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Upper semicontinuity of the spectrum function and automatic continuity in topological Q-algebras

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Abstract. In 1993, M. Fragoulopoulou applied the technique of Ransford and proved that if E and F are lmc algebras such that E is a Q-algebra, F is semisimple and advertibly complete, and (E, F) is a closed graph pair, then each surjective homomorphism $\varphi : E \longrightarrow F$ is continuous. Later on in 1996, it was shown by Akkar and Nacir that if E and F are both LFQ-algebras and F is semisimple then evey surjective homomorphism $\varphi : E \longrightarrow F$ is continuous. In this work we extend the above results by removing the lmc property from E.

We first show that in a topological algebra, the upper semicontinuity of the spectrum function, the upper semicontinuity of the spectral radius function, the continuity of the spectral radius function at zero, and being a Q-algebra, are all equivalent. Then it is shown that if A is a topological Q-algebra and B is an lmc semisimple algebra which is advertibly complete, then every surjective homomorphism $T : A \longrightarrow B$ has a closed graph. In particular, if A is a Q-algebra with a complete metrizable topology, and B is a semisimple Fréchet algebra, then every surjective homomorphism $T : A \longrightarrow B$ is automatically continuous.

Keywords: automatic continuity, topological algebra, Fréchet algebra, Q-algebra, spectrum function, spectral radius, upper semicontinuity, advertibly complete

MSC 2000 classification: 46H40, 46H05

1 Introduction

In 1989, Ransford presented a short proof of Johnson's uniqueness of norm theorem [7]. In 1993, M. Fragoulopoulou applied the technique of Ransford and extended the Johnson's theorem to diverse classes of semisimple lmc algebras,

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including the LFQ-algebras and Fréchet Q-algebras [4]. In 1996, it was shown by M. Akkar and C. Nacir that if E and F are both LFQ-algebras and F is semisimple then every surjective homomorphism $\varphi : E \longrightarrow F$ is continuous [1, Theorem 4]. They have also presented a proof for the upper semicontinuity of the spectral radius function $x \mapsto \rho_A(x)$, when A is a topological Q-algebra. However, as it has been mentioned in [1], this result is originally due to J. Esterle [3]. In both articles [4] and [1], authors assume that E is an lmc algebra, which is also a Q-algebra with some extra conditions.

In this paper we extend the above results by removing the lmc property from E.

We now present some definitions and known results. For further details one can refer, for example, to [2] or [5].

1 Definition. A locally multiplicatively convex (lmc) algebra is a topological algebra whose topology is defined by a separating family $\mathcal{P} = (p_\alpha)$ of submultiplicative seminorms. A complete metrizable lmc algebra is a Fréchet algebra.

The topology of a Fréchet algebra can be defined by an increasing sequence (p_n) of submultiplicative seminorms. We may assume, without loss of generality, that $p_n(e) = 1$ if A is unital with the unit e [5].

2 Definition. An F-algebra is a topological algebra whose underlying topological linear space is an F-space, or in other words, the topology of an F-algebra is defined by a complete invariant metric.

Note that a Fréchet algebra is an F-algebra which is also an lmc algebra.

3 Definition. A topological algebra A is a Q-algebra if the set of all quasi invertible elements of A ($q - \operatorname{Inv} A$) is open in A.

If A is unital then it is easy to see that A is a Q-algebra if and only if $\operatorname{Inv} A$, the set of all invertible elements of A, is open.

4 Definition. A topological algebra A is advertibly complete if a Cauchy net (x_α) in A converges in A whenever for some $y \in A$, $x_\alpha + y - x_\alpha.y$ converges to zero.

Note that a topological Q-algebra is advertibly complete [6, p. 45].

For a unital topological algebra A let $sp_A(x)$ denote the spectrum of $x \in A$ and $\rho_A(x)$ denote the spectral radius of $x \in A$. We take $\rho_A(x) = +\infty$ if $sp_A(x)$ is unbounded and $\rho_A(x) = 0$ if $sp_A(x) = \emptyset$.

5 Remark. Let A be an lmc algebra with the family of seminorms $\mathcal{P} = (p_\alpha)$. Let A_α denote the Banach algebra obtained by the completion of $A/\ker p_\alpha$ in the norm $p'_\alpha(x + \ker p_\alpha) = p_\alpha(x)$. Since $sp_{A_\alpha}(x + \ker p_\alpha) \subseteq sp_A(x)$ and $sp_{A_\alpha}(x+\ker p_\alpha) \neq \emptyset$, we have $sp_A(x) \neq \emptyset$. If, moreover, A is advertibly complete, then

$$\rho_A(x) = \sup_\alpha(\lim_{n\to\infty}(p_\alpha(x^n)^{\frac{1}{n}}) = \sup_\alpha \rho_{A_\alpha}(x + \ker p_\alpha)$$

[6, Theorem III, 6.1].

6 Definition. Let A be a topological algebra. The spectrum function $x \mapsto sp_A(x)$ is upper semicontinuous at $a \in A$ if, for every open set U containing $sp_A(a)$, there exists a neighbourhood V of a such that $sp_A(x) \subseteq U$ whenever $x \in V$.

We need the following known results in the next section. See, for example, [2, 1.5.32 and 5.1.8], or [7].

7 Lemma. *If A is a unital algebra then*

$$\operatorname{rad} A = \{x \in A : \forall y \in A, \rho_A(xy) = 0\},$$

where $\operatorname{rad} A$ *is the Jacobson radical of A.*

8 Lemma. *Let A be a Banach algebra and let $p(z)$, for $z \in \mathbb{C}$, be a polynomial with coefficients in A. Then for each $R > 0$ we have*

$$\rho_A^2(p(1)) \leq \sup_{|z|=R} \rho_A(p(z)). \sup_{|z|=\frac{1}{R}} \rho_A(p(z))$$

.

2 Main Results

In this section we assume that all algebras are unital with the unit e.

9 Theorem. *In a topological algebra A the following properties are equivalent:*

i) A is a Q-algebra.

ii) The spectrum function $x \mapsto sp_A(x)$ is upper semicontinuous on A.

iii) The spectral radius function $x \mapsto \rho_A(x)$ is upper semicontinuous on A.

iv) The spectral radius function $x \mapsto \rho_A(x)$ is continuous at zero.

Proof. i) $\Longrightarrow$ ii) Let V be a symmetric neighbourhood of zero in A such that $e + V \subseteq \operatorname{Inv} A$. Since scalar multiplication is continuous, for each $a \in A$ there exist a neighbourhood V_1 of zero in A and $\epsilon > 0$ such that $\lambda x \in V$ whenever $x \in a + V_1$ and $|\lambda| < \epsilon$. Hence $e - \frac{x}{\lambda} \in \operatorname{Inv} A$, whenever $|\lambda| > R = \frac{1}{\epsilon}$ and $x \in a + V_1$. This shows that $sp_A(x) \subseteq B(0, R)$ for all $x \in a + V_1$, where $B(0, R)$ is the closed disk with the center zero and radius R.

Suppose on the contrary that the spectrum function is not upper semicontinuous at a. Then there exist a neighbourhood U of $sp_A(a)$ and a net $(x_\alpha)_{\alpha\in\Lambda}$ converging to a such that for each $\alpha \in \Lambda$ there is $\lambda_\alpha \in sp_A(x_\alpha)\backslash U$. By the first part of the proof $sp_A(x_\alpha) \subseteq B(0,R)$ whenever $\alpha > \mu$, for some $\mu \in \Lambda$. Let $D = \{\lambda_\alpha : \alpha > \mu\}$. Then $\overline{D} \subseteq B(0,R) \setminus U$. For each $\lambda \in \overline{D}$ we have $\lambda.e - a \in \operatorname{Inv} A$. Since $\operatorname{Inv} A$ is open and the vector space operations are continuous, for each $\lambda \in \overline{D}$ there are neighbourhoods E and F of λ and a, respectively, such that $z.e - x \in \operatorname{Inv} A$ whenever $z \in E$ and $x \in F$. By the compactness of $\overline{D}$ there is a finite number of neighbourhoods $E_1, E_2, \ldots, E_m$ such that $\overline{D} \subseteq \bigcup_{i=1}^m E_i$. Take $F_1, F_2, \ldots, F_m$ to be the corresponding neighbourhoods of a and $W = \bigcap_{i=1}^m F_i$. Since $(x_\alpha)_{\alpha\in\Lambda}$ tends to a, there exists $\mu' \in \Lambda$ such that $\mu' > \mu$ and $x_\alpha \in W$ whenever $\alpha > \mu'$. So we must have $\lambda_\alpha.e - x_\alpha \in \operatorname{Inv} A$, for each $\alpha > \mu'$, which is in contrary with the fact that $\lambda_\alpha \in sp_A(x_\alpha)$. Consequently, for every neighbourhood U of $sp_A(a)$, there exists a neighbourhood V of a such that $sp_A(x) \subseteq U$ for all $x \in V$, that is, the spectrum function $x \mapsto sp_A(x)$ is upper semicontinuous at a.

ii)$\Longrightarrow$iii) It is obvious.

iii)$\Longrightarrow$iv) It is obvious.

iv) $\Longrightarrow$i) For every ϵ, $0 < \epsilon < 1$, there exists a symmetric neighbourhood V of zero such that $\rho_A(x) < \epsilon$ for every $x \in V$. Thus $sp_A(x) \subseteq B(0,\epsilon)$ for all $x \in V$ and hence $e + V \subseteq \operatorname{Inv} A$. Therefore, $\operatorname{Inv} A$ has a non-empty interior and so it is open in A by [2, p. 177]. Since A is a unital algebra we conclude that A is a Q-algebra. QED

To prove the next result we need the following elementary lemma.

10 Lemma. *Let f be an upper semicontinuous real-valued function on a topological space X, and K be a compact subset of X. Then f takes its maximum on K.*

Proof. We first show that $f(K)$ is bounded above.
Since $f(K) \subseteq \bigcup_{\beta\in\mathbb{R}}(-\infty,\beta) = \mathbb{R}$, we have $K \subseteq \bigcup_{\beta\in\mathbb{R}} f^{-1}(-\infty,\beta)$. By the definition of upper semicontinuity, $f^{-1}(-\infty,\beta)$ is open in X.
By the compactness of K there are finite numbers $\beta_1, \beta_2, \ldots, \beta_p \in \mathbb{R}$ such that $K \subseteq \bigcup_{k=1}^p f^{-1}(-\infty,\beta_K)$. Taking $\gamma = \max_{1\le k\le p}\beta_k$, we have $f(K) \subseteq (-\infty,\gamma)$. Let $\alpha = \sup_{x\in K} f(x)$. If $\alpha \notin f(K)$ then $f(K) \subseteq \bigcup_{n=1}^\infty(-\infty,\alpha-\frac{1}{n}) = (-\infty,\alpha)$. Following the same argument as above, we can find N, large enough, such that $f(K) \subseteq (-\infty,\alpha-\frac{1}{N})$, which is in contrary with $\alpha = \sup_{x\in K} f(x)$. Therefore, $\alpha \in f(K)$. QED

11 Theorem. *Let A be a topological Q-algebra and let B be an lmc semisimple algebra which is advertibly complete. If $T : A \longrightarrow B$ is a surjective homomorphism then T has a closed graph.*

PROOF. Let $\{q_\alpha\}_{\alpha\in J}$ be a family of seminorms on B and let $(a_i)_{i\in\Lambda}$ be a net in A such that $a_i \longrightarrow 0$ in A and $Ta_i \longrightarrow b$ in B. Since T is surjective, there exists $a \in A$ such that $Ta = b$.

For each $i \in \Lambda$ let $P_i(z) = zTa_i + T(a-a_i)$, for $z \in \mathbb{C}$ and let $g_i(z) = (z-1)a_i + a$, $z \in \mathbb{C}$. Since g_i is continuous and the function $x \mapsto \rho_A(x)$ is upper semicontinuous by Theorem 9, the composite function $f_i = \rho_A \circ g_i$ is a real-valued upper semicontinuous function on $\mathbb{C}$. Note that since $sp_B(Tx) \subseteq sp_A(x)$ and $sp_B(Tx) \neq \emptyset$, by Remark 5, we have $sp_A(x) \neq \emptyset$ for all $x \in A$.

By Lemma 10, for each $R > 0$ there exists $z_i \in \mathbb{C}$ such that $|z_i| = R$ and $\sup_{|z|=R} f_i(z) = f_i(z_i)$. Since $(z_i - 1)a_i + a \longrightarrow a$ and the spectral radius function is upper semicontinuous on A, for each $\epsilon > 0$ there exists $\mu \in \Lambda$ such that $\rho_A((z_i-1)a_i + a) < \rho_A(a) + \epsilon$ for each $i > \mu$. If B_α denotes the completion of $B/\ker q_\alpha$ in the norm $q'_\alpha(y + \ker q_\alpha) = q_\alpha(y)$, then by Remark 5 for every $i \in \Lambda$ we have $\rho_{B_\alpha}(P_i(z) + \ker q_\alpha) \leq \rho_B(P_i(z)) \leq \rho_A((z-1)a_i + a)$.

On the other hand, for each $i \in \Lambda$ we have

$$\rho_{B_\alpha}(P_i(z) + \ker q_\alpha) \leq q_\alpha(P_i(z)) = \\ q_\alpha(zTa_i + T(a - a_i)) \leq |z| q_\alpha(Ta_i) + q_\alpha(T(a - a_i)).$$

By Lemma 8 for each $i > \mu$ we have

$$\begin{aligned}
\rho^2_{B_\alpha}(b + \ker q_\alpha) &= \rho^2_{B_\alpha}(P_i(1) + \ker q_\alpha) \\
&\leq \sup_{|z|=R} \rho_{B_\alpha}(P_i(z) + \ker q_\alpha). \sup_{|z|=\frac{1}{R}} \rho_{B_\alpha}(P_i(z) + \ker q_\alpha) \\
&\leq \sup_{|z|=R} \rho_A((z-1)a_i + a). \sup_{|z|=\frac{1}{R}} (|z| q_\alpha(Ta_i) + q_\alpha(T(a - a_i))) \\
&\leq \rho_A((z_i - 1)a_i + a).(\frac{1}{R} q_\alpha(Ta_i) + q_\alpha(Ta - Ta_i)) \\
&< (\rho_A(a) + \epsilon).(\frac{1}{R} q_\alpha(Ta_i) + q_\alpha(Ta - Ta_i)).
\end{aligned}$$

Taking limit with respect to i we obtain

$$\rho^2_{B_\alpha}(b + \ker q_\alpha) \leq (\rho_A(a) + \epsilon).(\frac{1}{R} q_\alpha(b)).$$

Let $R \longrightarrow \infty$ to obtain $\rho^2_{B_\alpha}(b + \ker q_\alpha) = 0$ for all $\alpha \in J$. Since B is advertibly complete, by Remark 5 it follows that $\rho_B(b) = \sup_{\alpha\in J} \rho_{B_\alpha}(b + \ker q_\alpha) = 0$.

If $b' \in B$ then $b' = T(a')$ for some $a' \in A$. Moreover, $a'a_i \longrightarrow 0$ and $T(a'a_i) \longrightarrow b'b$. By repeating the same argument as above, we have $\rho_B(b'b) = 0$. Since b' is arbitrary, by Lemma 7, we conclude that $b \in \operatorname{rad} B$ and hence $b = 0$. Therefore, T has a closed graph. QED

12 Corollary. *Let A be an F-algebra which is also a Q-algebra and let B be a semisimple Fréchet algebra. Then every surjective homomorphism $T : A \longrightarrow B$ is automatically continuous.*

Proof. It is an immediate consequence of Theorem 11 and the Closed Graph Theorem. $\boxed{QED}$

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