Note di Matematica
Note Mat. 28 (2008), n. 2, 43-56
ISSN 1123-2536, e-ISSN 1590-0932
DOI 10.1285/i15900932v28n2p43

On curvatures of ruled surfaces in Minkowski 3-spaces

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Received: 04/12/2006; accepted: 18/07/2007.

Abstract. In this paper, we classify ruled surfaces of non-degenerate second fundamental form in Minkowski 3-spaces satisfying some algebraic equations in terms of the second mean curvature, the mean curvature and the Gaussian curvature.

Keywords: Gaussian curvature, second mean curvature, mean curvature, Minkowski surface, non-developable ruled surface.

MSC 2000 classification: 53B25, 53C50.

1 Introduction

The inner geometry with non-degenerate second fundamental form has been a popular research topic for ages. We will refer the term "non-developable," and by a non-developable surface we mean that a surface free of points of vanishing Gaussian curvature in a Euclidean 3-space. It is readily seen that the second fundamental form of a surface is non-degenerate if and only if a surface is non-developable. On such a surface M, we can regard the second fundamental form II of a surface M as a new Riemannian metric or pseudo-Riemannian metric on the Riemannian or pseudo-Riemannian manifold (M, II). In this case, we can define the Gaussian curvature and the mean curvature of non-degenerate second fundamental form, denoted by K_{II} and H_{II} respectively, these are nothing but the Gaussian curvature and the mean curvature of (M, II). By Briosch's formula in a Euclidean 3-space and a Minkowski 3-space we are able to computer K_{II} of M by replacing the components of the first fundamental form E, F, G by the components of the second fundamental form e, f, g, respectively. The curvature K_{II} is called the second Gaussian curvature (cf. [2, 3, 7, 11, 13, 14, 15, etc]).

[i]This work was supported by Korea Research Foundation Grant funded the Korean Government (MOEHRD, Basic Research Promotion Fund) (KRF-2006-331-C00027)

On the other hand, the mean curvature H_{II} of non-degenerate second fundamental form in a Minkowski 3-pace $\mathbb{R}^3_1$ is defined by ([7])

$$H_{II} = H + \frac{1}{2}\Delta_{II} \ln \sqrt{|K|}, \tag{1.1}$$

where K and H are the Gaussian curvature and the mean curvature respectively, and Δ_{II} denotes the Laplacian operator of non-degenerate second fundamental form, that is,

$$\Delta_{II} = -\frac{1}{\sqrt{|h|}} \sum_{i,j=1}^{2} \frac{\partial}{\partial x^i}\left(\sqrt{|h|}h^{ij}\frac{\partial}{\partial x^j}\right), \tag{1.2}$$

where $e = h_{11}, f = h_{12}, g = h_{22}, h = \det(h_{ij}), (h^{ij}) = (h_{ij})^{-1}$ and $\{x_i\}$ is rectangular coordinate system in $\mathbb{R}^3_1$. The curvature H_{II} is said to be the second mean curvature of a surface M in a Minkowski 3-space.

Several geometers have studied the above mentioned curvatures of surfaces in a Euclidean space and a Minkowski space and obtained many interesting results. In particular, the authors in [6, 7, 15, 18, 19] investigated the relationship between the mean curvature and the Gaussian curvature, and in [7, 11, 13, 19] investigated the relationship between the Gaussian curvature and the second Gaussian curvature. Also, the authors in [2, 3, 7, 11, 14, 17, 19] studied the relationship between the mean curvature and the second Gaussian curvature, and in [7, 8, 17] studied the relationship between the Gaussian curvature, the mean curvature and the second mean curvature.

Recently, Y. H. Kim and the present first author([12]) classified non-developable ruled surface in a Minkowski 3-space satisfying the equations

$$\begin{aligned} aH^2 + 2bHK_{II} + cK_{II}^2 &= \text{constant},\\ aK^2 + 2bKK_{II} + cK_{II}^2 &= \text{constant}, \end{aligned} \tag{1.3}$$

where a, b, c are constants.

In this article, we investigate a non-developable ruled surface in a Minkowski 3-space $\mathbb{R}^3_1$ satisfying the equations

$$aH^2 + 2bHH_{II} + cH_{II}^2 = \text{constant}, \tag{1.4}$$

$$aK^2 + 2bKH_{II} + cH_{II}^2 = \text{constant}, \tag{1.5}$$

along each ruling, where a, b, c are constant. If a surface satisfies the equations (1.4) and (1.5), then a surface is said to be a HH_{II}-quadric and KH_{II}-quadric, respectively.

2 Preliminaries

Let $\mathbb{R}^3_1$ be a Minkowski 3-space with the scalar product of index 1 given by $\langle \cdot, \cdot \rangle = -dx_1^2 + dx_2^2 + dx_3^2$, where (x_1, x_2, x_3) is a standard rectangular coordinate system of $\mathbb{R}^3_1$. A vector x of $\mathbb{R}^3_1$ is said to be space-like if $\langle x, x \rangle > 0$ or $x = 0$, time-like if $\langle x, x \rangle < 0$ and null if $\langle x, x \rangle = 0$ and $x \neq 0$. A time-like or null vector in $\mathbb{R}^3_1$ is said to be causal.

Now, we define a ruled surface M in $\mathbb{R}^3_1$. Let I and J be open intervals containing 0 in the real line $\mathbb{R}$. Let $\alpha = \alpha(s)$ be a curve of J into $\mathbb{R}^3_1$ and $\beta = \beta(s)$ a vector field along α. Then, a ruled surface M is defined by the parametrization given as follows:

$$x = x(s, t) = \alpha(s) + t\beta(s), \quad s \in J, \quad t \in I.$$

For such a ruled surface, α and β are called the base curve and the director vector field, respectively.

According to the causal character of α' and β, there are four possibilities:

(1) α' and β are non-null and linearly independent.

(2) α' is null and β is non-null with $\langle \alpha', \beta \rangle \neq 0$.

(3) α' is non-null and β is null with $\langle \alpha', \beta \rangle \neq 0$.

(4) α' and β are null with $\langle \alpha', \beta \rangle \neq 0$.

It is easily to see that, with an appropriate change of the curve α, cases (2) and (3) reduce to (1) and (4), respectively (For the details, see [1]).

First of all, we consider the ruled surface of the case (1). In this case, the ruled surface M is said to be cylindrical if the director vector field β is constant and non-cylindrical otherwise.

Let the base curve α and the director vector field β be non-null. Then, the base curve α can be chosen to be orthogonal to the director vector field β and β can be normalized satisfying $\langle \beta(s), \beta(s) \rangle = \varepsilon (= \pm 1)$ for all $s \in J$. In this case, according to the character of vector fields α' and β, we have ruled surfaces of five different kinds as follows: If the base curve α is space-like or time-like, then the ruled surface M is said to be of type M_+ or type M_-, respectively. Also, the ruled surface of type M_+ can be divided into three types. If the vector field β is space-like, it is said to be of type M_+^1 or M_+^2 if β' is non-null or null, respectively. When the vector field β is time-like, β' is space-like because of the causal character. In this case, M is said to be of type M_+^3. On the other hand, for the ruled surface of type M_-, the director vector field is always space-like. According as its derivative β' is non-null or null, it is also said to be of type

M^1_- or M^2_-, respectively (cf. [10]). The ruled surface M of the case (4) is called a null scroll. One of typical examples of null scrolls is B-scroll which is defined as follows:

Let $\alpha(s)$ be a null curve in $\mathbb{R}^3_1$ with Cartan frame $\{A, B, C\}$, i.e., A, B, C are vector fields along α in $\mathbb{R}^3_1$ satisfying the following conditions:

$$<A, A> = <B, B> = 0, \qquad <A, B> = -1,$$
$$<A, C> = <B, C> = 0, \qquad <C, C> = 1,$$

and

$$\alpha' = A,$$
$$C' = -aA - k(s)B,$$

where a is a constant and $k(s)$ a function vanishing nowhere.

Then the map

$$x : M \longrightarrow \mathbb{R}^3_1$$
$$(s, t) \to \alpha + tB(s)$$

defines a Lorentz surface M in $\mathbb{R}^3_1$ that L. K. Graves ([9]) called a B-scroll.

Throughout the paper, we assume the ruled surface M under consideration is connected unless stated otherwise.

On the other hand, many geometers have been interested in studying submanifolds of Euclidean and pseudo-Euclidean space in terms of the so-called finite type immersion ([4]). Also, such a notion can be extended to smooth maps on submanifolds, namely the Gauss map ([5]). In this regard, Y. H. Kim and the first author defined pointwise finite type Gauss map ([10]). In particular, the Gauss map G on a submanifold M of a pseudo-Euclidean space $\mathbb{E}^m_s$ of index s is said to be of *pointwise 1-type* if $\Delta G = fG$ for some smooth function f on M where Δ denotes the Laplace operator defined on M. In [10] the authors showed that minimal non-cylindrical ruled surfaces in a Minkowski 3-space have pointwise 1-type Gauss map. Based on this fact, the authors proved the following theorem which will be useful to prove our theorems in this paper.

1 Theorem ([10])**.** *Let M be a non-cylindrical ruled surface with space-like or time-like base curve in a Minkowski 3-space. Then, the Gauss map is of pointwise 1-type if and only if M is an open part of one of the following spaces: the space-like or time-like helicoid of the 1st, the 2nd and the 3rd kind, the space-like or time-like conjugate of Enneper's surface of the 2nd kind.*

3 Main Results

In this section we study ruled HH_{II}-quadric surface and KH_{II}-quadric surface M in a Minkowski 3-space $\mathbb{R}^3_1$. Thus the ruled surface M under consideration must have the non-degenerate second fundamental form which automatically implies that M is non-developable.

2 Theorem. *Let a, b, c be constants with $a^2+b^2+c^2 \neq 0, a-6b+9c \neq 0$. If M is a non-developable HH_{II}-quadric ruled surface with non-null base curve in a Minkowski 3-space. Then M is an open part of one of the following surfaces :*

(1) the helicoid of the 1st kind as space-like or time-like surface,

(2) the helicoid of the 2nd kind as space-like or time-like surface,

(3) the helicoid of the 3rd kind as space-like or time-like surface,

(4) the conjugate of Enneper's surfaces of the 2nd kind as space-like or time-like surface.

Proof. We consider two cases separately.

Case 1. Let M be a non-developable ruled surface of the three types M^1_+, M^3_+ or M^1_-. Then the parametrization for M is given by

$$x = x(s,t) = \alpha(s) + t\beta(s)$$

such that $\langle \beta, \beta \rangle = \varepsilon_1 (= \pm 1), \langle \beta', \beta' \rangle = \varepsilon_2 (= \pm 1)$ and $\langle \alpha', \beta' \rangle = 0$. In this case α is the striction curve of x, and the parameter is the arc-length on the (pseudo-)spherical curve β.

The first fundamental form of the surface M is given by $E = \langle \alpha', \alpha' \rangle + \varepsilon_2 t^2, F = \langle \alpha', \beta \rangle$ and $G = \varepsilon_1$. For later use, we define the smooth functions Q, J and D as follows:

$$Q = \langle \alpha', \beta \times \beta' \rangle \neq 0, \quad J = \langle \beta'', \beta' \times \beta \rangle, \quad D = \sqrt{|EG - F^2|}.$$

In terms of the orthonormal basis $\{\beta, \beta', \beta \times \beta'\}$ we obtain

$$\alpha' = \varepsilon_1 F \beta - \varepsilon_1 \varepsilon_2 Q \beta \times \beta', \tag{3.1}$$

$$\beta'' = \varepsilon_1 \varepsilon_2 (-\beta + J \beta \times \beta'), \tag{3.2}$$

$$\alpha' \times \beta = \varepsilon_2 Q \beta', \tag{3.3}$$

which imply $EG - F^2 = -\varepsilon_2 Q^2 + \varepsilon_1 \varepsilon_2 t^2$. And, the unit normal vector N is given by $N = \frac{1}{D}(\varepsilon_2 Q \beta' - t \beta \times \beta')$. Then, the components e, f and g of the second fundamental form are expressed as

$$e = \frac{1}{D}(\varepsilon_1 Q(F - QJ) - Q't + Jt^2), \quad f = \frac{Q}{D} \neq 0, \quad g = 0.$$

Therefore, the Gaussian curvature K and the mean curvature H are given by

$$K = \frac{Q^2}{D^4}, \tag{3.4}$$

$$H = \frac{1}{2D^3}(\varepsilon_1 J t^2 - \varepsilon_1 Q' t - QF - Q^2 J). \tag{3.5}$$

On the other hand, by (1.2) the Laplacian operator of non-degenerate second fundamental form II is

$$\begin{aligned}\Delta_{II} = &- \frac{2D}{Q}\frac{\partial^2}{\partial s \partial t} + \frac{1}{Q^2}(2JDt - Q'D)\frac{\partial}{\partial t} + \\ &+ \frac{D}{Q^2}(\varepsilon_1 QF - \varepsilon_1 Q^2 J - Q't + Jt^2)\frac{\partial^2}{\partial t^2}.\end{aligned} \tag{3.6}$$

Thus, by using (1.1), (3.4), (3.5) and (3.6) the second mean curvature H_{II} is given by

$$H_{II} = \frac{1}{2Q^2D^3}(-2Jt^4 + (2\varepsilon_1 QF + 5\varepsilon_1 Q^2 J)t^2 + 3\varepsilon_1 Q^2 Q' t + Q^3 F - 3Q^4 J). \tag{3.7}$$

First of all, we suppose that $Q^2 - \varepsilon_1 t^2 > 0$. We now differentiate H and H_{II} with respect to t, the results are

$$H_t = \frac{1}{2D^5}\left(Jt^3 - 2Q't^2 - \varepsilon_1 Q(3F + QJ)t - \varepsilon_1 Q^2 Q'\right), \tag{3.8}$$

and

$$\begin{aligned}(H_{II})_t = \frac{1}{2Q^2D^5}&(2\varepsilon_1 J t^5 + (2QF - 3Q^2 J)t^3 + \\ &+ 6Q^2 Q' t^2 + (7\varepsilon_1 Q^3 F + \varepsilon_1 Q^4 J)t + \varepsilon_1 Q^4 Q').\end{aligned} \tag{3.9}$$

Now, suppose that a non-developable ruled surface is HH_{II}-quadric surface. Then we have by (1.4)

$$aHH_t + b(H_t H_{II} + H(H_{II})_t) + cH_{II}(H_{II})_t = 0,$$

which implies we have

$$aQ^4 A_1 + bQ^2 B_1 + cC_1 = 0, \tag{3.10}$$

where we put

$$
\begin{aligned}
A_1 = {} & \varepsilon_1 J^2 t^5 - 3\varepsilon_1 J Q' t^4 + (4QJF - 2Q^2J^2 + 2\varepsilon_1 Q'^2)t^3 + \\
& + (2Q^2Q'J + 5QQ'F)t^2 + (Q^2Q'^2 + 4\varepsilon_1 Q^3 JF + \varepsilon_1 Q^4 J^2 + 3\varepsilon_1 Q^2 F^2)t + \\
& + \varepsilon_1 Q^3 Q'(F + QJ), \\
B_1 = {} & 2Q'Jt^6 + (8\varepsilon_1 QJF + 2\varepsilon_1 Q^2 J^2)t^5 + (-6\varepsilon_1 QQ'F + 4\varepsilon_1 Q^2 Q'J)t^4 + \\
& - (12\varepsilon_1 Q^2 Q'^2 + 8Q^2F^2 + 8Q^3JF + 4Q^4J^2)t^3 - (26Q^3Q'F + 2Q^4Q'J)t^2 + \\
& - (6Q^4Q'^2 + 10\varepsilon_1 Q^4 F^2 - 2\varepsilon_1 Q^6 J^2)t - 4\varepsilon_1 Q^5 Q'F, \\
C_1 = {} & - 4\varepsilon_1 J^2 t^9 + 16Q^2J^2t^7 - 6Q^2Q'Jt^6 + \varepsilon_1 Q^2(4F^2 8QJF - 23Q^2J^2)t^5 + \\
& + \varepsilon_1 Q^3 Q'(18F + 15QJ)t^4 + Q^4(18\varepsilon_1 Q' + 16F^2 + 28QJF + 14Q^2J^2)t^3 + \\
& + 23Q^5Q'Ft^2 + Q^6(9Q'^2 + 7\varepsilon_1 F^2 - 20\varepsilon_1 QJF - 3\varepsilon_1 Q^2J^2)t + \\
& + 3\varepsilon_1 Q^7 Q'F - 9\varepsilon_1 Q^8 Q'J.
\end{aligned}
\tag{3.11}
$$

The direct computation of the left-hand side of (3.10) gives a polynomial in t with functions of s as the coefficients and thus they must be zero. We can obtain that the coefficient of the highest order t^{16} of the equation (3.10) is

$$4c\varepsilon_1 J^2 = 0.$$

Therefore, one finds $J = 0$ since $c \neq 0$, which implies that the coefficient of t^5 is

$$4c\varepsilon_1 Q^2 F^2 = 0,$$

from this $F = 0$. Thus, from $J = F = 0$ we have

$$(a - 6b + 9c)Q'^2 = 0.$$

Since $a - 6b + 9c \neq 0$, one obtain $Q' = 0$. In this case the surface is minimal. Since $EG - F^2 = \varepsilon_1\varepsilon_2 t^2 - \varepsilon_2 Q^2$ and $Q^2 - \varepsilon_1 t^2 > 0$. Therefore, the surface is space-like or time-like when $\varepsilon_2 = -1$ or $\varepsilon_2 = 1$, respectively. But, $(\varepsilon_1, \varepsilon_2) = (-1, -1)$ is impossible because of the causal character. Let $(\varepsilon_1, \varepsilon_2) = (-1, 1)$. Then M is of the type M^3_+. Thus the surface is a helicoid of the 3rd kind according to Theorem 1. If $(\varepsilon_1, \varepsilon_2) = (1, \pm 1)$, then M is of the type M^1_+ or M^1_-. Hence the surface is a helicoid of the 1st kind or 2nd kind according to Theorem 1.

Next, we suppose that $Q^2 - \varepsilon_1 t^2 < 0$. By the similar discussion as above we can also obtain $J = F = 0$ and $Q' = 0$ when $a - 6b + 9c \neq 0$. Therefore, the surface is minimal. Since $EG - F^2 = -\varepsilon_2(Q^2 - \varepsilon_1 t^2)$ and $Q^2 - \varepsilon_1 t^2 < 0$. Consequently, M is space-like or time-like according to $\varepsilon_2 = 1$ or $\varepsilon_2 = -1$, respectively. In this case, $\varepsilon_1 = 1$. Therefore, M is of type M_+^1 or M_-^1 depending on $\varepsilon_2 = \pm 1$. Thus, the surface is a helicoid of the 1st kind and the 2nd kind according to Theorem 1.

Case 2. Let M be a non-developable ruled surface of type M_+^2 or M_-^2. Then, the surface M is parametrized by

$$x(s,t) = \alpha(s) + t\beta(s).$$

In this case, the base curve α is space-like or time-like and the director vector field β is space-like but β' is null. So, we may take α and β satisfying $\langle \alpha', \beta \rangle = 0$, $\langle \beta, \beta \rangle = 1$, $\langle \beta', \beta' \rangle = 0$ and $\langle \alpha', \alpha' \rangle = \varepsilon_1 (= \pm 1)$. We have put the non-zero functions q and R as follows:

$$q = ||x_s||^2 = \varepsilon \langle x_s, x_s \rangle = \varepsilon(\varepsilon_1 + 2Rt), \quad R = \langle \alpha', \beta' \rangle$$

where ε denotes the sign of x_s. Therefore, the components of the first fundamental form are $E = \varepsilon q$, $F = 0$ and $G = 1$. Since $\beta \times \beta'$ is a null vector field orthogonal to β', we can assume $\beta \times \beta' = \beta'$. Since β' is a null direction in the hyperboloid $\{\mathbf{x} \mid \langle \mathbf{x}, \mathbf{x} \rangle = 1\}$, β can be chosen as a straight line. Changing the parameter s (if necessary), we have $\beta'' = 0$.

Let $\{\alpha', \beta, \alpha' \times \beta\}$ be a moving frame along M. Then, β' can be written as

$$\beta' = \varepsilon_1 R(\alpha' - \alpha' \times \beta). \tag{3.12}$$

It follows that the function R never vanishes everywhere on M. Since $\beta'' = 0$, (3.12) implies

$$\alpha'' = -R\beta + \frac{R'}{R}\alpha' \times \beta. \tag{3.13}$$

On the other hand, the unit normal vector field of M is given by

$$N = \frac{1}{\sqrt{q}}(\alpha' \times \beta - t\beta'),$$

from which the components of the second fundamental form e, f, and g are obtained as

$$e = -\frac{\varepsilon}{\sqrt{q}R}(RR't + \varepsilon_1 R'), \quad f = \frac{\varepsilon}{\sqrt{q}}R, \quad g = 0.$$

Thus, the Gaussian curvature K, the mean curvature H and the second mean curvature H_{II} are given respectively by

$$K = \frac{R^2}{q^2}, \tag{3.14}$$

$$H = -\frac{\varepsilon}{2Rq^{3/2}}(RR't + \varepsilon_1 R'), \tag{3.15}$$

and

$$H_{II} = \frac{\varepsilon}{2Rq^{\frac{3}{2}}}(-RR't + \varepsilon_1 R'). \tag{3.16}$$

Suppose that the surface is HH_{II}-quadric surface. Similarly to Case 1, we have then

$$(a + 2b + c)RR'^2 = 0,$$
$$(3a - 2b - 5c)RR'^2 = 0,$$
$$(a - 3b + 2c)RR'^2 = 0$$

which imply $R' = 0$ because a, b, c are non-zero constants. Thus, from (3.15) M is minimal, that is, it is a conjugate of Enneper's surface of the 2nd kind as space-like or time-like surface according to Theorem 1. This completes the proof. QED

3 Remark. In Case 1 of Theorem 2, if $a - 6b + 9c = 0$, then, $J = F = 0$ with arbitrary Q'. By (3.5) and (3.7) we get the equation $H_{II} = -3H$. In this case, from (2) and (3.2) we have

$$\begin{aligned} \alpha' &= -\varepsilon_1\varepsilon_2 Q\beta \times \beta', \\ \beta'' &= -\varepsilon_1\varepsilon_2\beta. \end{aligned} \tag{3.17}$$

To solve the equation (3.17) we consider four cases separately.

1. $(\varepsilon_1, \varepsilon_2) = (1, 1)$. Without loss of generality, we may assume $\beta(0) = (0, 0, 1)$. Then we have

$$\beta(s) = (d_1 \sin s, d_2 \sin s, \cos s + d_3 \sin s)$$

for some constants d_1, d_2, d_3 satisfying $-d_1^2 + d_2^2 + d_3^2 = 1$. Since $\langle \beta, \beta \rangle = 1$, we have $-d_1^2 + d_2^2 = 1$ and $d_3 = 0$. From this we can obtain

$$\beta(s) = (d_1 \sin s, \pm\sqrt{1 + d_1^2} \sin s, \cos s),$$

for some constant d_1. Therefore, we have

$$\alpha(s) = (\mp\sqrt{1 + d_1^2}, -d_1, 0)f(s) + \mathbb{E},$$

where $f(s) = \int Q(s)ds$ and $\mathbb{E} = (e_1, e_2, e_3)$ is constant vector. Thus, the surface M has the parametrization of the form

$$\begin{aligned} x(s,t) =& (\mp\sqrt{1 + d_1^2} f(s) + t d_1 \sin s + e_1, \\ & - d_1 f(s) \pm t\sqrt{1 + d_1^2} \sin s + e_2, t\cos s + e_3), \end{aligned} \tag{3.18}$$

where d_1 is constant, $f(s) = \int Q(s)ds$ and (e_1, e_2, e_3) is constant vector. If $d_1 = 0$, then the surface M is a conoid of the 3rd kind (See [11]).

2. $(\varepsilon_1, \varepsilon_2) = (1, -1)$. Without loss of generality, we may assume $\beta(0) = (0, 0, 1)$. Then we have

$$\beta(s) = (d_1 \sinh s, \pm\sqrt{d_1^2 - 1} \sinh s, \cosh s),$$

where $d_1 \leq -1$ or $d_1 \geq 1$. Therefore, we have

$$\alpha(s) = (\mp\sqrt{d_1^2 - 1}, d_1, 0) f(s) + \mathbb{E},$$

where $f(s) = \int Q(s)ds$ and $\mathbb{E} = (e_1, e_2, e_3)$ is constant vector. Thus, the parametrization for the surface M is given by

$$\begin{aligned} x(s,t) =&(\mp\sqrt{d_1^2 - 1} f(s) + t d_1 \sinh s + e_1, \\ & d_1 f(s) \pm t\sqrt{d_1^2 - 1} \sinh s + e_2, t\cosh s + e_3), \end{aligned} \tag{3.19}$$

where $d_1 \leq -1$ or $d_1 \geq 1$, $f(s) = \int Q(s)ds$ and (e_1, e_2, e_3) is constant vector.

If $d_1 = \pm 1$, then the surface M is a conoid of the 1st kind (See [11]).

3. $(\varepsilon_1, \varepsilon_2) = (-1, 1)$. We may assume $\beta(0) = (1, 0, 0)$. Then we have

$$\beta(s) = (\cosh s, d_2 \sinh s, \pm\sqrt{1 - d_2^2} \sinh s),$$

where $-1 \leq d_2 \leq 1$. Therefore, we have

$$\alpha(s) = (0, \pm\sqrt{1 - d_2^2}, -d_2) f(s) + \mathbb{E},$$

where $f(s) = \int Q(s)ds$ and $\mathbb{E} = (e_1, e_2, e_3)$ is constant vector. Thus, the surface M is parametrized by

$$\begin{aligned} x(s,t) =&(t\cosh s + e_1, \pm\sqrt{1 - d_2^2} f(s) + t d_2 \sinh s + e_2, \\ & - d_2 f(s) \pm t\sqrt{1 - d_2^2} \sinh s + e_3), \end{aligned} \tag{3.20}$$

where $-1 \leq d_2 \leq 1$, $f(s) = \int Q(s)ds$ and (e_1, e_2, e_3) is constant vector.

If $d_2 = 0$ or $d_2 = \pm 1$, then the surface M is a conoid of the 2nd kind (See [11]).

4. $(\varepsilon_1, \varepsilon_2) = (-1, -1)$ is impossible because of the causal character.

4 Theorem. *Let a, b, c be constants with $c \neq 0$. If M is a non-developable KH_{II}-quadric ruled surface with non-null base curve in a Minkowski 3-space. Then M is an open part of one of the following surfaces:*

(1) the helicoid of the 1st kind as space-like or time-like surface,

(2) the helicoid of the 2nd kind as space-like or time-like surface,

(3) the helicoid of the 3rd kind as space-like or time-like surface,

(4) the conjugate of Enneper's surfaces of the 2nd kind as space-like or time-like surface.

Proof. In order to prove the theorem, we split it into two cases.

Case 1. As is described in Theorem 2 we assume that the non-developable ruled surface M of the three types M_+^1, M_+^3 or M_-^1 is parametrized by

$$x = x(s,t) = \alpha(s) + t\beta(s)$$

such that $\langle\beta,\beta\rangle = \varepsilon_1 (= \pm 1), \langle\beta',\beta'\rangle = \varepsilon_2(=\pm1)$ and $\langle\alpha',\beta'\rangle = 0$.

On the other hand, the Gaussian curvature K and the second mean curvature H_{II} are given by (3.4) and (3.7), respectively.

Suppose that the surface M is KH_{II}-quadric. Then the equation (1.5) implies

$$aKK_t + b(K_t H_{II} + K(H_{II})_t) + cH_{II}(H_{II})_t = 0. \tag{3.21}$$

First of all, we assume that $Q^2 - \varepsilon_1 t^2 > 0$. By differentiating (3.4) with respect to t

$$K_t = \frac{4\varepsilon_1 Q^2 t}{D^6}. \tag{3.22}$$

Then, by substituting (3.4), (3.7), (3.8) and (3.22) into (3.21) it follows that

$$4b^2 Q^8 D^2 A_2^2 = (16a\varepsilon_1 Q^8 t + cD^2 B_2)^2, \tag{3.23}$$

where we put

$$\begin{aligned} A_2 =& -10\varepsilon_1 J t^5 + (23Q^2J + 6QF)t^3 + 6Q^2Q't^2 + \\ & - (3\varepsilon_1 Q^3 F + 4\varepsilon_1 Q^4 J)t - 3\varepsilon_1 Q^4 Q', \\ B_2 =&\ 4\varepsilon_1 J^2 t^9 - 16Q^2J^2t^7 + 6Q^2Q'Jt^6 + (28\varepsilon_1 Q^3 JF - 4\varepsilon_1 Q^2F^2 + 23Q^4J^2)t^5 \\ & - (18\varepsilon_1 Q^3 Q' F + 15\varepsilon_1 Q^4 Q' J)t^4 + \\ & - (16Q^4F^2 + 18Q^5JF + 14Q^6J^2 + 18\varepsilon_1 Q^4 Q'^2)t^3 \\ & - 33Q^5Q'Ft^2 + (3\varepsilon_1 Q^8 J^2 + 20\varepsilon_1 Q^7 JF - 7\varepsilon_1 Q^6 F^2 - 9Q^6Q'^2)t \\ & - 3\varepsilon_1 Q^7 Q' F + 9\varepsilon_1 Q^8 Q' J. \end{aligned} \tag{3.24}$$

From (3.24) we obtain that the coefficient of the highest order of the equation (3.23) is

$$16c^2J^4 = 0.$$

It follows $J = 0$ since $c \neq 0$, which implies that the coefficient of t^{14} is

$$16cQ^4F^4 = 0,$$

from this $F = 0$. Thus, from $J = F = 0$ we can obtain $Q' = 0$. Consequently, the mean curvature H is identically zero.

Next, we suppose that $Q^2 - \varepsilon_1 t^2 < 0$. In this case, by using (3.21) we can also show that the surface M is minimal. Consequently, by the proof of Theorem 2 the surface M is an open part of one of the helicoid of the 1st kind, 2nd kind and 3rd kind as space-like or time-like surface.

Case 2. Let M be a non-developable ruled surface of type M^2_+ or M^2_-. In this case, the curve α is space-like or time-like and β space-like but β' is null. We will also use the notations given in Theorem 2. Then, the Gaussian curvature K and the second mean curvature H_{II} are given by (3.14) and (3.16), respectively.

Suppose that the surface M is KH_{II}-quadric. Then, by the equation (3.14), (3.16) and (3.21), and by the similar discussion of Case 1 in Theorem 2, we can also obtain $R' = 0$ because $c \neq 0$, it follows the mean curvature H is identically zero. Consequently, by the proof of Theorem 2 the surface M is a conjugate of Enneper's surface of the 2nd kind as space-like or time-like surface. This completes the proof. QED

Finally, we investigate the relations between the second mean curvature, the Gaussian curvature and the mean curvature of null scrolls in $\mathbb{R}^3_1$.

Let $\alpha = \alpha(s)$ be null curve in $\mathbb{R}^3_1$ and $B = B(s)$ be null vector field along α. Then, the null scroll M is parametrized by

$$x = x(s,t) = \alpha(s) + tB(s)$$

such that $\langle \alpha', \alpha' \rangle = 0$, $\langle B, B \rangle = 0$ and $\langle \alpha', B \rangle = -1$. Furthermore, without loss of generality, we may choose α as a null geodesic of M. We then have $\langle \alpha'(s), B'(s) \rangle = 0$ for all s. By putting, $C = \alpha' \times B$, then $\{\alpha', B, C\}$ is an orthonormal basis along α in $\mathbb{R}^3_1$. In terms of the basis, we have

$$\begin{aligned} \alpha'' &= vC, \\ B' &= -uC, \\ C' &= -u\alpha' + vB \end{aligned} \tag{3.25}$$

where we put $u = \langle B, C' \rangle$ and $v = \langle \alpha'', C \rangle$. The induced Lorentz metric on M is given by $E = u^2t^2$, $F = -1$, $G = 0$ and the unit normal vector N is obtained by

$$N = C + tB' \times B.$$

Thus, the component functions of the second fundamental form are given by

$$e = \langle \alpha'' + tB'', N\rangle = u^3t^2 - u't + v, \quad f = \langle B', C\rangle = -u, \quad g = 0,$$

which imply $H = u$ and $K = u^2$.

On the other hand, by (1.2) the Laplacian operator of non-degenerate second fundamental form II is

$$\Delta_{II} = \frac{1}{u}\frac{\partial^2}{\partial s\partial t} + \frac{1}{u^2}(2u^3t - u')\frac{\partial}{\partial t} + \frac{1}{u^2}(u^3t^2 - u't + v)\frac{\partial^2}{\partial t^2}, \tag{3.26}$$

it follows that the second mean curvature H_{II} is given by

$$H_{II} = u. \tag{3.27}$$

Thus, we have the following:

5 Theorem. *Let M be null scrolls in a Minkowski 3-space. Then, M satisfies the equations $K = u^2, H = u, H_{II} = u$.*

6 Theorem. *Let a, b, c, d be constants with $a + 2b + c \neq 0$. B-scrolls over null curves are the only null scrolls with non-degenerate second fundamental form in a Minkowski 3-space satisfying $aH^2 + 2bHH_{II} + cH_{II}^2 = d$ along each ruling.*

Proof. Let M be a null scroll with non-degenerate second fundamental form in a Minkowski 3-space. Then by Theorem 4 $u^2(a + 2b + c) = d$, it follows that the function u is a constant when $a + 2b + c \neq 0$. Thus, a null scroll M is a B-scroll. QED

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