basic example.

The paper is organized as follows. In Sec. 1, the method of monotonous iterations for the solutions u^t of the boundary-value problem $\mathcal{P}$ is described. Here $t \in \mathbb{N}$. This method is based on the theory of differential inequalities [11]. In Sec. 2, we construct a method for solving the boundary-value problems that involves the evolution representation of the equation $\mathcal{E}$ under study. We shall see that this interpretation is admissible in a sufficiently general situation [2,12]. Then, in Sec. 3 we consider various methods based on the deformations $\dot{f}$ of the boundary conditions f and simultaneous invariance of the equation $\mathcal{E}$:

$$\dot{\mathcal{P}} = (\mathcal{E}, \mathcal{D}, \dot{f}).$$

In Sec. 4 we describe the relaxation method based on the substitution

$$\mathcal{P} \mapsto \mathcal{P}' = (\mathcal{E}', \mathcal{D} \times \mathbb{R}_+, f \otimes \text{id}_t)$$

of the boundary-value problem such that solutions u of the problem $\mathcal{P}$ are stable stationary solutions for the new problem $\mathcal{P}'$ with respect to the evolution equation $\mathcal{E}'$. The deformations

$$\dot{\mathcal{P}} = (\dot{\mathcal{E}}, \mathcal{D}, f),$$

which are considered in Sec. 5, are in some sense antipodal to ones described in Sec. 3. Now, the boundary condition f is invariant and the equation $\mathcal{E}(t)$ moves. The "planting" of a nonlinearity could be an example. In the final section, we discuss the results of a computer experiment in practical application of all these methods.

1 Remark. In what follows, we assume that there is a unique classical solution to the boundary-value problem at hand (see [4]). Also, we recall that the transformation laws of solutions with respect to transformations of the independent variables are known for some equations, *e.g.*, the conformally invariant equations, see [5] and references therein. Therefore, the domain $\mathcal{D}$, where the problem $\mathcal{P}$ is solved, can be chosen relatively regular (usually we set $\mathcal{D} \sim B_0^1$).

2 Remark. First, we point out a practically useful way to construct a set of solutions (that can be large enough) of a boundary-value problem $\mathcal{P} = (\mathcal{E}, \mathcal{D}, f)$. Assume that the equation $\mathcal{E}$ is invariant with respect to a vector field X. Denote its generating section by φ_X and suppose that A_t is the flow of the field X such that

$$A_t(\partial\mathcal{D}) = \partial\mathcal{D} \text{ and } A_t(j_k(f)) = j_k(f),$$

that is, the boundary condition is invariant with respect to the symmetry X of the equation $\mathcal{E}$ at hand. Then problem $\mathcal{P}$ can be reduced to finding solutions of the boundary-value problem

$$\mathcal{P}' = (\mathcal{E} \cap \{\varphi_X = 0\}, \mathcal{D}, f)$$

that are invariant with respect to X at any point in $\mathcal{D}$. If the problem $\mathcal{P}'$ has a solution, then, in general, this solution may be not unique.

3 Example ([6]). Consider the Dirichlet boundary-value problem

$$\Big\{u_{xx} + u_{yy} = \exp(2u), \qquad u\big|_{r=1} = f, \quad f \in C^{0,\lambda}\Big\} \tag{1}$$

in the unit disc B_0^1 and assume that the boundary condition is homogeneous and trivial, $f \equiv 0$. The solutions for the Liouville equation $\mathcal{E}_{\text{Liou}}$ that are invariant with respect to its point symmetries are [9]

$$u = \frac{1}{2}\log\frac{v_x^2 + v_y^2}{\sinh^2 v}, \tag{2a}$$

$$u = \frac{1}{2}\log\frac{v_x^2 + v_y^2}{v^2}, \tag{2b}$$

$$u = \frac{1}{2}\log\frac{v_x^2 + v_y^2}{\sin^2 v}, \tag{2c}$$

where v is a harmonic function,

$$\Delta v = 0.$$

Consider the radial symmetry

$$X = -y\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial y}.$$

Then the solutions $u(r) = u(\sqrt{x^2 + y^2})$ of problem (1) are the following:

$$u_1(r) = \log \frac{2(1+\sqrt{2})}{(1+\sqrt{2})^2 - r^2}, \tag{3a}$$

$$u_2(r) = -\log r - \log(1 - \log r), \tag{3b}$$

$$u_3(r) = \frac{1}{2} \log \frac{\alpha^2 r^{-2}}{\sin^2(\alpha\beta - \alpha \log r)}, \qquad \beta = \frac{\pm 1}{\alpha} \arcsin \alpha,\ \alpha \neq 0. \tag{3c}$$

A unique classical solution u_1 has no singularity at the point 0. The solution u_2 has an integrable singularity at $r = 0$, and its gradient on the boundary vanishes. The solution u_3 has the denumerable set of logarithmic singularities along the radius r. These singularities accumulate as $r \to 0$.

4 Remark. Suppose that the equation $\mathcal{E}$ at hand is Euler. Then, obviously, one can search solutions of the Dirichlet boundary-value problem by using a direct minimization of the action functional. We emphasize that the projective methods can be used in addition to the lattice discretization methods.

1 The monotonous iterations method

The monotonous iterations method [11] is a useful instrument in the theory of differential inequalities. This method allows to construct solutions of the Dirichlet boundary-value problems

$$\Big\{\Delta u = h(u, x), \quad u|_{\partial\mathcal{D}} = f, \qquad \partial\mathcal{D} \in C^{1+\varepsilon}, \quad f, h \in C^{0,\lambda}\Big\} \tag{4}$$

and then check the local uniqueness of these solutions.

5 Definition. A function $\alpha \in C(\overline{\mathcal{D}}) \cap C^2(\mathcal{D})$ (resp, β) is a *lower* (*upper*) solution of problem (4) is the following two conditions hold:

(1) $\Delta\alpha - h(\alpha,\, x) \geq 0$ in $\mathcal{D}$ and

(2) $f(x) \geq \alpha(x)$ on $\partial\mathcal{D}$

(respectively, $\leq$). Suppose further $\alpha(x) \leq \beta(x)$ for each $x \in \overline{\mathcal{D}}$. Then we set $\alpha \preceq \beta$.

6 Proposition ([11]). *Assume that a lower and an upper solutions of problem* (4) *are ordered,* $\alpha \preceq \beta$*. Suppose there is a nonnegative constant* $C \in \mathbb{R}$ *such that*

$$h(u_1, x) - h(u_2, x) \leq C \cdot (u_1 - u_2)$$

for any $x \in \overline{\mathcal{D}}$*,* u_1*, and* u_2 *provided that* $\alpha \leq u_2 \leq u_1 \leq \beta$*. Consider two sequences* $\underline{U} = \{\underline{u}^k\}$ *and* $\overline{U} = \{\overline{u}^k\}$ *of solutions of the problem*

$$\Delta u^k - C\,u^k = h(u^{k-1}, x) - C\,u^{k-1}, \quad u^k|_{\partial\mathcal{D}} = f, \qquad x \in \mathcal{D},\ k \in \mathbb{N}. \tag{5}$$

Here the initial values are $\underline{u}^0 = \alpha$ *and* $\overline{u}^0 = \beta$*, respectively. Then the monotonously nondecreasing (non increasing) sequence* $\underline{U}$ $(\overline{U})$ *converges to a solution* $\underline{u}$ *(resp.,* $\overline{u}$*) for problem* (4)*. Moreover, we have* $\underline{u} \le \overline{u}$ *in* $\overline{\mathcal{D}}$ *and the bound* $\underline{u} \le u_* \le \overline{u}$ *holds for any solution* $u_* \in [\alpha,\ \beta]$*.*

Suppose further that h is monotonous with respect to u: $h(u_1, x) - h(u_2, x) \ge 0$ *provided that* $\alpha \preceq u_2 \preceq u_1 \preceq \beta$*. Then the solution for problem* (4) *is unique on* $[\alpha, \beta]$*:* $\underline{u} = \overline{u}$*.*

The notions of a lower and upper solution, which were introduced in Sec. 1, and the method of proving the local uniqueness for solutions of problem (4) will be used in Sec. 4. The relaxation method will be described there. In that case, the evolution of the superscript k is continuous unlike in problems (5).

2 Evolutionary representation of differential equations

The following remarkable result was in fact obtained in the papers concerning the formal theory of differential equations (see [2,12] and references therein): any equation that satisfies the assumptions of the '2-line theorem' for the Vinogradov's $\mathcal{C}$-spectral sequence [2, 7] admits a representation in the form of an evolution equation. In practice, this means that any equation that does not have gauge symmetries (unlike the Maxwell, the Yang–Mills, and the Einstein equations and similar systems) admits a set of coordinate transformations and differential substitutions that map it to an evolution equation (possibly, the new equation is imposed on a larger set of dependent variables). The class of equations subject to the assumptions of the '2-line theorem' is really wide. Of course, fixed-precision integrating of evolution equations is simpler that integrating of arbitrarily chosen equations of mathematical physics. Therefore, solving the initial boundary-value problem is divided to two steps. First, we find an evolution representation of the equation $\mathcal{E}$. Then we reconstruct the boundary conditions for the auxiliary dependent variables if there appears a necessity to introduce them.

7 Example. Consider the boundary-value problem

$$\begin{gathered} u_{xx} \pm u_{yy} = \exp(2u), \qquad \mathcal{D} = \{(x,y), |x| < 1, |y| < 1\}, \\ u(1,y) = f_1(y),\ u(x,1) = f_2(x),\ u(-1,y) = f_3(y),\ u(x,-1) = f_4(x) \end{gathered} \tag{6}$$

for the elliptic (resp., hyperbolic) Liouville equation $\mathcal{E}_{\text{Liou}}$. Introduce the additional dependent variable $v = u_y$. Then the equation $\mathcal{E}_{\text{Liou}}$ has the form

$$\frac{\partial}{\partial y}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \mp u_{xx} \pm \exp(2u) \end{pmatrix}$$

and the conditions on the bound $\partial\mathcal{D}$ are split to the initial and the boundary conditions:

$$\begin{array}{lll} u(x,-1) = f_4(x), & u(1,y) = f_1(y), & u(-1,y) = f_3(y) \\ v(x,-1) = v[f_2], & v(1,y) = df_1(y)/dy, & v(-1,y) = df_3(y)/dy. \end{array}$$

Thence, solving boundary-value problem (6) is reduced to reconstruction of the initial section $v(x,-1)$ by the terminal section $u(x,1) = f_2(x)$. In practice, this can be done by using the conjugated gradient method.

3 Evolution of boundary conditions

Recall that any symmetry $\varphi \in \operatorname{sym}\mathcal{E}^\infty$ of an equation $\mathcal{E} = \{\vec{F} = 0\}$ is an element of the kernel $\ker \bar{\ell}_F$ of the linearization

$$\ell_F = \left\| \sum_\sigma \frac{\partial F_i}{\partial u^j_\sigma} D_\sigma \cdot \mathbf{1}_{ij} \right\|$$

restricted onto $\mathcal{E}$. The evolutionary vector field

$$\mathbf{Э}_\varphi = \sum_{j,\sigma} \bar{D}_\sigma(\varphi^j)\,\frac{\partial}{\partial u^j_\sigma}$$

commutes with the total derivatives D_i and is tangent to the infinite prolongation

$$\mathcal{E}^\infty = \{D_\sigma(F) = 0,\ |\sigma| \geq 0\} \subset J^\infty(\pi).$$

The manifold $\mathcal{E}^\infty$ that is defined by the closed algebra of smooth functions is in fact infinite-dimensional, therefore, formally, the field $\mathbf{Э}_\varphi$ has no flow. Indeed, there is no Cauchy theorem for the initial–value problem

$$\dot{u}_\sigma = \bar{D}_\sigma(\varphi), |\sigma| \geq 0, \qquad u(t=0) = u \tag{7}$$

composed by the cardinal set of equations. The integral trajectories for problem (7) exist provided that $\varphi(x, u, D_i(u^j))$ is a contact symmetry. In particular, the point symmetries that are linear with respect to the derivatives suit well. From Eq. (7) it follows that the functions φ describe the correlated evolution of the dependent variables and their derivatives. This evolution preserves solutions of the equation $\mathcal{E}$ if $\varphi \in \ker \bar{\ell}_F$.

3.1 Deformation of the boundary data

Suppose a solution u_0 of some boundary-value problem $\mathcal{P}_0 = (\mathcal{E},\ \mathcal{D},\ f_0)$ for an equation $\mathcal{E}$ is known. Then, deform the problem $\mathcal{P} = (\mathcal{E}, \mathcal{D}, f)$ such that $f(t = 0) = f_0$ and $f(t = 1) = f$. Here by $\dot{f}(t)$ we denote the deformation velocity of the boundary condition on $\partial\mathcal{D}$ and by $\varphi(t)$ the corresponding deformation of the solution. One easily checks that $\varphi \in \ker \bar{\ell}_F$. Suppose further that the problem

$$\varphi \in \ker \bar{\ell}_F, \qquad \varphi|_{\partial\mathcal{D}} = \dot{f} \tag{8}$$

is soluble at each $t \in [0, 1)$. Then (see [1] for the analytic case),

$$u = u_0 + \int_0^1 \varphi(t)\, dt$$

is the required solution for the problem $\mathcal{P}$.

The condition $\varphi \in \ker \bar{\ell}_F$ is the decomposition of the deformation $\dot{u} = \varphi$ to a power series in t. The analysis of the deformation equations that are coefficients of the higher powers of t is interesting by itself, see [8]. For example, insolubility of the equation at t^2 implies impossibility to solve the problem $\mathcal{P}$ by using this deformation method.

8 Remark. (*A. V. Samokhin, private communication.*) The Liouville equation $\mathcal{E}_{\text{Liou}}$ has the following peculiar feature. Its general solution [9], which depends on an arbitrary holomorphic function $v(z)$, is always invariant with respect to a point symmetry X whose coefficients are related with $v(z)$ by the Abel transformation. The curve $X(t)$ in the space $\operatorname{sym}\mathcal{E}_{\text{Liou}}$ is assigned to the deformation $\dot{f}$ of the boundary condition. This curve is such that the solution $u(t')$ is invariant with respect to $X(t')$ for any $t' \in [0, 1]$. Therefore, the solution of boundary-value problem (1) can be reduced to analysis of the equations that define the curve $X(t)$.

3.2 Substitutions and the induced boundary-value problems

Suppose that the substitution $u = u[v]$ maps solutions $v \in \operatorname{Sol}\tilde{\mathcal{E}}$ of an equation $\tilde{\mathcal{E}}$ to solutions $u \in \operatorname{Sol}\mathcal{E}$. Many examples are well known: the Cole–Hopf substitution

$$u = \frac{v_x}{v}, \tag{9}$$

the Miura transformation $u = v_x - v^2$, and the formula

$$u = \frac{1}{2}\log[4\partial v \cdot \bar{\partial}\bar{v}/(1 - v\bar{v})^2] \tag{10}$$

by Liouville [9] that relate the Burgers and the heat equations, the Korteweg–de Vries equation and the modified Korteweg–de Vries equation, and the Liouville and the Cauchy–Riemann equations, respectively. We recall that all three expressions (2) are transformed to Eq. (10) by an appropriate change of variables [9]. Also, let the condition f be fixed. Then there can be several substitutions $u[v]$ that solve the boundary-value problem $\mathcal{P}$. Therefore, the reconstruction problem for the function v in the whole domain $\mathcal{D}$ is incorrect by Hadamars. Still, assume that a class of the substitutions $v \mapsto u$ is fixed. Then one easily obtains the coordinate expressions for the equations

$$\psi \equiv \dot{v} \in \ker \tilde{\ell}_{\tilde{\mathcal{E}}},$$

which are analogous to Eq. (8), plus the condition

$$\tilde{\ni}_\psi(u[v]) = \dot{f}$$

defined by the boundary functions f and f_0, and the quadrature

$$v(1) = v(0) + \int_0^1 \psi(t)\,dt.$$

Thence we construct the solution u of the initial boundary-value problem $\mathcal{P}$ in the whole domain $\mathcal{D}$ by using the exact formula $u = u[v]$ and the solution $v(1)$ of the equation $\tilde{\mathcal{E}}$. We note that the condition $\tilde{\ni}_\psi(u[v]) = \dot{f}$ may not even be defined by an operator with directional derivative if the function u depends on the gradient grad v explicitly.

Nowadays, there exist regular algorithmic methods [2] that allow obtaining and classification of the transformations between equations of mathematical physics. These algorithms are already available as environments [10] for the symbolic transformations software.

3.3 One-dimensional reductions by using the Poisson kernel

Now we consider in more details solving the Dirichlet boundary-value problem, see Eq. (1), for the Liouville equation $\mathcal{E}_{\text{Liou}}$ within the class of substitutions (2b) by using two methods described above. Recall that the harmonic functions $v(t)$ admit the representations $P[g]$ via the Poisson kernel P (see [4]) by their boundary value $g \equiv v|_{\partial\mathcal{D}}$. Therefore, the initial boundary-value problem $\mathcal{P}$ which is solved by using the homotopy $\dot{f}$ of the solution u such that $f(t)$ are its values on $\partial\mathcal{D}$ is reduced to the integral equation [6]

$$\frac{d}{dx}P[g]\cdot\frac{d}{dx}P[\dot{g}] + \frac{d}{dy}P[g]\cdot\frac{d}{dy}P[\dot{g}] = (g\dot{g} + g^2\cdot(f - f_0))\cdot\exp(2f)$$

with respect to the deformation $\dot{g} = \psi|_{\partial\mathcal{D}}$ of the boundary value for the harmonic function v. Suppose the boundary value for v is obtained, then an approximation of v on a lattice in $\mathcal{D}$ can be obtained by using multiple averaging of the values of v in the neighbouring interpolation points (see also Sec. 4 below).

In Remark 6.2 on page 109, the results of a computer experiment in application of this method for solving boundary-value problem (1) are discussed.

3.4 Initial approximation supplied by solving another equation

Finally, we note that solutions of some other equation $\mathcal{E}'$ (in general, distinct from the equation $\mathcal{E}$ at hand) can be used for construction of the initial sections u_0 that correspond to the boundary values f_0 on $\partial\mathcal{D}$. Here we assume that the solutions $u(t)$ satisfy a degenerate equation $\mathcal{E}'$ as $t \to 0$.

9 Example ([6]). Suppose $f \rightrightarrows -\infty$ such that $\max f - \min f \le \text{const} < \infty$. Then a solution u of boundary-value problem (1) in the disc B_0^1 is approximated with respect to the norm $\|\cdot\|_{C^0(\bar{B}_0^1)}$ by the harmonic function $P[f]$ as closely as desired. Let

$$\Delta u(t) = \exp(2u(t)), \qquad f(t) = f + \log t, \quad 0 < t \le 1,$$

then in the same notation we have

$$\Delta\varphi(t) - 2\exp(2u(t))\varphi(t) = 0, \quad \dot{f}(t) = t^{-1}, \quad u(t) = u_{t_{\min}} + \int_{t_{\min}}^{t} \varphi(\tau)d\tau,$$

where $t_{\min} \to +0$ and $u_{t_{\min}} = P[f + \log t_{\min}]$ is the initial approximation. Now suppose t is small. We expand the solution $u(t)$ by using the Hadamars lemma and obtain $u(t) = \log t + P[f] + t^2 \cdot U$. If $t \to +0$, then the function U satisfies the homogeneous problem

$$\Delta U = \exp(2P[f]), \qquad U|_{\partial B_0^1} = 0$$

for the Poisson equation. Its solution is

$$U(p) = \int_{B_0^1} \exp(2P[f])(q) \cdot G(p,q)\,dq,$$

where G is the Green function [4] for the Laplace operator in the unit disc.

The method described in Sec. 3.4 is based on the simultaneous deformation $\dot{\mathcal{P}}$ of the boundary condition f and the equation $\mathcal{E}$ itself. Indeed, we have $\mathcal{E}(0) = \mathcal{E}'$ and $\mathcal{E}(t) = \mathcal{E}$ for $t \in (0, 1]$ by construction. In Sec. 4, we consider an opposite situation: an auxiliary equation $\mathcal{E}'$ is used at any value of the parameter t. Here we assume that solutions of $\mathcal{E}'$ tend to solutions of the problem $\mathcal{P}$ as t increases.

4 The relaxation method

Complement the mixed boundary-value problem, which is a generalization of problem (4), by the relaxation term $\partial u/\partial t$. Assume that α and β are the lower and the upper solutions of the initial stationary problem, respectively. Then, extend the boundary value f onto $\partial\mathcal{D}\times\mathbb{R}_+$ and fix an initial approximation u_0 such that $\alpha \preceq u_0 \preceq \beta$: hence we obtain

$$\frac{\partial u}{\partial t} - \Delta u = -h(u,x), \quad a\,\frac{\partial u}{\partial \vec{n}} + b\,u = f, \quad u(x,0) = u_0(x). \tag{11}$$

10 Proposition ([11]). *Let the above assumptions hold. Assume further that $h \in C^1_u[\alpha,\beta]$. Then the following two statements are equivalent*:

(1) the stationary solution u_s of problem (11) *is unique in* $[\alpha,\beta]$;

(2) the solution $u_s \in [\alpha,\beta]$ is asymptotically stable such that the stability domain is $[\alpha,\beta]$.

Here we discussed a method based on the replacement of the elliptic equation $\mathcal{E}$ in problem (4) by the evolution equation $\mathcal{E}'$. This method is an alternative to the evolutionary representation method for the initial equation $\mathcal{E}$ (see Sec. 2).

5 Deformation of the equation

Now we consider the deformations $\dot{\mathcal{P}}$ of the boundary-value problem $\mathcal{P}(t)$ that are induced by the evolution $\dot{\mathcal{E}}$ of the equation $\mathcal{E}(t) = \{F(t) = 0\}$ at hand. Here we assume that the boundary value f remains invariant. Namely. consider the homotopy

$$\mathcal{E}(t) = (1 - r(t))\cdot\mathcal{E}_0 + r(t)\cdot\mathcal{E},$$

where $r(0) = 0$ and $r(1) = 1$. Then the velocity φ of deformation of the solution $u(t)$ is subject to the boundary-value problem

$$\bar{\ell}_{F(t)}(\varphi) + r'(t)\cdot(F - F_0) = 0, \qquad \varphi|_{\partial\mathcal{D}} = 0.$$

Therefore, a solution of the problem $\mathcal{P}$ is defined by the quadrature $u = u_0 + \int_0^1 \varphi(t)\,dt$. We emphasize that this method provides reliable solution approximations in computations, although solutions may be lost if solution to a boundary value problem is not unique (see [3, Lecture 46]).

6 Discussion and practical hints

6.1 The step-like homotopy function

Consider the deformation $(1 - r(t)) \cdot \mathcal{P}_0 + r(t) \cdot \mathcal{P}$ of the boundary values (see Sec. 3) or of the equation itself (see Sec. 5). Then the smooth step-like homotopy function

$$r(t) = \exp(-\mathrm{ctg}^2(\pi t/2)), \qquad 0 < t < 1,$$

is more preferable than the linear motion $\dot{\mathcal{P}} = \mathcal{P} - \mathcal{P}_0$. Then, the final stage of solving the boundary-value problem $\mathcal{P}$ is in fact equivalent to the Newton method that starts with the approximation obtained in the initial computations.

6.2 Numerical experiment

The comparative analysis of practical computations by using the algorithms described in this appendix was carried out in the diploma papers by A. V. Punina (Chair of Higher Mathematics, ISPU; Master thesis (2002) "Comparative analysis of methods for solving the elliptic Liouville equation by using homotopies"), N. P. Cheluhoeva (Chair of Higher Mathematics, ISPU; Master thesis (2002) "Integral equations on the boundary method applied for solving the elliptic Liouville equation"), and N. V. Sergeeva (Chair of Higher Mathematics, ISPU; Master thesis (2004) "Analysis of properties of the homotopy based methods for solving boundary-value problems for nonlinear differential equations"). The following assertion is in order.

- The method of deforming the equation $\mathcal{E}$ (see Sec.5) is twice more precise with respect to the absolute deviation from known exact solutions than the method based on the deformation of the boundary function, see Sec. 3, although the latter method converges 3-5 times faster.

- The relaxation method is the simplest among the algorithms that assume t be continuous. The iterative method is also preferable for solving the Laplace equation that appeared in Sec. 3 since the use of the Poisson kernel requires greater time and the Gauss method cannot be speeded up owing to the strongly sparse matrix of the discrete Laplace operator Δ. In Sec. 3, we described the reduction of the Dirichlet boundary-value problem to the one-dimensional problem and the integral equation. This method is very sensitive with respect to smoothness of the boundary conditions $f(t)$ since the Poisson kernel has a singularity on ∂B_0^1 and the harmonic functions are less smooth on ∂B_0^1 than in $\mathcal{D}$ (where they are infinitely smooth). We also see that a solution of the Liouville equation, when moving along t, can be “attracted” to a nonclassical (of type (3b) or (3c)) solution which satisfies the same boundary conditions. This can happen if the time step is sufficiently large.

Summarizing, we conclude that all these methods demonstrated comparable precisions. Therefore, the choice of an algorithm should be based on the actual properties of the problem at hand.

6.3 Conservation laws in control of precision

Conservation laws for the analysed equation $\mathcal{E}$ serve an important instrument for the precision control in computations. Nowadays, there exist regular methods [2, 7, 16] of reconstruction of the exhaustive set of conservation laws for the equations subject to the ‘2-line theorem’. These methods are realized in the form of the computer software [10] for systems of analytic transformations.

6.4 The classical approach

Application of the methods of solving boundary-value problems described in this appendix does neither neglect nor underesteem the standard check of continuity of the resulting solutions, their Hölder or the Sobolev space properties an so on. Meanwhile, we hope that these practical ideas will compliment the everyday set of instruments for numerical analysis of nonlinear equations of mathematical physics.

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