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On rarely δs-continuous functions

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Abstract. The notion of rare continuity introduced by Popa [12]. In this paper, we introduce a new class of functions called rarely δs-continuous functions and investigate some of its fundamental properties. This type of continuity is a generalization of super continuity [10].

Keywords: Rare set, δ-semiopen, rarely δs-continuous, rarely almost compact.

MSC 2000 classification: primary 54B05, 54C08, secondary 54D05

1 Introduction

Levine [7] defined semiopen sets which are weaker than open sets in topological spaces. After Levine's semiopen sets, mathematicians gave in several papers different and interesting new open sets as well as generalized open sets. In 1968, Veličko [13] introduced δ-open sets, which are stronger than open sets, in order to investigate the characterization of H-closed spaces. In 1997, Park et al. [11] have introduced the notion of δ-semiopen sets which are stronger than semiopen sets but weaker than δ-open sets and investigated the relationships between several types of open sets. In 1979, Popa [12] introduced the useful notion of rare

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continuity as a generalization of weak continuity [6]. The class of rarely continuous functions has been further investigated by Long and Herrington [8] and Jafari [3] and [4].

The purpose of the present paper is to introduce the concept of rare δs-continuity in topological spaces as a generalization of super continuity. We also investigate several properties of rarely δs-continuous functions. The notion of $I.\delta s$-continuity is also introduced which is weaker than super-continuity and stronger than rare δs-continuity. It is shown that when the codomain of a function is regular, then the notions of rare δs-continuity and $I.\delta s$-continuity are equivalent.

2 Preliminaries

Throughout this paper, (X,τ) and (Y,σ) (or simply, X and Y) denote topological spaces on which no separation axioms are assumed unless explicitly stated. If A is any subset of a space X, then $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure and the interior of A, respectively.

A subset A of X is called regular open (resp. regular closed) if $A = \mathrm{Int}(\mathrm{Cl}(A))$ (resp. $A = \mathrm{Cl}(\mathrm{Int}(A))$). Recall that a subset A of X is called semi-open [7] if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$. The complement of a semi-open sets is called semi-closed. A rare or codense set is a set A such that $\mathrm{Int}(A) = \emptyset$, equivalently, if the complement $X \setminus A$ is dense. A point $x \in X$ is called a δ-cluster [13] of A if $A \cap U \neq \emptyset$ for each regular open set U containing x. The set of all δ-cluster points of A is called the δ-closure of A and is denoted by $\mathrm{Cl}_\delta(A)$. A subset A is called δ-closed if $\mathrm{Cl}_\delta(A) = A$. The complement of a δ-closed set is called δ-open. The δ-interior of a subset A of a space (X,τ), denoted by $\mathrm{Int}_\delta(A)$, is the union of all regular open sets of (X,τ) contained in A. A topological space (X,τ) is said to be semi-regular [2] if for each semi-closed set A and any point $x \in X \setminus A$, there exist disjoint semi-open sets U and V such that $A \subset U$ and $x \in V$.

A subset A of a topological space X is said to be δ-semiopen sets [11] if there exists a δ-open set U of X such that $U \subset A \subset \mathrm{Cl}(U)$, equivalently if $A \subset \mathrm{Cl}(\mathrm{Int}_\delta(A))$. The complement of a δ-semiopen set is called a δ-semiclosed set. A point $x \in X$ is called the δ-semicluster point of A if $A \cap U \neq \emptyset$ for every δ-semiopen set U of X containing x. The set of all δ-semicluster points of A is called the δ-semiclosure of A, denoted by $\mathrm{sCl}_\delta(A)$ and The δ-semiinterior of A, denoted by $\mathrm{sInt}_\delta(A)$, is defined as the union of all δ-semiopen sets contained in A. We denote the collection of all δ-semiopen (resp. δ-semiclosed, δ-open, regular open and open) sets by $\delta\mathrm{SO}(X)$ (resp. $\delta\mathrm{SC}(X)$, $\delta O(X)$, $RO(X)$ and $O(X)$). We set $\delta SO(X,x) = \{U \mid x \in U \in \delta SO(X)\}$, $\delta O(X,x) = \{U \mid x \in U \in \delta O(X)\}$, $RO(X,x) = \{U \mid x \in U \in RO(X)\}$ and $O(X,x) = \{U \mid x \in U \in O(X)\}$.

1 Lemma. *The intersection (resp. union) of an arbitrary collection of δ-semiclosed (resp. δ-semiopen) sets in (X, τ) is δ-semiclosed (resp. δ-semiopen)*

2 Corollary. *Let A be a subset of a topological space (X, τ). Then the following properties hold:*

(1) $\mathrm{sCl}_\delta(A) = \cap\{F \in \delta SC(X, \tau) : A \subset F\}$.

(2) $\mathrm{sCl}_\delta(A)$ *is δ-semiclosed.*

(3) $\mathrm{sCl}_\delta(\mathrm{sCl}_\delta(A)) = \mathrm{sCl}_\delta(A)$.

3 Lemma ([1])**.** *For subsets A and A_i $(i \in I)$ of a space (X, τ), the following hold:*

(1) $A \subset \mathrm{sCl}_\delta(A)$.

(2) If $A \subset B$, then $\mathrm{sCl}_\delta(A) \subset \mathrm{sCl}_\delta(B)$.

(3) $\mathrm{sCl}_\delta(\cap\{A_i : i \in I\}) \subset \cap\{\mathrm{sCl}_\delta(A_i) : i \in I\}$.

(4) $\mathrm{sCl}_\delta(\cup\{A_i : i \in I\}) = \cup\{\mathrm{sCl}_\delta(A_i) : i \in I\}$.

(5) A is δ-semiclosed if and only $A = \mathrm{sCl}_\delta(A)$.

4 Lemma (Park et al. [11])**.** *For a subset A of a space (X, τ), the following hold:*

(1) A is a δ-semiopen set if and only if $A = \mathrm{sInt}_\delta(A)$.

(2) $X - \mathrm{sInt}_\delta(A) = \mathrm{sCl}_\delta(X - A)$ *and* $\mathrm{sInt}_\delta(X - A) = X - \mathrm{sCl}_\delta(A)$.

(3) $\mathrm{sInt}_\delta(A))$ *is a δ-semiopen set.*

5 Definition. A function $f : X \to Y$ is called:

1) Weakly continuous [6] (resp. almost weakly-δs-continuous) if for each $x \in X$ and each open set G containing $f(x)$, there exists $U \in O(X, x)$ (resp. $U \in \delta SO(X, x)$) such that $f(U) \subset \mathrm{Cl}(G)$.

2) Rarely continuous [12] if for each $x \in X$ and each $G \in O(Y, f(x))$, there exist a rare set R_G with $G \cap \mathrm{Cl}(R_G) = \emptyset$ and $U \in O(X, x)$ such that $f(U) \subset G \cup R_G$.

3) super-continuous [10] if the inverse image of every open set in Y is δ-open in X.

3 Rare δs-continuity

6 Definition. A function $f : X \to Y$ is called rarely δs-continuous if for each $x \in X$ and each $G \in O(Y, f(x))$, there exist a rare set R_G with $G \cap \mathrm{Cl}(R_G) = \emptyset$ and $U \in \delta SO(X, x)$ such that $f(U) \subset G \cup R_G$.

7 Example. Let $X = Y = \{a, b, c\}$ and $\tau = \sigma = \{X, \emptyset, \{a\}\}$. Then the identity function $f : (X, \tau) \to (Y, \sigma)$ is rare δs-continuous.

Question 1 Is there any nontrivial example of a rarely δs-continuous function?

8 Theorem. *The following statements are equivalent for a function $f : X \to Y$:*

(1) f is rarely δs-continuous at $x \in X$.

(2) For each set $G \in O(Y, f(x))$, there exists $U \in \delta SO(X, x)$ such that $\mathrm{Int}[f(U) \cap (Y \setminus G)] = \emptyset$.

(3) For each set $G \in O(Y, f(x))$, there exists $U \in \delta SO(X, x)$ such that $\mathrm{Int}[f(U)] \subset \mathrm{Cl}(G)$.

(4) For each $G \in O(Y, f(x))$, there exists a rare set R_G with $G \cap \mathrm{Cl}(R_G) = \emptyset$ such that $x \in \mathrm{sInt}_\delta(f^{-1}(G \cup R_G))$.

(5) For each $G \in O(Y, f(x))$, there exists a rare set R_G with $\mathrm{Cl}(G) \cap R_G = \emptyset$ such that $x \in \mathrm{sInt}_\delta(f^{-1}(\mathrm{Cl}(G) \cup R_G))$.

(6) For each $G \in RO(Y, f(x))$, there exists a rare set R_G with $G \cap \mathrm{Cl}(R_G) = \emptyset$ such that $x \in \mathrm{sInt}_\delta(f^{-1}(G \cup R_G))$.

Proof. (1) $\to$ (2) : Let $G \in O(Y, f(x))$. By $f(x) \in G \subset \mathrm{Int}(\mathrm{Cl}(G))$ and the fact that $\mathrm{Int}(\mathrm{Cl}(G)) \in O(Y, f(x))$, there exist a rare set R_G with $\mathrm{Int}(\mathrm{Cl}(G)) \cap \mathrm{Cl}(R_G) = \emptyset$ and a δ-semiopen set $U \subset X$ containing x such that $f(U) \subset \mathrm{Int}(\mathrm{Cl}(G)) \cup R_G$. We have $\mathrm{Int}[f(U) \cap (Y - G)] = \mathrm{Int}[f(U)] \cap \mathrm{Int}(Y - G) \subset \mathrm{Int}[\mathrm{Cl}(G) \cup R_G] \cap (Y - \mathrm{Cl}(G)) \subset (\mathrm{Cl}(G) \cup \mathrm{Int}(R_G)) \cap (Y - \mathrm{Cl}(G)) = \emptyset$.
(2) $\to$ (3) : It is straightforward.
(3) $\to$ (1) : Let $G \in O(Y, f(x))$. Then by (3), there exists $U \in \delta SO(X, x)$ such that $\mathrm{Int}[f(U)] \subset \mathrm{Cl}(G)$. We have $f(U) = [f(U) - \mathrm{Int}(f(U))] \cup \mathrm{Int}(f(U)) \subset [f(U) - \mathrm{Int}(f(U))] \cup \mathrm{Cl}(G) = [f(U) - \mathrm{Int}(f(U))] \cup G \cup (\mathrm{Cl}(G) - G) = [(f(U) - \mathrm{Int}(f(U))) \cap (Y - G)] \cup G \cup (\mathrm{Cl}(G) - G)$.
Set $R^* = [f(U) - \mathrm{Int}(f(U))] \cap (Y - G)$ and $R^{**} = (\mathrm{Cl}(G) - G)$. Then R^* and R^{**} are rare sets. More $R_G = R^* \cup R^{**}$ is a rare set such that $\mathrm{Cl}(R_G) \cap G = \emptyset$ and $f(U) \subset G \cup R_G$. This shows that f is rarely-δs-continuous.
1) $\to$ 4) : Suppose that $G \in O(Y, f(x))$. Then there exists a rare set R_G with

$G \cap \mathrm{Cl}(R_G) = \emptyset$ and $U \in \delta SO(X, x)$ such that $f(U) \subset G \cup R_G$. It follows that $x \in U \subset f^{-1}(G \cup R_G)$. This implies that $x \in \mathrm{sInt}_\delta(f^{-1}(G \cup R_G))$.
4) $\to$ 5): Suppose that $G \in O(Y, f(x))$. Then there exists a rare set R_G with $G \cap \mathrm{Cl}(R_G) = \emptyset$ such that $x \in \mathrm{sInt}_\delta(f^{-1}(G \cup R_G))$. Since $G \cap \mathrm{Cl}(R_G) = \emptyset$, $R_G \subset Y - G$, where $Y - G = (Y - \mathrm{Cl}(G)) \cup (\mathrm{Cl}(G) - G)$. Now, we have $R_G \subset (R_G \cup (Y - \mathrm{Cl}(G)) \cup (\mathrm{Cl}(G) - G))$. Set $R^* = R_G \cap (Y - \mathrm{Cl}(G))$. It follows that R^* is a rare set with $\mathrm{Cl}(G) \cap R^* = \emptyset$. Therefore $x \in \mathrm{sInt}_\delta[f^{-1}(G \cup R_G)] \subset \mathrm{sInt}_\delta[f^{-1}(\mathrm{Cl}(G) \cup R^*)]$.
5) $\to$ 6) : Assume that $G \in RO(Y, f(x))$. Then there exists a rare set R_G with $\mathrm{Cl}(G) \cap R_G = \emptyset$ such that $x \in \mathrm{sInt}_\delta[f^{-1}(\mathrm{Cl}(G) \cup R_G)]$. Set $R^* = R_G \cup (\mathrm{Cl}(G) - G)$. It follows that R^* is a rare set and $G \cap \mathrm{Cl}(R^*) = \emptyset$. Hence
$x \in \mathrm{sInt}_\delta[f^{-1}(\mathrm{Cl}(G) \cup R_G)] = \mathrm{sInt}_\delta[f^{-1}(G \cup (\mathrm{Cl}(G) - G) \cup R_G)] = \mathrm{sInt}_\delta[f^{-1}(G \cup R^*)]$.
6) $\to$ 2): Let $G \in O(Y, f(x))$. By $f(x) \in G \subset \mathrm{Int}(\mathrm{Cl}(G))$ and the fact that $\mathrm{Int}(\mathrm{Cl}(G)) \in RO(Y)$, there exist a rare set R_G and $\mathrm{Int}(\mathrm{Cl}(G)) \cap \mathrm{Cl}(R_G) = \emptyset$ such that $x \in \mathrm{sInt}_\delta[f^{-1}(\mathrm{Int}(\mathrm{Cl}(G)) \cup R_G)]$. Let $U = \mathrm{sInt}_\delta[f^{-1}(\mathrm{Int}(\mathrm{Cl}(G)) \cup R_G)]$. Hence, $U \in \delta SO(X, x)$ and, therefore $f(U) \subset \mathrm{Int}(\mathrm{Cl}(G)) \cup R_G$. Hence, we have $\mathrm{Int}[f(U) \cap (Y - G)] = \emptyset$. $\boxed{QED}$

9 Theorem. *A function $f : X \to Y$ is rarely δs-continuous if and only if for each open set $G \subset Y$, there exists a rare set R_G with $G \cap \mathrm{Cl}(R_G) = \emptyset$ such that $f^{-1}(G) \subset \mathrm{sInt}_\delta[f^{-1}(G \cup R_G)]$.*

Proof. It follows from Theorem 8. $\boxed{QED}$

It is shown in [10] that a function $f : X \to Y$ is super-continuous if and only if for each $x \in X$ and each $G \in O(Y, f(x))$, there exists $U \in \delta O(X, x)$ such that $f(U) \subset G$.

We define the following notion which is a new generalization of super-continuity.

10 Definition. A function $f : X \to Y$ is $I.\delta s$-continuous at $x \in X$ if for each set $G \in O(Y, f(x))$, there exists $U \in \delta SO(X, x)$ such that $\mathrm{Int}[f(U)] \subset G$. If f has this property at each point $x \in X$, then we say that f is $I.\delta s$-continuous on X.

11 Remark. It should be noted that super-continuity implies $I.\delta s$-continuity and $I.\delta s$-continuity implies rare δs-continuity. But the converses are not true as shown by the following examples.

12 Example. Let $X = Y = \{a, b, c\}$ and $\tau = \sigma = \{X, \emptyset, \{a\}\}$. Then a function $f : (X, \tau) \to (Y, \sigma)$ defined by $f(a) = f(b) = a$ and $f(c) = c$, is $I.\delta s$-continuous. Since f is not continuous, then it is not super continuous.

13 Example. Let (X, τ) and (Y, σ) be the same spaces as in the above Example. Then the identity function $f : (X, \tau) \to (Y, \sigma)$ is rare δs-continuous but it is not $I.\delta s$-continuous.

14 Theorem. *Let Y be a regular space. Then a function $f : X \to Y$ is $I.\delta s$-continuous on X if and only if f is rarely δs-continuous on X.*

Proof. We prove only the sufficient condition since the necessity condition is evident.

Let f be rarely δs-continuous on X and $x \in X$. Suppose that $f(x) \in G$, where G is an open set in Y. By the regularity of Y, there exists an open set $G_1 \in O(Y, f(x))$ such that $\mathrm{Cl}(G_1) \subset G$. Since f is rarely δs-continuous, then there exists $U \in \delta SO(X, x)$ such that $\mathrm{Int}[f(U)] \subset \mathrm{Cl}(G_1)$ (Theorem 8). This implies that $\mathrm{Int}[f(U)] \subset G$ and therefore f is $I.\delta s$-continuous on X. QED

We say that a function $f : X \to Y$ is δs-semiopen if the image of a δ-semiopen set is semiopen.

15 Theorem. *If $f : X \to Y$ be a δs-semiopen rarely δs-continuous function, then f is almost weakly δs-continuous.*

Proof. Suppose that $x \in X$ and $G \in O(Y, f(x))$. Since f is rarely δs-continuous, there exists $U \in \delta SO(X, x)$ such that $\mathrm{Int}(f(U)) \subset \mathrm{Cl}(G)$. Since f is δs-semiopen, $f(U)$ is semiopen and hence $f(U) \subset \mathrm{Cl}(\mathrm{Int}(f(U))) \subset \mathrm{Cl}(G)$. This shows that f is weakly δs-continuous. QED

16 Theorem. *Let X be a semi-regular space. If $f : X \to Y$ is rarely δs-continuous function, then the graph function $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$ for every x in X, is rarely δs-continuous.*

Proof. Suppose that $x \in X$ and W is any open set containing $g(x)$. It follows that there exist open sets U and V in X and Y, respectively, such that $(x, f(x)) \in U \times V \subset W$. Since f is rarely δs-continuous, there exists $G \in \delta SO(X, x)$ such that $\mathrm{Int}[f(G)] \subset \mathrm{Cl}(V)$. Let $E = U \cap G$. Since X is semi-regular, U is δ-open in X and it follows from Lemma 2.4 of [5] that $E \in \delta SO(X, x)$ and we have $\mathrm{Int}[g(E)] \subset \mathrm{Int}(U \times f(G)) \subset U \times \mathrm{Cl}(V) \subset \mathrm{Cl}(W)$. Therefore, g is rarely δs-continuous. QED

17 Definition. Let $A = \{G_i\}$ be a class of subsets of X. By rarely union sets [3] of A we mean $\{G_i \cup R_{G_i}\}$, where each R_{G_i} is a rare set such that each of $\{G_i \cap \mathrm{Cl}(R_{G_i})\}$ is empty.

Recall that a subset B of X is said to be rarely almost compact relative to X [3] if every cover of B by open sets of X, there exists a finite subfamily whose rarely union sets cover B. A topological space X is said to be rarely almost compact if the set X is rarely almost compact relative to X.
A subset K of a space X is said to be δSO-compact relative to X if every cover

of K by δ-semiopen sets in X has a finite subcover. A space X is said to be δSO-compact if X is δSO-compact relative to X.

18 Theorem. *Let $f : X \to Y$ be rarely δs-continuous and K a δSO-compact relative to X. Then $f(K)$ is rarely almost compact relative to Y.*

Proof. Suppose that Ω is an open cover of $f(K)$. Let B be the set of all V in Ω such that $V \cap f(K) \neq \emptyset$. Then B is an open cover of $f(K)$. Hence for each $k \in K$, there is some $V_k \in B$ such that $f(k) \in V_k$. Since f is rarely δs-continuous, there exist a rare set R_{V_k} with $V_k \cap \mathrm{Cl}(R_{V_k}) = \emptyset$ and a δ-semiopen set U_k containing k such that $f(U_k) \subset V_k \cup R_{V_k}$. Hence there is a finite subfamily $\{U_k\}_{k \in \triangle}$ which covers K, where $\triangle$ is a finite subset of K. The subfamily $\{V_k \cup R_{V_k}\}_{k \in \triangle}$ also covers $f(K)$. QED

19 Theorem. *Let $f : X \to Y$ be rarely continuous and X be a semi-regular space. Then f is rarely δs-continuous.*

Proof. Suppose that $x \in X$ and $G \in O(Y, f(x))$. Since f is rarely continuous, by Theorem 1 of [8] exists $U \in O(X, x)$ such that $\mathrm{Int}(f(U)) \subset \mathrm{Cl}(G)$. Since X is semi-regular, U is δ-open and hence $U \in \delta SO(X, x)$. It follows from Theorem 8 that f is rarely δs-continuous. QED

20 Lemma. *(Long and Herrington [8]). If $g : Y \to Z$ is continuous and one-to-one, then g preserves rare sets.*

21 Theorem. *If $f : X \to Y$ is rarely δs-continuous and $g : Y \to Z$ is a continuous injection, then $g \circ f : X \to Z$ is rarely δs-continuous.*

Proof. Suppose that $x \in X$ and $(g \circ f)(x) \in V$, where V is an open set in Z. By hypothesis, g is continuous, therefore $G = g^{-1}(V)$ is an open set Y containing $f(x)$ such that $g(G) \subset V$. Since f is rarely δs-continuous, there exists a rare set R_G with $G \cap \mathrm{Cl}(R_G) = \emptyset$ and a δ-semiopen set U containing x such that $f(U) \subset G \cup R_G$. It follows from Lemma 20 that $g(R_G)$ is a rare set in Z. Since R_G is a subset of $Y \setminus G$ and g is injective, we have $\mathrm{Cl}(g(R_G)) \cap V = \emptyset$. This implies that $(g \circ f)(U) \subset V \cup g(R_G)$. Hence we obtain the result. QED

A function $f : X \to Y$ is called pre-δs-open if $f(U)$ is δ-semiopen in Y for every δ-semiopen set U of X.

22 Theorem. *Let $f : X \to Y$ be a pre-δs-open surjection and $g : Y \to Z$ a function such that $g \circ f : X \to Z$ is rarely δs-continuous. Then g is rarely δs-continuous.*

Proof. Let $y \in Y$ and $x \in X$ such that $f(x) = y$. Let $G \in O(Z, (g \circ f)(x))$. Since $g \circ f$ is rarely δs-continuous, there exists a rare set R_G with $G \cap \mathrm{Cl}(R_G) = \emptyset$ and $U \in \delta SO(X, x)$ such that $(g \circ f)(U) \subset G \cup R_G$. But $f(U)$ (say V) is a δ-semiopen set containing $f(x)$. Therefore, there exists a rare set R_G with

$G \cap \mathrm{Cl}(R_G) = \emptyset$ and $V \in \delta SO(Y, y)$ such that $g(V) \subset G \cup R_G$, i.e., g is rarely δs-continuous. $\boxed{QED}$

23 Definition. A space X is called
(1) r-separate [4] if for every pair of distinct points x and y in X, there exist open sets U_x and U_y containing x and y, respectively, and rare sets R_{U_x}, R_{U_y} with $U_x \cap \mathrm{Cl}(R_{U_x}) = \emptyset$ and $U_y \cap \mathrm{Cl}(R_{U_y}) = \emptyset$ such that $(U_x \cup R_{U_x}) \cap (U_y \cup R_{U_y}) = \emptyset$,
(2) semi-Hausdorff [9] if for any distinct pair of points x and y in X, there exist semiopen sets U and V in X containing x and y, respectively, such that $U \cap V = \emptyset$.

24 Theorem. *If Y is r-separate and $f : X \to Y$ is a rarely δs-continuous injection, then X is semi-Hausdorff.*

Proof. Since f is injective, then $f(x) \neq f(y)$ for any distinct points x and y in X. Since Y is r-separate, There exist open sets G_1 and G_2 in Y containing $f(x)$ and $f(y)$, respectively, and rare sets $R_{G_1}$and R_{G_2} with $G_1 \cap \mathrm{Cl}(R_{G_1}) = \emptyset$ and $G_2 \cap \mathrm{Cl}(R_{G_2}) = \emptyset$ such that $(G_1 \cup R_{G_1}) \cap (G_2 \cup R_{G_2}) = \emptyset$. Therefore $\mathrm{sInt}_\delta[f^{-1}(G_1 \cup R_{G_1})] \cap \mathrm{sInt}_\delta[f^{-1}(G_2 \cup R_{G_2})] = \emptyset$. By Theorem 9, we have $x \in f^{-1}(G_1) \subset \mathrm{sInt}_\delta[f^{-1}(G_1 \cup R_{G_1})]$ and $y \in f^{-1}(G_2) \subset \mathrm{sInt}_\delta[f^{-1}(G_2 \cup R_{G_2})]$. Since $\mathrm{sInt}_\delta[f^{-1}(G_1 \cup R_{G_1})]$ and $\mathrm{sInt}_\delta[f^{-1}(G_2 \cup R_{G_2})]$ are δ-semiopen and as every δ-semiopen subset is semiopen , then X is a semi-Hausdorff space. $\boxed{QED}$

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