Note di Matematica
Note Mat. 28 (2008), n. 2, 153-162
ISSN 1123-2536, e-ISSN 1590-0932
DOI 10.1285/i15900932v28n2p153

Best Simultaneous L^p Approximation in the "Sum" Norm

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Received: 15/08/2007; accepted: 28/11/2007.

Abstract. In this paper we consider best simultaneous approximation by algebraic polynomials respect to the norm $\sum_{j=1}^{k} \|f_j - P\|_p$, $1 \leq p < \infty$. We prove an interpolation property of the best simultaneous approximations and we study the structure of the set of cluster points of the best simultaneous approximations on the interval $[-\epsilon, \epsilon]$, as $\epsilon \to 0$.

Keywords: Simultaneous approximation, Algebraic polynomials, L^p-Norm

MSC 2000 classification: primary 41A28, secondary 41A10

Introduction

Let X be the space of measurable Lebesgue real functions defined on the interval $[-1, 1]$. If $h \in X$ and $0 < \epsilon \leq 1$ we denote

$$\|h\|_{p,\epsilon} = \left(\int_{-\epsilon}^{\epsilon} |h(x)|^p dx \right)^{\frac{1}{p}} \quad 1 \leq p < \infty.$$

Let $\Pi^n \subset X$ be the space of polynomials of degree at most n. Given $h_i \in X$, $1 \leq i \leq k$, we consider the norm

$$\rho_{p,\epsilon}(h_1, \ldots, h_k) = \sum_{i=1}^{k} \|h_i\|_{p,\epsilon}. \tag{1}$$

We say that $P_\epsilon \in \Pi^n$ *is a* $\rho_{p,\epsilon}$*-best simultaneous approximation(*$\rho_{p,\epsilon}$*-b.s.a.) in* Π^n *of the functions* $f_i \in X$*,* $1 \leq i \leq k$*, respect to* $\rho_{p,\epsilon}$, if

$$\rho_{p,\epsilon}(f_1 - P_\epsilon, \ldots, f_k - P_\epsilon) = \inf_{Q \in \Pi^n} \rho_{p,\epsilon}(f_1 - Q, \ldots, f_k - Q). \tag{2}$$

[i]This work was supported by Universidad Nacional de Rio Cuarto and Conicet.
[ii]This work was supported by Universidad Nacional de Rio Cuarto.

In [3] the authors proved that the best approximation to $\frac{1}{n}\sum_{i=1}^{k} f_i$ in Π^n with the norm $\|.\|_{2,\epsilon}$ are identical with the best simultaneous approximation to $\{ f_1, \ldots, f_k \}$, with the measure $\sum_{i=1}^{k} \|h_i\|_{2,\epsilon}^2$. In this case, there is uniqueness of the b.s.a., however it is easy to see that if $f_1,\ f_2 \in \Pi^n$, then any convex combination of them is a $\rho_{p,\epsilon}$-b.s.a.. Further, even for $p = 2$, the previous equivalence is not true, an example is showed in ([4]).

We prove in this paper that if $1 < p < \infty$, any $\rho_{p,\epsilon}$-b.s.a. in Π^n of two continuous functions f and g in X, interpolates some convex combination of f and g in at least $n + 1$ points. If $p = 2$, a similar result is obtained for $\rho_{2,\epsilon}$-b.s.a. of k continuous functions. For $p = 1$ other necessary condition over the $\rho_{1,\epsilon}$-b.s.a. of k continuous functions is established.

For $1 < p < \infty$, if we assume that f and g have continuous derivatives up to order n in a neighborhood of 0, we show that for any net of $\rho_{p,\epsilon}$-b.s.a. in Π^n, P_ϵ, $\epsilon \to 0$, there exists a subsequence which converges to some convex combination of the Taylor's polynomials of f and g. We get an analogous result for k functions and $p = 2$.

We give an example which shows that, in general, the set of cluster points of P_ϵ, $\epsilon \to 0$, is not unitary, even if we have uniqueness of the $\rho_{p,\epsilon}$-b.s.a. for each $0 < \epsilon$.

Finally, if $1 < p < \infty$, $k = 2$, or $p = 2$, $k \geq 2$, we prove that the set of cluster points of P_ϵ, as $\epsilon \to 0$, is a compact and convex set in Π^n with the uniform norm.

1 Interpolating of best simultaneous approximations

We recall a Lemma proved in [6].

1 Lemma. *Let M be a linear subspace of X, and $f \in X \setminus \overline{M}$. Then $g^* \in M$ is a best approximation of f in M if and only if*

$$\tau_+(f - g^*, g) \geq 0,$$

for all $g \in M$, where $\tau_+(f, g) = \lim_{t\to 0^+} \frac{\|f+tg\|-\|f\|}{t}$.

Given k functions $f_1, \ldots, f_k$, let P_ϵ be a $\rho_{p,\epsilon}$-b.s.a. of them. If $\|f_j - P_\epsilon\|_{p,\epsilon} \neq 0$ for all $1 \leq j \leq k$, we consider the numbers

$$\alpha_j = \frac{\|f_j - P_\epsilon\|_{p,\epsilon}^{-1}}{\sum_{i=1}^{k} \|f_i - P_\epsilon\|_{p,\epsilon}^{-1}},\ 1 \leq j \leq k.$$

With this notation we have

2 Theorem. *Let $f_1, \ldots, f_k \in X$ be continuous functions and let P_ϵ be a $\rho_{p,\epsilon}$-b.s.a. in Π^n of the functions f_i, $1 \leq i \leq k$. Then*

a) *If $p = 2$, there is j, $1 \leq j \leq k$, such that $P_\epsilon = f_j$ on $[-\epsilon, \epsilon]$ or P_ϵ interpolates $\sum_{j=1}^k \alpha_j f_j$ in at least $n+1$ points of $[-\epsilon, \epsilon]$.*

b) *If $1 < p < \infty$ and $k = 2$, there is j, $1 \leq j \leq 2$, such that $P_\epsilon = f_j$ on $[-\epsilon, \epsilon]$ or P_ϵ interpolates $\alpha_1 f_1 + \alpha_2 f_2$, in at least $n+1$ points of the interval $[-\epsilon, \epsilon]$.*

c) *If $p = 1$, there is j, $1 \leq j \leq k$, such that $P_\epsilon = f_j$ on a positive measure subset of $[-\epsilon, \epsilon]$, or there are at least $n+1$ points $x_i \in [-\epsilon, \epsilon]$ such that $\sum_{j=1}^k sgn(f_j - P_\epsilon)(x_i) = 0$.*

Proof. For simplicity we omit everywhere the indexes ϵ and p.
If $\|f_j - P\| = 0$ for some j the Theorem follows immediately. So, we suppose that $\|f_j - P\| \neq 0$ for all j. First we assume $p > 1$. By a straightforward computation and Lemma 1, we obtain

$$\lim_{t \to 0^+} \frac{\rho((f_1, \ldots, f_k) - (P, \ldots, P) + t(Q, \ldots, Q)) - \rho((f_1, \ldots, f_k) - (P, \ldots, P))}{t}$$
$$= \tau_+(f_1 - P, Q) + \cdots + \tau_+(f_k - P, Q) = \int h(x)Q(x)dx \geq 0, \tag{3}$$

for all $Q \in \Pi^n$, where

$$h(x) := \sum_{j=1}^k \frac{1}{\|f_j - P\|^{p-1}} |(f_j - P)(x)|^{p-1} \text{sgn}(f_j - P)(x). \tag{4}$$

Suppose that $x_0, \ldots, x_m \in [-\epsilon, \epsilon]$ are the points where the function h changes of sign. We observe that $m \geq n$. In fact, if $m < n$ we can find a polynomial $Q \in \Pi^n$ which changes of sign exactly in these points, so $h(x)Q(x) \leq 0$ on the interval $[-\epsilon, \epsilon]$ and $h(x)Q(x) < 0$ on some subset of positive measure. It contradicts (3). Henceforth we suppose $h(x_i) = 0$, where $x_i \in [-\epsilon, \epsilon]$, $0 \leq i \leq n$.

a) If $p = 2$, from (3) and (4) we get

$$P(x_i) = \sum_{j=1}^k \alpha_j f_j(x_i), \quad 0 \leq i \leq n. \tag{5}$$

b) Suppose $k = 2$, and let $x \in [-\epsilon, \epsilon]$ be such that $h(x) = 0$. If $(f - P)(x)(g - P)(x) \geq 0$, then $f(x) = P(x) = g(x)$, while $(f - P)(x)(g - P)(x) < 0$ implies $P(x) = (\alpha_1 f_1 + \alpha_2 f_2)(x)$. Therefore, in either case we have $P(x) = (\alpha_1 f_1 + \alpha_2 f_2)(x)$.
In consequence, $P(x_i) = (\alpha_1 f_1 + \alpha_2 f_2)(x_i)$, $0 \leq i \leq n$. This proves b).

c) Assume $p = 1$. By (3) we get

$$\sum_{j=1}^{k} \int_{\{f_j \neq P\}} \operatorname{sgn}(f_j - P)(x) Q(x) dx + \int_{\{f_j = P\}} |Q(x)| dx \geq 0, \qquad (6)$$

for all $Q \in \Pi^n$. If there is j, $1 \leq j \leq k$, such that $P = f_j$ on a positive measure subset, the result is obvious. Suppose that $|\{ x \in [-\epsilon, \epsilon] \mid P(x) = f_j(x) \}| = 0$ for all $1 \leq j \leq k$. From (6) we get $\int h(x) Q(x) dx \geq 0$, for all $Q \in \Pi^n$, where

$$h(x) := \sum_{j=1}^{k} \operatorname{sgn}(f_j - P)(x). \qquad (7)$$

By the proof of part a), there are at least $n+1$ points x_i such that $h(x_i) = 0$, $0 \leq i \leq n$. This proves c).

QED

We recall the Newton's divided difference formula for the interpolation polynomial (see [1]): The polynomial interpolating $h(x)$ of degree n at $x_0, \dots, x_n$ is

$$P(x) = h(x_0) + (x - x_0) h[x_0, x_1] + \dots + (x - x_0) \dots (x - x_{n-1}) h[x_0, \dots, x_n], \quad (8)$$

where $h[x_0, \dots, x_n]$ denotes the nth-order Newton divided difference. Also, it is well known that

$$h[x_0, \dots, x_m] = \frac{h^{(m)}(\xi)}{m!}, \qquad (9)$$

for some ξ in the smallest interval containing $x_0, \dots, x_m$.

Henceforth we denote $T(f)$ the Taylor's polynomial of f at 0 of degree n.

3 Theorem. *Let $1 < p < \infty$ and let $0 < \epsilon_j \leq 1$ be a sequence such that $\epsilon_j \downarrow 0$. Suppose that $f_1, \dots, f_k \in X$ are functions with continuous derivatives up to order n and let P_{ϵ_j} be a $\rho_{p,\epsilon}$-b.s.a. in Π^n of $f_1, \dots, f_k$. Then*

a) *If $p = 2$, there exist a subsequence ϵ_{j_s} and $\gamma_l \in [0, 1]$, $1 \leq l \leq k$, such that $\sum_{l=1}^{k} \gamma_l = 1$ and $P_{\epsilon_{j_s}} \to \sum_{l=1}^{k} \gamma_l T(f_l)$, as $s \to \infty$.*

b) *If $k = 2$, there exist a subsequence ϵ_{j_s} and $\gamma_0 \in [0, 1]$ such that $P_{\epsilon_{j_s}} \to \gamma_0 T(f_1) + (1 - \gamma_0) T(f_2)$, as $s \to \infty$.*

Here the convergence is uniform on any compact subset of $\mathbb{R}$.

Proof. We only prove b), the proof of a) is analogous. Suppose that $k = 2$. By Theorem 2, b), for each ϵ_j there exist $x_i = x_i(\epsilon_j) \in [-\epsilon_j, \epsilon_j]$, $0 \leq i \leq n$, such that P_{ϵ_j} interpolates $h_j := \gamma_j f_1 + (1 - \gamma_j) f_2$ in x_i, $0 \leq i \leq n$, where $\gamma_j \in [0, 1]$.

Since $\{\gamma_j\}$ is bounded, there exists a convergent subsequence γ_{j_s}. Suppose that $\gamma_{j_s} \to \gamma_0 \in [0,1]$ as $s \to \infty$. From (8) and (9) follows that

$$P_{\epsilon_{j_s}}(x) = h_{j_s}(x_0) + (x - x_0) h_{j_s}^{(1)}(\xi(s,1)) + \cdots$$
$$\cdots + (x - x_0) \cdots (x - x_{n-1}) \frac{h_{j_s}^{(n)}(\xi(s,n))}{n!}, \quad (10)$$

where $\xi(s,i) \in [-\epsilon_{j_s}, \epsilon_{j_s}]$, $1 \le i \le n$, $s \in \mathbb{N}$. Taking limit for $s \to \infty$ in (10) and using the continuity of the derivatives of the functions f_1 and f_2 we get the Theorem. QED

Given $f_1, \dots, f_k \in X$ we consider the set $\mathcal{H}(\rho_p) = \mathcal{H}(\rho_p; f_1, \dots, f_k)$, defined by

$$\{\, Q \in \Pi^n \mid \exists \text{ a sequence of } \rho_{p,\epsilon_m}\text{-b.s.a. to } f_j, 1 \le j \le k,$$
$$P_{\epsilon_m} \to Q, \ \text{ as } \epsilon_m \downarrow 0 \,\}. \quad (11)$$

If there exist $T(f_1), \dots, T(f_k)$, we write

$$T(f_1, \dots, f_k) = \left\{ \sum_{j=1}^{k} \beta_j T(f_j) \middle| \sum_{j=1}^{k} \beta_j = 1, \beta_j \ge 0, 1 \le j \le k \right\}. \quad (12)$$

With this notation we immediately get the following Corollary of the Theorem 2.

4 Corollary. *Let $n \in \mathbb{N} \cup \{0\}$, $k \in \mathbb{N}$, and let $f_1, \dots, f_k \in X$ be functions with continuous derivatives up to order n in a neighborhood of the origin. We have*

a) $\varnothing \ne \mathcal{H}(\rho_2; f_1, \dots, f_k) \subset T(f_1, \dots, f_k)$.

b) *If $1 < p < \infty$, then $\varnothing \ne \mathcal{H}(\rho_p; f_1, f_2) \subset T(f_1, f_2)$.*

2 The structure of the set $\mathcal{H}(\rho_p)$

In this Section we study the structure of the set $\mathcal{H}(\rho_p)$. As we observe in the Introduction, if f, $g \in \Pi^n$ then for all $0 < \epsilon \le 1$ the set of $\rho_{p,\epsilon}$-b.s.a. is the segment $\overline{fg} := \{\, \alpha f + (1-\alpha) g \mid \alpha \in [0,1] \,\}$. So, $\mathcal{H}(\rho_p) = \overline{fg}$. Here, we will give an example where there is uniqueness of the $\rho_{p,\epsilon}$-b.s.a for all $\epsilon > 0$, but the set $\mathcal{H}(\rho_p)$ is not a unitary set.

We introduce some notation. Let $0 < a < b < c < d \le 1$ and let f_1, g_1 be bounded and even measurable Lebesgue real functions defined on $[-d, d]$. Set

$\overline{h_1}(x)$ the linear function defined on $[a,b]$, which joins the points $(b, -b+\frac{a+d}{2}+1)$ and $(a,1)$, and $h_1(x)$ the linear function on $[c,d]$, which joins the points $(c, -c+\frac{a+d}{2}+1)$ and $(d,1)$. We define two functions f and g on $[-d,d]$ by:

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in [0,a] \\ \overline{h_1}(x) & \text{if } x \in [a,b] \\ -x + \frac{a+d}{2} + 1 & \text{if } x \in [b,c] \\ h_1(x) & \text{if } x \in [c,d], \end{cases} \tag{13}$$

$$g(x) = \begin{cases} g_1(x) & \text{if } x \in [0,a] \\ 0 & \text{if } x \in [a,d], \end{cases} \tag{14}$$

and $f(x) = f(-x)$, $g(x) = g(-x)$ if $x \in [-d, 0]$.

We need the following auxiliary Lemma.

5 Lemma. *Let $d > 0$ and $\lambda > 0$. Then there are real numbers a, b, c with $0 < a < b < c < d$ such that any $\rho_{2,d}$-b.s.a. by constants of the functions f and g, defined by (13) and (14), is at most λ.*

Proof. Let $E(\gamma) := \|f-\gamma\|_d + \|g-\gamma\|_d$, $\gamma \geq \lambda$. We have

$$\begin{aligned} \|f-\gamma\|_d = (&\int_0^a 2(f_1-\gamma)^2(x)dx + \int_a^b 2(\overline{h_1}(x)-\gamma)^2 dx \\ &+ \int_b^c 2(x + \frac{a+d}{2} + 1 - \gamma)^2 dx + \int_c^d 2(h_1(x)-\gamma)^2 dx)^{1/2} \\ =: &(B_1(a,\gamma) + B_2(a,b,\gamma) + B_3(a,b,c,\gamma) + B_4(c,\gamma))^{1/2}, \end{aligned} \tag{15}$$

and

$$\begin{aligned} \|g-\gamma\|_d &= \left(\int_0^a 2(g_1-\gamma)^2(x)dx + \int_a^d 2\gamma^2 dx\right)^{1/2} \\ &=: \left(B_5(a,\gamma) + 2\gamma^2(d-a)\right)^{1/2}. \end{aligned} \tag{16}$$

We estimate the derivative of the error function $E(\gamma)$.

$$\begin{aligned} E'(\gamma) = \frac{1}{2}&(B_1+B_2+B_3+B_4)^{-1/2}(B_1'+B_2'+B_3'+B_4') \\ &+ \frac{1}{2}(B_5 + 2\gamma^2(d-a))^{-1/2}(B_5' + 4\gamma(d-a)). \end{aligned} \tag{17}$$

Since f_1 and g_1 are bounded on $[-d,d]$, it follows that f and g are uniformly bounded, with bound independent on the values a, b and c.

Suppose that $|f(x)| \leq \Gamma$ and $|g(x)| \leq \Gamma$ for all $x \in [-d,d]$ and for all choice of a, b and c. Therefore, the $\rho_{2,\epsilon}$-b.s.a. constant of f and g verifies $|\gamma| \leq \Gamma$. We shall prove that there are a, b and c such that $E'(\gamma) > 0$ for all $\gamma \in [\lambda, \Gamma]$. Since f_1, g_1, h_1, and $\overline{h_1}$ are uniformly bounded, with bound independent on the values a, b and c, we get

$$\lim_{a\to 0} B_1 = \lim_{a,b\to 0} B_2 = \lim_{c\to d} B_4 = \lim_{a\to 0} B_5 = 0, \tag{18}$$

$$\lim_{a\to 0} B_1' = \lim_{a,b\to 0} B_2' = \lim_{c\to d} B_4' = \lim_{a\to 0} B_5' = 0, \tag{19}$$

$$\lim_{a,b\to 0, c\to d} B_3 = \frac{d^3}{6} + 2(1-\gamma)^2 d, \text{ and } \lim_{a,b\to 0, c\to d} B_3' = -4(1-\gamma)d, \tag{20}$$

uniformly on $\gamma \in [\lambda, \Gamma]$.
From (18), (19) and (20) we get

$$\lim_{a,b\to 0, c\to d} E'(\gamma) = (2d)^{1/2} \left(\left(\frac{d^2}{12} + (1-\gamma)^2 \right)^{-1/2} (\gamma - 1) + 1 \right), \tag{21}$$

uniformly on $\gamma \in [\lambda, \Gamma]$.
Consider the function $S(x) = -x(A + x^2)^{-1/2} + 1$ with $A > 0$. It is easy to see that $S(x) \geq 1 - (A+1)^{-1/2}$ on the interval $(-\infty, 1]$. In fact, if $x \leq 0$, $S(x) \geq 1$. If $0 < x \leq 1$, $S(x)$ is a decreasing function and $S(1) = 1 - (A+1)^{-1/2}$. From (21) with $A = \frac{d^2}{12}$ and $x = 1 - \gamma$, we obtain

$$\lim_{a,b\to 0, c\to d} E'(\gamma) \geq (2d)^{1/2} \left(1 - \left(\frac{d^2}{12} + 1 \right)^{-1/2} \right), \tag{22}$$

for all $\gamma \in [\lambda, \Gamma]$.
From (22) immediately follows that there exist a, b, and c such that $E'(\gamma) > 0$, for all $\gamma \in [\lambda, \Gamma]$. As a consequence any constant $\rho_{2,d}$-b.s.a., say γ, of f and g defined by (13) and (14) for those values of a, b and c, verifies $\gamma \leq \lambda$. QED

6 Remark. Similarly to Lemma 5, given $d > 0$ and $0 < \lambda < 1$, we can find real numbers a, b, c with $0 < a < b < c < d$ such that any constant $\rho_{2,d}$-b.s.a. on the interval $[-d,d]$ of the functions $f - 1$ and $g + 1$, where f and g are given by (13) and (14) respectively, is greater or equal than $1 - \lambda$.

The following Lemma was proved in [5], Theorem 4, (a) in a more general way.

7 Lemma. *Let $1 < p < \infty$, $0 < d \leq 1$, and let $f_1, \ldots, f_k \in \mathcal{C}([-d,d], \mathbb{R})$. Then the set S_d of $\rho_{p,d}$-b.s.a. of f_j, $1 \leq j \leq k$, from Π^n, is a unitary set or there exists $i, 1 \leq i \leq k-1$, such that $f_j \in \{ \alpha f_{i+1} + (1-\alpha) f_i \mid \alpha \geq 1 \}$, $i+1 \leq j \leq k$, $f_j \in \{ \alpha f_{i+1} + (1-\alpha) f_i \mid \alpha \leq 0 \}$, $1 \leq j \leq i$, and S_d is the segment $\overline{f_i f_{i+1}}$.*

Now, we are in conditions to give the example mentioned at begin of this Section.

8 Example. Let ϵ_k, η_k, $\overline{\eta}_k$, δ_k, and $\overline{\delta}_k$, $k \in \mathbb{N}$ be five sequences of real numbers satisfying

(1) $\epsilon_1 = 1$,

(2) $\epsilon_{2k} < \overline{\eta}_{2k-1} < \eta_{2k-1} < \epsilon_{2k-1}$,

(3) $\epsilon_{2k+1} < \overline{\delta}_{2k} < \delta_{2k} < \epsilon_{2k}$,

(4) $\epsilon_k \downarrow 0$.

We consider two functions f and g defined on $[-1,1]$ by:

$$f(x) = \begin{cases} 1 & \text{if } x = 1 \\ 1 & \text{if } x \in [\epsilon_{2k+1}, \epsilon_{2k}] \\ \overline{h}_{2k-1}(x) & \text{if } x \in [\epsilon_{2k}, \overline{\eta}_{2k-1}] \\ -x + \frac{\epsilon_{2k}+\epsilon_{2k-1}}{2} + 1 & \text{if } x \in [\overline{\eta}_{2k-1}, \eta_{2k-1}] \\ h_{2k-1}(x) & \text{if } x \in [\eta_{2k-1}, \epsilon_{2k-1}], \end{cases} \tag{23}$$

$$g(x) = \begin{cases} 0 & \text{if } x = 0 \\ 0 & \text{if } x \in [\epsilon_{2k}, \epsilon_{2k-1}] \\ \overline{l}_{2k}(x) & \text{if } x \in [\epsilon_{2k+1}, \overline{\delta}_{2k}] \\ -x + \frac{\epsilon_{2k}+\epsilon_{2k+1}}{2} + 1 & \text{if } x \in [\overline{\delta}_{2k}, \delta_{2k}] \\ l_{2k}(x) & \text{if } x \in [\delta_{2k}, \epsilon_{2k}], \end{cases} \tag{24}$$

where h_{2k-1}, $\overline{h}_{2k-1}$, l_{2k} and $\overline{l}_{2k}$ are linear functions chosen in a such way that f and g be continuous functions on $[0,1]$. Finally, we put $f(x) = f(-x)$, $g(x) = g(-x)$ if $x \in [-1,0]$. We can choose the sequences $\epsilon_k, \eta_k, \overline{\eta}_k, \delta_k$, and $\overline{\delta}_k$, such that any constant $\rho_{2,\epsilon_{2k+1}}$-b.s.a. is at most $\frac{1}{3}$, and any constant $\rho_{2,\epsilon_{2k}}$-b.s.a. is greater or equal than $\frac{2}{3}$. In fact, it is sufficient to apply the Lemma 5 and the Remark 6 alternatively with $d = \epsilon_k, k \in \mathbb{N}$, and $\lambda = \frac{1}{3}$.

9 Remark. Since $f \notin \Pi^n$ and $g \notin \Pi^n$ on $[-\epsilon, \epsilon]$, for all $0 < \epsilon \leq 1$, the Lemma 7 implies uniqueness of the $\rho_{2,\epsilon}$-b.s.a. by constants for all $0 < \epsilon \leq 1$.

Next, we give the main Theorem of this Section.

10 Theorem. *Let $n \in \mathbb{N} \cup \{0\}, k \in \mathbb{N}$, and $1 < p < \infty$. Let $f_1, \ldots, f_k \in X$ be functions with continuous derivatives up to order n. Then $\mathcal{H}(\rho_2; f_1, \ldots, f_k)$ and $\mathcal{H}(\rho_p; f_1, f_2)$ are convex and compact sets in Π^n with the uniform norm.*

PROOF. Let $P_j \in \mathcal{H}(\rho_2; f_1, \dots, f_k)$ $(P_j \in \mathcal{H}(\rho_p; f_1, f_2))$ be a sequence such that $P_j \to P_0 \in \Pi^n$, as $j \to \infty$. For each $j \in \mathbb{N}$ there exists ϵ_j such that $\|P_{\epsilon_j} - P_j\| < \frac{1}{j}$. We can choose ϵ_j such that $\epsilon_{j+1} < \frac{\epsilon_j}{2}$, then $\epsilon_j \to 0$ and

$$\|P_{\epsilon_j} - P_0\| \leq \|P_{\epsilon_j} - P_j\| + \|P_j - P_0\| \to 0 \text{ as } j \to \infty.$$

It follows that $P_0 \in \mathcal{H}(\rho_2; f_1, \dots, f_k)$ $(P_0 \in \mathcal{H}(\rho_p; f_1, f_2))$. So, these sets are closed.

By Corollary 4 we have $\mathcal{H}(\rho_2; f_1, \dots, f_k) \subset T(f_1, \dots, f_k)$, and $\mathcal{H}(\rho_p; f_1, f_2) \subset T(f_1, f_2)$. This proves that the sets are bounded, so they are compact.

Next we prove the convexity of the set $\mathcal{H}(\rho_p; f_1, f_2)$.

Let S_d be as in Lemma 7. If for some $0 < d \leq 1$, S_d is not unitary set, the Lemma 7 implies that $S_d = \overline{f_1 f_2}$. It is easy to see that $S_\epsilon = \overline{f_1 f_2}$ for all $0 < \epsilon \leq d$. So, $\mathcal{H}(\rho_p; f_1, f_2) = \overline{f_1 f_2}$.

Now suppose that S_ϵ is a unitary set for all $0 < \epsilon \leq 1$. We write $S(\epsilon) = P_\epsilon$. The function $S : (0, 1] \to \Pi^n$ is continuous. In fact, if $0 < a_j \leq 1$, $j \in \mathbb{N}$, is a real number sequence such that $a_j \to a > 0$, as $j \to \infty$, then $\|h\|_{p,a_j} \to \|h\|_{p,a}$ for all continuous function $h \in X$. Thus $\rho_{p,a_j}(h_1, h_2) \to \rho_{p,a}(h_1, h_2)$, as $j \to \infty$ for all pair of continuous functions in X. Since there exists a unique $\rho_{p,a}$-b.s.a. of f_1 and f_2, the Polya's algorithm, (see [2]), implies that $S(a_j) \to S(a)$, as $j \to \infty$.

Let P_1, $P_2 \in \mathcal{H}(\rho_p)$, $P_1 \neq P_2$ and $P_3 = \alpha P_1 + (1 - \alpha) P_2$, with $0 < \alpha < 1$. By definition of $\mathcal{H}(\rho_p; f_1, f_2)$ there exist two sequences $\epsilon_j \to 0$ and $\epsilon'_j \to 0$ such that

$$P_{\epsilon_j} \to P_1, \quad P_{\epsilon'_j} \to P_2, \text{as } j \to \infty. \tag{25}$$

Without loss generality, we can suppose that $\epsilon_1 > \epsilon'_1 > \epsilon_2 > \epsilon'_2 > \dots$. Let U be a hyperplane in Π^n orthogonal to the segment $\overline{P_1 P_2}$, with respect to the inner product in Π^n, which contains to P_3, i.e.,

$$U = \{\, Q + P_3 \mid Q \in \Pi^n \text{ and } Q \cdot (P_1 - P_2) = 0 \,\}.$$

Since U is a closed set the distance of P_1 to U and the distance of P_2 to U are both positive. Thus (25) implies that there exists N such that for $j > N$, $S(\epsilon_j)$ and $S(\epsilon'_j)$ live in different semi-planes respect to U. Let $j > N$. As $S(x)$ is a continuous function, $S((\epsilon'_j, \epsilon_j))$ is a connected arc set in Π^n. Therefore $U \cap S((\epsilon'_j, \epsilon_j)) \neq \emptyset$. In consequence, we can find ϵ''_j, $\epsilon'_j < \epsilon''_j < \epsilon_j$, such that $P_{\epsilon''_j} \in U$. On the other hand, Theorem 3 implies that there exist a subsequence of $\{\, \epsilon''_j \,\}$, which we denote again by ϵ''_j, and $0 \leq \beta \leq 1$ such that

$$S(\epsilon''_j) \to \beta T(f_1) + (1 - \beta) T(f_2).$$

Since $S(\epsilon_j'') \in U$, $j > N$, and U is a closed set, then $\beta T(f_1) + (1-\beta)T(f_2) \in U$. In addition, $U \cap T(f_1, f_2) = \{ P_3 \}$, so we get $P_3 = \beta T(f_1) + (1-\beta)T(f_2)$, i.e., $P_3 \in \mathcal{H}(\rho_p; f_1, f_2)$.

The convexity of $\mathcal{H}(\rho_2; f_1, \ldots, f_k)$ follows analogously. The proof is complete.

QED

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