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Non-existence of non-trivial generic warped product in Kaehler manifolds

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Abstract. Warped product manifolds provide excellent setting to model space time near black holes or bodies with large gravitational force [7]. Recently there have been studies to explore the existence, of warped products in certain settings [5], [9]. To continue the sequel, non existence of generic warped product submanifolds in a Kaehler manifold is established extending the results of B. Y. Chen [5] and B. Sahin [9].

Keywords: Generic Submanifold, purely real, totally real and holomorphic distributions, slant submanifold, warped product submanifold.

MSC 2000 classification: 53C40, 53C42, 53C15

1 Introduction

The study of geometry of warped product manifolds was introduced by R. L. Bishop and B. O'Neill [2]. These manifolds emerge in a natural manner. For instance the best relativistic model of the Schwarzschild space time that describes the outer space around a massive star or a black hole is a warped product manifold. Bishop and O'Neill obtained various fundamentally important results for warped product manifolds. Recently the study got impetus with B. Y. Chen's work on warped product CR-submanifolds of a Kaehler manifold (cf. [5], [6], [7]). He studied CR-Submanifolds of a Kaehler manifold which are warped products of a holomorphic and totally real submanifolds N_T and $N^{\perp}$ respectively. He proved that a warped product submanifold $N_{\perp} \times_f N_T$ in a Kaehler manifolds is simply a CR-product whereas the warped product $N_T \times_f N_{\perp}$ is nontrivial. B. Sahin [9] extended the study by proving that the semi-slant warped product submanifolds $N_T \times_f N_\theta$ and $N_\theta \times_f N_T$ in Kaehler manifolds are trivial in the sense that they are simply Riemannian product of N_T and N_θ where N_θ denotes a proper slant submanifold of the underlying Kaehler manifold. Our aim in the present note is to extend the study by considering the warped products $N_T \times_f N^0$ and $N^0 \times_f N_T$ of a Kaehler manifold $\overline{M}$ where N^0 is an arbitrary submanifold of $\overline{M}$.

2 Preliminaries

Let $\overline{M}$ be a Kaehler manifold with an almost complex structure J and Levi-Civita connection $\overline{\nabla}$. If M is a submanifold of $\overline{M}$, then the Gauss and Weingarten formulae are given respectively by

$$\overline{\nabla}_U V = \nabla_U V + h(U,V), \tag{1}$$

$$\overline{\nabla}_U \xi = -A_\xi U + \nabla_U^\perp \xi, \tag{2}$$

for any vector fields U, V tangent to M and ξ normal to M, where ∇ and $\nabla^\perp$ denote the induced connections on the tangent bundle TM and the normal bundle $T^\perp M$ respectively. h is the second fundamental form and A is the shape operator of the immersion of M into $\overline{M}$. The two are related by

$$g(A_\xi U, V) = g(h(U,V), \xi), \tag{3}$$

where g denotes the metric on $\overline{M}$ as well as the one induced on M.

For any vector field U tangent to M, we put

$$JU = PU + FU, \tag{4}$$

where PU and FU are respectively the tangential and normal components of JU.

The covariant differentiation of the tensor P is defined by

$$(\overline{\nabla}_U P)V = \nabla_U PV - P \nabla_U V. \tag{5}$$

As $\overline{M}$ is Kaehler, by using (1), (2), (4) and (5), we obtain

$$(\overline{\nabla}_U P)V = A_{FV} U + th(U,V). \tag{6}$$

1 Definition. Let M be a submanifold of a Kaehler manifold $\overline{M}$ and for $x \in M$, $D_x = T_x(M) \cap JT_x(M)$ be the maximal complex subspace of the tangent space $T_x(M)$. If $D : x \longrightarrow D_x$ defines a C^∞-distribution on M, known as holomorphic distribution, then M is called a *generic submanifold* (cf. [4]).

For a generic submanifold M of a Kaehler manifold, the tangent bundle TM, can be decomposed as

$$TM = D \oplus D^0,$$

where D^0 denotes the orthogonal complementary distribution of D and is known as *purely real distribution.* A generic submanifold is a holomorphic submanifold if $D^0 = \{0\}$ and is called a *purely real submanifold* if $D = \{0\}$. Further, if D^0 is totally real, the generic submanifold is a CR-submanifold. Thus, a generic submanifold provides a generalization of holomorphic, totally real, purely real and a CR-submanifold. It can also be observed that a purely real distribution D^0 on a submanifold M, is a slant distribution if the angle $\theta(Z) \in [0, \pi/2]$ between D^0_x and JZ is constant for each $Z \in D^0_x$ and $x \in M$. A purely distribution D^0 on M is called a *proper purely real distribution* if $\theta(Z) \neq \pi/2$ for any $Z \in D^0$. On a generic submanifold of a Kaehler manifold,

$$a)\ PD = D, \qquad b)\ PD^0 \subset D^0, \qquad c)\ FD = \{0\}.$$

2 Theorem. *[4] Let M be a generic submanifold of a Kaehler manifold $\overline{M}$. Then the holomorphic distribution D on M is integrable if and only if*

$$g(h(JX, Y), FZ) = g(h(X, JY), FZ),$$

for each $X, Y \in D$ and $Z \in D^0$.

3 Definition. Let B and F be two Riemannian manifolds with Riemannian metric g_B and g_F respectively and $f > 0$ a smooth function on B. Consider the product manifold $B \times F$ with its projections $\pi : B \times F \longrightarrow B$ and $\eta : B \times F \longrightarrow F$. The *warped product* $B \times_f F$ is the manifold $B \times F$ equipped with the Riemannian metric such that

$$||U||^2 = ||d\pi U||^2 + f^2(\pi(x)||d\eta U||^2,$$

for any tangent vector U on $B \times F$. In other words, the Riemannian metric g on a warped product manifold $B \times_f F$ is given by

$$g = g_B + f^2 g_F.$$

The function f is called the *warping function* of the warped product. For a warped product $N \times_f N^0$, we may consider D and D^0 the distributions determined by the vectors tangent to the leaves and fibres respectively. That is, D is obtained from tangent vectors of N via the horizontal lift and D^0 is obtained by tangent vectors of N^0 via the vertical lift.

A warped product $N \times_f N^0$ is said to be trivial if its warping function f is constant. A trivial generic warped product $N_T \times_f N^0$ is nothing but a generic product $N_T \times N_f^0$, where N_f^0 is the manifold with metric $f^2 g_{N^0}$ which is homothetic to the original metric g_{N^0} on N^0.

R. L. Bishop and B. O'Neill [2] obtained the following basic results for warped product manifolds.

4 Theorem. *[2] Let $M = B \times_f F$ be a warped product manifold. Then for any $X, Y \in D$ and $V, W \in D^0$,*

(i) $\nabla_X Y \in D$,

(ii) $\nabla_X V = \nabla_V X = (X \ln f)V$,

(iii) $\nabla_V W = -\frac{g(V,W)}{f} \nabla f$.

∇f *is the gradient of* f *and is defined as*

$$g(\nabla f, U) = Uf.$$

3 Warped product generic submanifolds in a Kaehler manifold

In this section, we study generic submanifolds of a Kaehler manifold $\overline{M}$ which are warped products of the form $N^0 \times_f N_T$ and $N_T \times_f N^0$ where N_T is a holomorphic submanifold and N^0 is a proper purely real submanifold of $\overline{M}$.

5 Theorem. *A Kaehler manifold does not admit non-trivial warped products with one of the factors a holomorphic submanifold.*

Proof. Let N_T be a holomorphic submanifold of a Kaehler manifold $\overline{M}$ and N^0 an arbitrary submanifold.

Consider the warped product $M = N^0 \times_f N_T$ in a Kaehler manifold $\overline{M}$. Then by Theorem 4

$$\nabla_X Z = \nabla_Z X = (Z \ln f)X, \tag{7}$$

for each $X \in TN_T$ and $Z \in TN^0$. Thus

$$g(X, \nabla_{JX} Z) = 0.$$

Making use of (1), (2), (4) and the fact that $\overline{M}$ is Kaehler, we deduce from the above equation that

$$0 = g(JX, \overline{\nabla}_{JX} JZ) = g(JX, \nabla_{JX} PZ) - g(h(JX, JX), FZ),$$

which on applying formula (7) yields that

$$g(h(JX, JX), FZ) = (PZ \ln f)||X||^2. \tag{8}$$

Now, by (5), (6) and (7), we obtain

$$(PZ \ln f)X - (Z \ln f)PX = A_{FZ}X + th(X, Z).$$

On taking inner product with $Y \in TN_T$, the above equation gives

$$(Z \ln f)g(X, Y) - (PZ \ln f)g(PX, Y) = g(h(JX, Y), FZ). \tag{9}$$

Interchanging X and Y in the above equation and adding the resulting equation in (9) while taking account of Theorem 2, we obtain

$$(Z \ln f)g(X, Y) = g(h(JX, Y), FZ). \tag{10}$$

In particular, we have

$$g(h(JX, JX), FZ) = 0. \tag{11}$$

Now, by (8) and (11), it follows that

$$PZ \ln f = 0,$$

for each $Z \in TN^0$. This shows that f is constant and thus M is a Riemannian product of N^0 and N_T.

Let now M be the warped product submanifold $N_T \times_f N^0$ of $\overline{M}$. Then for any $X \in TN_T$ and $Z \in TN^0$, by Theorem 4

$$\triangledown_X Z = \triangledown_Z X = (X \ln f)Z, \tag{12}$$

and therefore

$$(\triangledown_X P)Z = 0,$$
$$(\triangledown_Z P)X = (PX \ln f)Z - (X \ln f)PZ.$$

The above equations, in view of (6) yield

$$A_{FZ}X + th(X, Z) = 0, \tag{13}$$

and,

$$(PX \ln f)Z - (X \ln f)PZ = th(X, Z). \tag{14}$$

Thus, we have

$$A_{FZ}X = (X \ln f)PZ - (PX \ln f)Z. \tag{15}$$

Taking inner product with PZ in (15) and making use of (13), we obtain

$$g(h(X, PZ), FZ) = g(h(X, Z), FPZ) = (X \ln f)||PZ||^2. \tag{16}$$

On the other hand as $\overline{M}$ is Kaehler and formula (13) holds we have

$$g(\overline{\nabla}_{PZ}JZ, JX) = 0.$$

Which on using (1), (2) and (4) yields

$$g(\nabla_{PZ}PZ, JX) = g(A_{FZ}PZ, JX).$$

The above equation on taking account of (5) gives

$$g(h(X, PZ), FZ) = -(X \ln f)||PZ||^2. \tag{17}$$

By (16), (17) and the assumption that N^0 is a proper purely real submanifold, we get

$$X \ln f = 0.$$

This proves that the product $N_T \times_f N^0$ is trivial. $\boxed{QED}$

References

[1] K. Abe, M. Magid: *Relative nullity foliation and indefinite isometric immersions*, Pacific Journ. of Math., **124** no. 1, (1986), 1–20.

[2] R. L. Bishop, B. O'Neill: *Manifolds of negative curvature*, Trans. Amer. Math. Soc., **145** (1969), 1–49.

[3] B. Y. Chen: *CR-Submanifolds of a Kaehler manifold*, J. Diff. Geom., **16** (1981), 305–322.

[4] B. Y. Chen: *Differential geometry of real submanifolds in a Kaehler manifold*, Monatsh. Math., **91** (1981), 252–275.

[5] B. Y. Chen: *Geometry of warped product CR-Submanifolds in a Kaehler manifold*, Monatsh. Math., **133** (2001), 177–195.

[6] B. Y. Chen: *Geometry of warped product CR-Submanifolds in a Kaehler manifold II*, Monatsh. Math., **134** (2001), 103–119.

[7] B. Y. Chen: *Geometry of warped products as Riemannian submanifolds and related problems*, Soochow J. Math., **28** no. 2 (2002), 125–156.

[8] N. Papaghiue: *Semi-slant submanifolds of a Kaehlerian manifold*, Ann. St. Univ. Iasi, **tom XL, S.I. 9(f_1)** (1994), 55–61.

[9] B. Sahin: *Non-existence of warped product semi-slant submanifolds of Kaehler manifolds*, Geom. Dedicata, **117** no. 1 (2006), 195–202.