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Conformally recurrent (κ, μ)-contact manifolds

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Abstract. In this paper it is shown that a conformally recurrent (κ, μ)-contact space M^{2n+1}, $(n > 1)$ is locally isometric to either (i) unit sphere $S^{2n+1}(1)$ or (ii) $E^{n+1} \times S^n(4)$

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1 Introduction

A Riemannian manifold (M, g) is said to be conformally recurrent [1] if there exist a 1-form π such that the conformal curvature C satisfies $\nabla C = \pi \otimes C$, where ∇ is the Levi-Civita connection of g. This type of manifold appears as a generalization of conformally symmetric manifold, introduced and studied by Chaki and Gupta [7]. The aim of this paper is to study a conformally recurrent (κ, μ)-contact space. By a (κ, μ)-contact space we mean a contact metric manifold $M^{2n+1}(\eta, \xi, \varphi, g)$ in which the curvature tensor R satisfies

$$R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hY - \eta(X)hY),$$

for some constants κ and μ on M and $2h = L_\xi\varphi$. Such class of space was introduced in [5] and studied in depth by Boeckx in [6]. Actually this class of space was obtained through D-homothetic deformation [11] to a contact metric manifold whose curvature satisfying $R(X, Y)\xi = 0$. There exist contact metric manifolds for which $R(X, Y)\xi = 0$. For instance the tangent sphere bundle of flat Riemannian manifold admits such structure. Further it is well known that(see [5]) the tangent sphere bundle T_1M of a Riemannian manifold of constant curvature c is a (κ, μ)-contact metric space where $\kappa = c(2-c)$ and $\mu = -2c$. Thus in onehand there exists examples of (κ, μ)-contact manifolds in all dimensions and on the other this class is invariant under D-homothetic deformation. It is evident that the class of (κ, μ)-contact manifolds contains the class of

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Sasakian manifolds, in which $\kappa = 1$. In [8] the author and Sharma proved that a conformally recurrent Sasakian manifold is locally isometric to a unit sphere $S^{2n+1}(1)$. Generalizing this for a (κ, μ)-contact manifold we prove

1 Theorem. *Let M^{2n+1},$(n > 1)$ be a conformally recurrent (κ, μ) -contact space. Then M^{2n+1} is locally isometric to either (i) unit sphere $S^{2n+1}(1)$ or (ii) $E^{n+1} \times S^n(4)$.*

Preliminaries A differential 1-form η on a $(2n + 1)$dimensional differential manifold M is called a contact form if it satisfies $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. By a contact manifold (M, η) we mean a manifold M together with a contact form η. For a contact form η there exists a unique vector field ξ, called the characteristic vector field, such that $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$, for any vector field X on M. Moreover , it is well known that there exists a Riemannian metric g and a (1-1) tensor field φ satisfying $d\eta(X, Y) = g(X, \varphi Y)$, $\eta(X) = g(X, \xi)$, $\varphi^2 X = -X + \eta(X)\xi$. From these we have $\varphi\xi = 0$, $\eta o \varphi = 0$, $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$. The manifold M equipped with the structure (η, ξ, φ, g) is called a contact metric manifold. Denoting by L the Lie differentiation and R the curvature tensor of M, we define the operator h and l by $h = \frac{1}{2} L_\xi \varphi$ and $l = R(., \xi)\xi$. The (1-1) tensors h and l are self adjoint and satisfy $h\xi = 0$, $l\xi = 0$, $Tr\varphi = Trh = Tr\varphi h = 0$, $h\varphi = -\varphi h$. For a contact metric manifold we also have (see [2] and [4])

$$\nabla_X \varphi = -\varphi X - \varphi h X, \tag{2.1}$$

$$\nabla_\xi h = \varphi - \varphi l - \varphi h^2, \tag{2.2}$$

A contact metric manifold is K-contact (ξ is Killing) if and only if $h = 0$. Further a contact metric manifold is Sasakian if and only if

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y.$$

We now give the definition of a (κ, μ)-contact space. By a (κ, μ)-nullity distribution on a contact metric manifold $M^{2n+1}(\eta, \xi, \varphi, g)$ for the pair $(\kappa, \mu) \in R^2$ is a distribution

$$N(\kappa, \mu) : p \to N_p(\kappa, \mu) = \{Z \in T_pM | R(X, Y)Z = \\ = \kappa(g(Y, Z)X - g(X, Z)Y) + \mu(g(Y, Z)hX - g(X, Z)hY)\}.$$

A contact metric manifold M is said to be a (κ, μ)-contact space if ξ belongs to (κ, μ)-nullity distribution of M i.e. (see [5])

$$R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hY - \eta(X)hY). \tag{2.3}$$

If Q denote the Ricci operator and r denote the scalar curvature of M, then the following relations are known for a (κ, μ)-contact space. For details we refer [5].

$$h^2 = (\kappa - 1)\varphi^2, \kappa \leq 1, \tag{2.4}$$

and $\kappa = 1$ if and only if M is Sasakian.

$$Q\xi = (2n\kappa)\xi, \tag{2.5}$$

$$\begin{aligned}(\nabla_X h)Y - (\nabla_Y h)X =&(1-\kappa)[2g(X,\varphi Y)\xi + \eta(X)\varphi Y - \eta(Y)\varphi X]+ \\ &+ (1-\mu)[\eta(X)\varphi hY - \eta(Y)\varphi hX],\end{aligned} \tag{2.6}$$

$$QX = [2(n-1) - n\mu]X + [2(n-1)+\mu]hX + [2(1-n) + n(2\kappa+\mu)]\eta(X)\xi, \tag{2.7}$$

$$r = 2n[2(n-1) + \kappa - n\mu], \tag{2.8}$$

The Weyl conformal curvature tensor C on a $(2n+1)$,$(n > 1)$ dimensional Riemanniam manifold is defined by

$$\begin{aligned}C(X,Y)Z =&R(X,Y)Z - \frac{1}{2n-1}[g(QY,Z)X - g(QX,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QX] \\ &+ \frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y].\end{aligned} \tag{2.9}$$

From this we also have (see [10])

$$\begin{aligned}(divC)(X,Y)Z = &\frac{2(n-1)}{2n-1}[g(\nabla_X Q)Y,Z) - g(\nabla_Y Q)X,Z) \\ &- \frac{1}{4n}\{(X.r)g(Y,Z) - (Y.r)g(X,Z)\}.\end{aligned} \tag{2.10}$$

Finally,we recall the notion of a D-homothetic deformation [11] on a contact metric manifold $M^{2n+1}(\eta,\xi,\varphi,g)$. By a D-homothetic deformation we mean a change of structure tensors of the form $\overline{\eta} = a\eta$, $\overline{\xi} = \frac{1}{a}\xi$, $\overline{g} = ag + a(a-1)\eta\otimes\eta$, where a is a positive constant. It is well known that $M^{2n+1}(\overline{\eta},\overline{\xi},\overline{\varphi},\overline{g})$ is also contact metric manifold. A D-homothetic deformation with constant a transforms a (κ,μ) -contact space into a $(\overline{\kappa},\overline{\mu})$-contact space(see [5]), where $\overline{\kappa} = \frac{\kappa+a^2-1}{a^2}$ and $\overline{\mu} = \frac{\mu+2a-2}{a}$.

Proof. [3 Proof Theorem 1]

Lemma. For a (κ,μ)-contact space, $\nabla_\xi h = \mu h\varphi$.

Proof. Setting $Y = \xi$ in (2.3), and by definition of l and $h\xi = 0$ we have $lX = \kappa(X - \eta(X)\xi) + \mu hX$. Using this in (2.2) and recalling (2.4) we get the required result. QED

We now prove our main theorem.
Since M is (κ,μ)-contact metric space we have $\kappa \leq 1$. For $\kappa = 1$ the manifold

becomes Sasakian and the result follows from [8]. So we assume that $\kappa < 1$. By hypothesis we have

$$(\nabla_W C)(X,Y)Z = \pi(W)C(X,Y)Z. \tag{3.1}$$

Contracting (3.1) over W provides

$$(divC)(X,Y)Z = g(C(X,Y)Z,P). \tag{3.2}$$

Where P is the recurrence vector metrically associated to the recurrence form π. Since κ and μ is constant, from (2.8), we see that r is also constant. Applying this consequence in (2.10), (3.2) reduces to

$$[g(\nabla_X Q)Y,Z) - g(\nabla_Y Q)X,Z)] = \frac{2n-1}{2(n-1)} g(C(X,Y)Z,P). \tag{3.3}$$

Next, differentiating covariantly (2.5) along an arbitrary vector field X and using (2.1) we get

$$(\nabla_X Q)\xi = Q\varphi X + Q\varphi hX - 2n\kappa(\varphi X + \varphi hX). \tag{3.4}$$

Setting $Z = \xi$ in (3.3) and using (3.4) we find that

$$\begin{aligned} g(Q\varphi X &+ \varphi QX + Q\varphi hX - h\varphi QX - (4n\kappa)X, Y) \\ &= \frac{2n-1}{2(n-1)} g(C(X,Y)\xi, P). \end{aligned} \tag{3.5}$$

Replacing X by φX , Y by φY and Z by ξ in (2.10) and by virtue of (2.3), (2.5), it follows that $C(\varphi X, \varphi Y)\xi = 0$. Thus setting $X = \varphi X$, $Y = \varphi Y$ in (3.5) and making use of (2.5) ,(2.7) and the last equality we obtain

$$2\kappa + \mu - \mu\kappa + n\kappa = 0. \tag{3.6}$$

Taking the covariant differentiation of (2.7) and using (2.1) gives

$$\begin{aligned} (\nabla_X Q)Y &= [2(n-1) + \mu](\nabla_X h)Y \\ &- [2(1-n) + n(2\kappa+\mu)][g(\varphi X - \varphi hX, Y)\xi + \eta(Y)(\varphi X + \varphi hX)]. \end{aligned} \tag{3.7}$$

Interchanging X and Y in (3.7) and subtracting the resulting equation from (3.7) and by virtue of (2.6) and (3.6) we find that

$$\begin{aligned} g(\nabla_X Q)Y,Z) &- g(\nabla_Y Q)X,Z) \\ &= (3\mu - n\mu - \mu^2 + 2n\kappa)[\eta(X)g(\varphi hY,Z) - \eta(Y)g(\varphi hX,Z)]. \end{aligned} \tag{3.8}$$

Thus through (3.8), (3.3) reduces to

$$\frac{2(n-1)}{2n-1}g(C(X,Y)Z,P) = (3\mu - n\mu - \mu^2 + 2n\kappa)[\eta(X)g(\varphi hY,Z) - \eta(Y)g(\varphi hX,Z)]. \quad (3.9)$$

Setting $Z = P$ and $X = \xi$ in (3.9) yields

$$(3\mu - n\mu - \mu^2 + 2n\kappa)g(\varphi hY,P) = 0.$$

So we have the two possible cases:

$$(i) 3\mu - n\mu - \mu^2 + 2n\kappa = 0, \quad (3.10)$$

$$(ii) h\varphi P = 0. \quad (3.11)$$

Case (i) Solving (3.6) and (3.10) we obtain the following solutions $\kappa = \mu = 0$, $\kappa = \mu = n+3$ or $\kappa = \frac{n^2-1}{n}$, $\mu = 2(1-n)$.
When $\kappa = \mu = 0$, we have from (2.3) $R(X,Y)\xi = 0$, and applying Blair's theorem (see [3]) we see that M is locally isometric to the product $E^{n+1} \times S^n(4)$. Since $\kappa < 1$ and $n > 1$, the last two solutions are not possible.
Case (ii) Operating (3.11) by h and in view of (2.4) it follows that $P = \pi(\xi)\xi$. Use of this in (3.1) provides $(\nabla_W C)(X,Y)Z = \pi(\xi)\eta(W)C(X,Y)Z$. Next, replacing W by $\varphi^2 W$ and then contracting over W the last equality gives

$$(divC)(X,Y)Z = g((\nabla_\xi C)(X,Y)Z,\xi). \quad (3.12)$$

Setting $X = \xi$ in (3.7) and using $\varphi\xi = h\xi = 0$ and through the lemma yields

$$(\nabla_\xi Q)Y = \mu[2(n-1)+\mu]h\varphi X. \quad (3.13)$$

Further taking the covariant differentiation of (2.3) along ξ and applying the **lemma** provides

$$(\nabla_\xi R)(X,Y)\xi = \mu^2\{\eta(Y)h\varphi X - \eta(X)h\varphi Y\}. \quad (3.14)$$

On the otherhand from (2.9) and together with the help of (3.13), (3.14) and making use of the fact that the scalar curvature is constant we have

$$g((\nabla_\xi C)(X,Y)Z,\xi) = \frac{2\mu(\mu-1)(n-1)}{2n-1}\{\eta(Y)g(h\varphi X,Z) - \eta(X)g(h\varphi Y,Z)\}. \quad (3.15)$$

Comparing (3.15) with (3.12) and then the use of (2.10) yields

$$\frac{2(n-1)}{2n-1}[g(\nabla_Y Q)X, Z) - g(\nabla_X Q)Y, Z) = \frac{2\mu(\mu-1)(n-1)}{2n-1}\{\eta(Y)g(h\varphi X, Z) - \eta(X)g(h\varphi Y, Z)\}. \quad (3.16)$$

Finally, setting $Y = \xi$ in (3.16) and recalling (3.4) and (3.13) we obtain

$$Q\varphi X + Q\varphi hX - 2n\kappa(\varphi X + \varphi hX) - (2n-1)\mu h\varphi X = 0.$$

Hence in view of (2.7) the last equation implies that

$$2\mu + 2n\kappa - n\mu = 0. \quad (3.17)$$

Solving (3.6) and (3.17) it follows that
$\kappa = \mu = 0$ or $\kappa = \frac{(n+1)^2-3}{n}$, $\mu = \frac{2\{(n+1)^2-3\}}{n-2}$ (in the last solution $n \neq 2$, because if $n = 2$, then from (3.17) it follows that $\kappa = 0$ and hence $\mu = 0$). The formar shows that M must be locally isometric to the product $E^{n+1} \times S^n(4)$, and the later leads to a contradiction as $\kappa < 1$.This completes the proof. $\boxed{QED}$

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