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Poisson-Gradient Dynamical Systems with Bounded Non-Linearity

Constantin Udrişte
Department of Mathematics, University Politehnica of Bucharest, Splaiul Independentei 313, Bucharest 060042, Romania
udriste@mathem.pub.ro

Iulian Duca
Department of Mathematics, University Politehnica of Bucharest, Splaiul Independentei 313, Bucharest 060042, Romania

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Abstract. We study the multi-periodical solutions of a Poisson-gradient $PDEs$ system with bounded non-linearity, showing that suitable properties of the potential imply properties of solutions.

Section 1 introduces the basic spaces, the type of functionals and the Poisson-gradient PDEs. Section 2 studies the weak differential of a function and establishes an integral inequality satisfied by a suitable integrable function. Section 3 formulates some conditions under which a given action functional is continuously differentiable. Section 4 analyzes the Poisson-gradient systems and some conditions that ensure multi-periodical solutions.

Keywords: variational methods, elliptic PDEs, multi-periodical solutions, weak derivatives, potentials.

MSC 2000 classification: primary 35J50, secondary 35J55

1 Poisson-gradient PDEs

We consider the point $T = \left(T^1, \dots, T^p\right)$ and the parallelepiped $T_0 = \left[0, T^1\right] \times \dots \times [0, T^p]$ in R^p. We denote by $W_T^{1,2}$ the Sobolev space of the functions $u \in L^2\left[T_0, R^n\right]$ which have weak derivatives $\partial u/\partial t \in L^2\left[T_0, R^n\right]$. The index T from the notation $W_T^{1,2}$ comes from the fact that the weak derivatives are defined using the space C_T^∞ of all indefinitely differentiable multiple T-periodic functions from R^p into R^n. We denote by H_T^1 the Hilbert space $W_T^{1,2}$. The norm used in H_T^1 is induced by the scalar product

$$\langle u, v\rangle = \int_{T_0} \left(\delta_{ij} u^i\left(t\right) v^j\left(t\right) + \delta_{ij}\delta^{\alpha\beta}\frac{\partial u^i}{\partial t^\alpha}\left(t\right)\frac{\partial v^j}{\partial t^\beta}\left(t\right)\right) dt^1 \wedge \dots \wedge dt^p.$$

In other words, on the multiphase space R^{n+np}, we use the Riemannian metric

$$G = \begin{pmatrix} \delta_{ij} & 0 \\ 0 & \delta^{\alpha\beta}\delta_{ij} \end{pmatrix}$$

and its associated norm. Also, we recall that the Euclidean space R^n is endowed with the scalar product $(u, v) = \delta_{ij} u^i v^j$ and the norm $|u| = \sqrt{\delta_{ij} u^i u^j}$. Let $t = \left(t^1, \ldots, t^p\right)$ be a generic point in R^p. Then the opposite faces of the parallelepiped T_0 can be described by the equations

$$S_i^- : t^i = 0, S_i^+ : t^i = T^i$$

for each $i = 1, \ldots, p$. We shall study the minimum of the action

$$\varphi(u) = \int_{T_0} L\left(t, u(t), \frac{\partial u}{\partial t}\right) dt^1 \wedge \cdots \wedge dt^p,$$

determined by the Lagrangian

$$L\left(t, u(t), \frac{\partial u}{\partial t}\right) = \frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2 + F(t, u(t))$$

on the space H_T^1, considering that the potential function F has the property of bounded non-linearity. We use the method of the minimizing sequences and the coercitivity condition $\int_{T_0} F(t, u(t))\, dt^1 \wedge \cdots \wedge dt^p \to \infty$ when $|u| \to \infty$. The extremals of the action φ verifies the Euler-Lagrange equations with the boundary conditions

$$u\,|_{S_i^-} = u\,|_{S_i^+}, \frac{\partial u}{\partial t}\,|_{S_i^-} = \frac{\partial u}{\partial t}\,|_{S_i^+}, i = 1, \ldots, p.$$

Due to the particularity of the Lagrangian L, the Euler-Lagrange equations reduce to a $PDEs$ system of the Poisson-gradient type

$$\Delta u(t) = \nabla F(t, u(t)).$$

The aim of this paper is to discuss the existence of solutions of this PDEs system with suitable boundary conditions. More precisely, we extend the theory in [2] from single-time to multi-time field theory, developing the ideas in the papers [6], [7], [9]. In this way we find positive answers for the existence of multi-periodical solutions of Euler-Lagrange equations that are Poisson-gradient PDEs with bounded non-linearity. The results can be applied to the PDEs involved in multi-time geometric dynamics ([5], [8], [10]-[12]).

2 The weak differential and an integral inequality

We consider C_T^∞ the space of the indefinitely differentiable functions multiple periodical with the period $T = \left(T^1, \ldots, T^p\right)$, defined on R^p taking values in R^n. We know that $C_T^\infty \subset W_T^{1,2}$. We establish some conditions satisfied by a function $u \in L^1\left[T_0, R^n\right]$ which has a weak differential.

1 Theorem. *Let $u \in L^1[T_0, R^n]$. Suppose $v_\alpha \in L^1[T_0, R^n]$ are such that the vector form $v_\alpha dt^\alpha = \left(v_\alpha^1 dt^\alpha, \ldots, v_\alpha^n dt^\alpha\right)$ is integrable. Denote by $\overset{\frown}{OT}$ an arbitrary curve from T_0, having the endings at $O = (0, \ldots, 0)$ and $T = \left(T^1, \ldots, T^p\right)$.*

If

$$\int_{\overset{\frown}{OT}} (u, df) = -\int_{\overset{\frown}{OT}} (v_\alpha dt^\alpha, f), \tag{1}$$

for any $f \in C_T^\infty$, then $\int_{\overset{\frown}{OT}} v_\alpha dt^\alpha = 0$ and it exists $c \in R^n$ such that $u(t) = \int_{\overset{\frown}{Ot}} v_\alpha ds^\alpha + c$. Also $u(0) = u(T)$.

Proof. We choose $f = e^i = (0, \ldots, 0, 1, 0, \ldots, 0)$, with the value 1 on the position i. From the relation (1) we have $0 = -\int_{\overset{\frown}{OT}} v_\alpha^i dt^\alpha$ and hence $\int_{\overset{\frown}{OT}} v_\alpha dt^\alpha = 0$.

We define $w \in C(T_0, R^n)$ by $w(t) = \int_{\overset{\frown}{Ot}} v_\alpha ds^\alpha, t \in \overset{\frown}{OT}$. By Fubini Theorem, the function w satisfies the relation

$$\int_{\overset{\frown}{OT}} (w, df) = \int_{\overset{\frown}{OT}} \left(\int_{\overset{\frown}{Ot}} v_\alpha ds^\alpha, df\right) = \int_{\overset{\frown}{OT}} \left(\int_{\overset{\frown}{sT}} (v_\alpha, df)\, ds^\alpha\right)$$
$$= \int_{\overset{\frown}{OT}} (v_\alpha, f(T) - f(s))\, ds^\alpha = -\int_{\overset{\frown}{OT}} (v_\alpha, f(s))\, ds^\alpha = \int_{\overset{\frown}{OT}} (u, df).$$

This means that

$$\int_{\overset{\frown}{OT}} (u - w, df) = 0. \tag{2}$$

We consider now $\gamma : [a, b] \to T_0, \gamma(\xi) = \left(t^1(\xi), \ldots, t^p(\xi)\right), \gamma(a) = O,\ \gamma(b) = T$, a parameterization of the curve $\overset{\frown}{OT}$. The equality (2) becomes

$$\int_a^b \left(u(t(\xi)) - w(t(\xi)), \left(\frac{\partial f^1}{\partial t^\alpha}\frac{dt^\alpha}{d\xi}, \ldots, \frac{\partial f^n}{\partial t^\alpha}\frac{dt^\alpha}{d\xi}\right)\right) d\xi = 0,$$

for any $f \in C_T^\infty$. We will particularize for the function sequences

$$f_j^{(k)}(t) = \left\{\begin{array}{c}\cos\\ \sin\end{array}\right\}\left(\frac{2k\pi t^j}{T^j}\right) e^j,\ k \in N \setminus \{0\},\ 1 \le j \le n$$

and we observe that (see the Fourier series theory) $u(t) - w(t) = c$, $c \in R^n$ almost everywhere in T_0 (the constant is the only function orthogonal to the previous sequences). By replacing $w(t)$, we find that $u(t) = \int_{\overset{\frown}{Ot}} v_\alpha ds^\alpha + c$ for any $t \in \overset{\frown}{OT}$. The function u satisfies $u(0) = c$ and $u(T) = \int_{\overset{\frown}{OT}} v_\alpha ds^\alpha + c = c$, so $u(0) = u(T)$. On the other side, the relation $u(t) - u(\tau) = \int_{\overset{\frown}{\tau t}} v_\alpha ds^\alpha$ implies that $u(t) = \int_{\overset{\frown}{\tau t}} v_\alpha ds^\alpha + u(\tau)$. The 1-form $v_\alpha dt^\alpha$ is called *weak differential* of the function u. By a Fourier series argument, the weak differential, if it exists, is unique. The weak differential of u will be denoted by du. The existence of du implies $u(0) = u(T)$. QED

2 Theorem. *If* $u = \left(u^1, \dots, u^n\right)$ *is a function in* $L^1[T_0, R^n] \cap L^2[T_0, R^n]$, *and* $|u(t)|^2 = \delta_{ij} u^i(t) u^j(t)$, *then*

$$\left|\int_{T_0} u(t)\, dt^1 \wedge \cdots \wedge dt^p\right| \le \left(nT^1 \dots T^p\right)^{\frac{1}{2}} \left(\int_{T_0} |u(t)|^2\, dt^1 \wedge \cdots \wedge dt^p\right)^{\frac{1}{2}}.$$

Proof. Successively we have the relations

$$\begin{aligned}
\left|\int_{T_0} u(t)\, dt^1 \wedge \cdots \wedge dt^p\right| &= \left|\int_{T_0} \left(u^1(t), \dots, u^n(t)\right) dt^1 \wedge \cdots \wedge dt^p\right| \\
&= \left|\left(\int_{T_0} u^1(t)\, dt^1 \wedge \cdots \wedge dt^p, \dots, \int_{T_0} u^n(t)\, dt^1 \wedge \cdots \wedge dt^p\right)\right| \\
&= \left(\left(\int_{T_0} u^1(t)\, dt^1 \wedge \cdots \wedge dt^p\right)^2 + \cdots + \left(\int_{T_0} u^1(t)\, dt^1 \wedge \cdots \wedge dt^p\right)^2\right)^{\frac{1}{2}} \\
&\le \left|\int_{T_0} u^1(t)\, dt^1 \wedge \cdots \wedge dt^p\right| + \cdots + \left|\int_{T_0} u^n(t)\, dt^1 \wedge \cdots \wedge dt^p\right| \\
&\le \int_{T_0} \left(\left|u^1(t)\right| + \cdots + \left|u^n(t)\right|\right) dt^1 \wedge \cdots \wedge dt^p \\
&= \int_{T_0} \left(\left(\left|u^1(t)\right|, \dots, \left|u^n(t)\right|\right), (1, \dots, 1)\right) dt^1 \wedge \cdots \wedge dt^p.
\end{aligned}$$

Using the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
&\left|\int_{T_0} u(t)\, dt^1 \wedge \cdots \wedge dt^p\right| \\
&\le \left(\int_{T_0} \left(\left|u^1(t)\right|^2 + \cdots + \left|u^n(t)\right|^2\right) dt^1 \wedge \cdots \wedge dt^p\right)^{\frac{1}{2}} \left(\int_{T_0} n dt^1 \wedge \cdots \wedge dt^p\right)^{\frac{1}{2}} \\
&= \left(nT^1 \dots T^p\right)^{\frac{1}{2}} \left(\int_{T_0} |u(t)|^2\, dt^1 \wedge \cdots \wedge dt^p\right)^{\frac{1}{2}}.
\end{aligned}$$

QED

3 A continuously differentiable action

The next theorem establishes some conditions in which the action

$$\varphi : W_T^{1,2} \to R, \varphi(u) = \int_{T_0} L\left(t, u(t), \frac{\partial u}{\partial t}(t)\right) dt^1 \wedge \cdots \wedge dt^p$$

is continuously differentiable. In this way we extend the particular case $p = 1$, studied in [3, Theorem 1.4].

3 Theorem. *We consider* $L : T_0 \times R^n \times R^{np} \to R, (t, x, y) \to L(t, x, y)$, *a measurable function in* t *for any* $(x, y) \in R^n \times R^{np}$ *and with the continuous partial derivatives in* x *and* y *for any* $t \in T_0$. *If here exist* $a \in C^1(R^+, R^+)$ *with the derivative* a' *bounded from above,* $b \in C(T_0, R^+)$ *such that for any* $t \in T_0$ *and any* $(x, y) \in R^n \times R^{np}$ *to have*

$$\begin{aligned} &|L(t, x, y)| \le a\left(|x| + |y|^2\right) b(t), \\ &|\nabla_x L(t, x, y)| \le a(|x|) b(t), \\ &|\nabla_y L(t, x, y)| \le a(|y|) b(t), \end{aligned} \tag{3}$$

then, the functional φ *has continuous partial derivatives in* $W_T^{1,2}$ *and his gradient derives from the formula*

$$\begin{aligned} (\nabla\varphi(u), v) &= \textstyle\int_{T_0} \left[\left(\nabla_x L\left(t, u(t), \frac{\partial u}{\partial t}\right), v(t)\right)\right. \\ &+ \left.\left(\nabla_y L\left(t, u(t), \frac{\partial u}{\partial t}(t)\right), \frac{\partial v}{\partial t}(t)\right)\right] dt^1 \wedge \cdots \wedge dt^p. \end{aligned} \tag{4}$$

Proof. It is enough to prove that φ has the derivative $\varphi'(u) \in (W_T^{1,2})^*$ given by the relation (4) and the function $\varphi' : W_T^{1,2} \to (W_T^{1,2})^*$, $u \to \varphi'(u)$ is continuous. We consider $u, v \in W_T^{1,2}$, $t \in T_0$, $\lambda \in [-1, 1]$. We build the functions

$$F(\lambda, t) = L\left(t, u(t) + \lambda v(t), \frac{\partial u}{\partial t}(t) + \lambda \frac{\partial v}{\partial t}(t)\right),$$

$$\Psi(\lambda) = \int_{T_0} F(\lambda, t)\, dt^1 \wedge \cdots \wedge dt^p.$$

Because the derivative a' is bounded from above, there exists $M > 0$ such that $\frac{a(|u|) - a(0)}{|u|} = a'(c) \le M$. This means that $a(|u|) \le M|u| + a(0)$. On the other side,

$$\begin{aligned} \frac{\partial F}{\partial \lambda}(\lambda, t) &= \left(\nabla_x L\left(t, u(t) + \lambda v(t), \frac{\partial u}{\partial t}(t) + \lambda \frac{\partial v}{\partial t}(t)\right), v(t)\right) \\ &\quad + \left(\nabla_y L\left(t, u(t) + \lambda v(t), \frac{\partial u}{\partial t}(t) + \lambda \frac{\partial v}{\partial t}(t)\right), \frac{\partial v}{\partial t}(t)\right) \\ &\le a(|u(t) + \lambda v(t)|) b(t) |v(t)| + a\left(\left|\frac{\partial u}{\partial t}(t) + \lambda \frac{\partial v}{\partial t}(t)\right|\right) b(t) \left|\frac{\partial v}{\partial t}(t)\right| \\ &\le b_0 (M(|u(t)| + |v(t)|) + a(0)) |v(t)| \\ &\quad + b_0 \left(M\left(\left|\frac{\partial u}{\partial t}(t)\right| + \left|\frac{\partial v}{\partial t}(t)\right|\right) + a(0)\right) \left|\frac{\partial v}{\partial t}(t)\right|, \end{aligned}$$

where

$$b_0 = \max_{t\in T_0} b\left(t\right).$$

Then, we have $\left|\frac{\partial F}{\partial \lambda}\left(\lambda, t\right)\right| \leq d\left(t\right) \in L^1\left(T_0, R^+\right)$. Then Leibniz formula of differentiation under integral sign is applicable and

$$\begin{aligned}
\frac{\partial \Psi}{\partial \lambda}\left(0\right) &= \int_{T_0} \frac{\partial F}{\partial \lambda}\left(0, t\right) dt^1 \wedge \cdots \wedge dt^p \\
&= \int_{T_0} \left[\left(\nabla_x L\left(t, u\left(t\right), \frac{\partial u}{\partial t}\left(t\right)\right), v\left(t\right)\right)\right. \\
&\quad \left. + \left(\nabla_y L\left(t, u\left(t\right), \frac{\partial u}{\partial t}\left(t\right)\right), \frac{\partial v}{\partial t}\left(t\right)\right)\right] dt^1 \wedge \cdots \wedge dt^p.
\end{aligned}$$

Moreover,

$$\left|\nabla_x L\left(t, u\left(t\right), \frac{\partial u}{\partial t}\left(t\right)\right)\right| \leq b_0\left(M\left|u\left(t\right)\right| + \left|a\left(0\right)\right|\right) \in L^1\left(T_0, R^+\right)$$

and

$$\left|\nabla_y L\left(t, u\left(t\right), \frac{\partial u}{\partial t}\left(t\right)\right)\right| \leq b_0\left(M\left|\frac{\partial u}{\partial t}\left(t\right)\right| + \left|a\left(0\right)\right|\right) \in L^2\left(T_0, R^+\right).$$

That is why

$$\begin{aligned}
&\int_{T_0} \left[\left(\nabla_x L\left(t, u\left(t\right), \frac{\partial u}{\partial t}\left(t\right)\right), v\left(t\right)\right)\right. \\
&\quad \left. + \left(\nabla_y L\left(t, u\left(t\right), \frac{\partial u}{\partial t}\left(t\right)\right), \frac{\partial v}{\partial t}\left(t\right)\right)\right] dt^1 \wedge \cdots \wedge dt^p \\
&\leq \int_{T_0} \left|\nabla_x L\left(t, u\left(t\right), \frac{\partial u}{\partial t}\left(t\right)\right)\right| \left|v\left(t\right)\right| dt^1 \wedge \cdots \wedge dt^p \\
&\quad + \int_{T_0} \left|\nabla_y L\left(t, u\left(t\right), \frac{\partial u}{\partial t}\left(t\right)\right)\right| \left|\frac{\partial v}{\partial t}\left(t\right)\right| dt^1 \wedge \cdots \wedge dt^p \\
&\leq b_0 \int_{T_0} \left(M\left|u\left(t\right)\right| + \left|a\left(0\right)\right|\right) \left|v\left(t\right)\right| dt^1 \wedge \cdots \wedge dt^p \\
&\quad + b_0 \int_{T_0} \left(M\left|\frac{\partial u}{\partial t}\left(t\right)\right| + \left|a\left(0\right)\right|\right) \left|\frac{\partial v}{\partial t}\left(t\right)\right| dt^1 \wedge \cdots \wedge dt^p.
\end{aligned}$$

By using the Cauchy-Schwartz inequality, we find

$$\begin{aligned}
\left|\frac{\partial\Psi}{\partial\lambda}(0)\right| &\le b_0\left(\int_{T_0}(M\,|u(t)|+|a(0)|)^2\,dt^1\wedge\cdots\wedge dt^p\right)^{\frac{1}{2}}\cdot\\
&\quad\cdot\left(\int_{T_0}|v(t)|^2\,dt^1\wedge\cdots\wedge dt^p\right)^{\frac{1}{2}}\\
&\quad+b_0\left(\int_{T_0}\left(M\left|\frac{\partial u}{\partial t}(t)\right|+|a(0)|\right)^2 dt^1\wedge\cdots\wedge dt^p\right)^{\frac{1}{2}}\cdot\\
&\quad\cdot\left(\int_{T_0}\left|\frac{\partial v}{\partial t}(t)\right|^2 dt^1\wedge\cdots\wedge dt^p\right)^{\frac{1}{2}}\\
&\le C_1\left(\int_{T_0}|v(t)|^2\,dt^1\wedge\cdots\wedge dt^p\right)^{\frac{1}{2}}+C_2\left(\int_{T_0}\left|\frac{\partial v}{\partial t}(t)\right|^2 dt^1\wedge\cdots\wedge dt^p\right)^{\frac{1}{2}}\\
&\le\max\{C_1,C_2\}\,2^{\frac{1}{2}}\left(\int_{T_0}\left(|v(t)|^2+\left|\frac{\partial v}{\partial t}(t)\right|^2\right)dt^1\wedge\cdots\wedge dt^p\right)^{\frac{1}{2}}\\
&=C\,\|v\|\,.
\end{aligned}$$

By consequence, the action φ has the derivative $\varphi'\in\left(W_T^{1,2}\right)^*$ given by (4). The Krasnoselski theorem and the hypothesis (3) imply the fact that the application $u\to\left(\nabla_x L\left(\cdot,u,\frac{\partial u}{\partial t}\right),\nabla_y L\left(\cdot,u,\frac{\partial u}{\partial t}\right)\right)$, from $W_T^{1,2}$ to $L^1\times L^2$, is continuous, so φ' is continuous from $W_T^{1,2}$ to $\left(W_T^{1,2}\right)^*$ and the proof is complete. QED

4 Poisson-gradient systems and their periodical solutions

4.1 Multi-time Euler-Lagrange PDEs

We consider the multi-time variable $t=\left(t^1,\dots,t^p\right)\in R^p$, the functions $x^i:R^p\to R,\left(t^1,\dots,t^p\right)\to x^i\left(t^1,\dots,t^p\right),i=1,\dots n$, and the partial velocities $x^i_\alpha=\frac{\partial x^i}{\partial t^\alpha},\alpha=1,\dots,p$. The Lagrangian

$$L:R^{p+n+np}\to R,\left(t^\alpha,x^i,x^i_\alpha\right)\to L\left(t^\alpha,x^i,x^i_\alpha\right)$$

determines the Euler-Lagrange equations

$$\frac{\partial}{\partial t^\alpha}\frac{\partial L}{\partial x^i_\alpha}=\frac{\partial L}{\partial x^i},\ i=1,\dots,n,\ \alpha=1,\dots p$$

(second order $PDEs$ system in the n-dimensional space). We remark that in the left hand member we have a summation after the index α (trace).

4.2 An action that produces Poisson-gradient systems

Let $\alpha = 1, \ldots, p$, $i = 1, \ldots, n$,

$$u^i : T_0 \to R, t = \left(t^1, \ldots, t^p\right) \to u^i\left(t^1, \ldots, t^p\right),$$

$$u : T_0 \to R^n, u\left(t\right) = \left(u^1\left(t\right), \ldots, u^n\left(t\right)\right), u^i_\alpha = \frac{\partial u^i}{\partial t^\alpha}, \frac{\partial u}{\partial t} = \left(u^i_\alpha\right).$$

We consider the Lagrangian

$$L : T_0 \times R^n \times R^{np} \to R, \left(t^\alpha, u^i, u^i_\alpha\right) \to L\left(t^\alpha, u^i, u^i_\alpha\right),$$

$$L\left(t^\alpha, u^i, u^i_\alpha\right) = \frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2 + F(t, u\left(t\right)).$$

A function u (field) that realizes the minimum of the action

$$\varphi\left(u\right) = \int_{T_0} L\left(t, u\left(t\right), \frac{\partial u}{\partial t}\left(t\right)\right) dt^1 \wedge \cdots \wedge dt^p,$$

verifies a $PDEs$ system of Poisson-gradient type (Euler-Lagrange equations on H^1_T)

$$\Delta u\left(t\right) = \nabla F\left(t, u\left(t\right)\right),$$

together with the boundary conditions

$$u \mid_{S_i^-} = u \mid_{S_i^+}, \frac{\partial u}{\partial t} \mid_{S_i^-} = \frac{\partial u}{\partial t} \mid_{S_i^+}, i = 1, \ldots, p.$$

4.3 Periodical solutions of Poisson-gradient dynamical systems with bounded non-linearity

Let us formulate some conditions on the potential function that ensure the existence of extremals.

4 Theorem. *Suppose the potential function* $F : T_0 \times R^n \to R, (t, u) \to F\left(t, u\right)$ *satisfies four properties:*

(1) $F\left(t, u\right)$ *is measurable in* t *for any* $u \in R^n$ *and it is continuously differentiable in* u *for any* $t \in T_0$

(2) There exist the function $a \in C^1\left(R^+, R^+\right)$, *with the derivative* a' *bounded from above, and the function* $b \in C\left(T_0, R^+\right)$ *such that for any* $t \in T_0$ *and any* $u \in R^n$ *to have* $\left|F\left(t, u\right)\right| \leq a\left(\left|u\right|\right) b\left(t\right)$ *and* $\left|\nabla_u F\left(t, u\right)\right| \leq a\left(\left|u\right|\right) b\left(t\right)$

(3) There exists $g \in C^1(T_0, R)$ such that for any $t \in T_0$ and any $u \in R^n$, to have

$$|\nabla_u F(t,u)| \le g(t).$$

(4) The action $\varphi_1(u) = \int_{T_0} F(t,u(t))\, dt^1 \wedge \cdots \wedge dt^p$ is weakly lower semi-continuous. If $\int_{T_0} F(t,u)\, dt^1 \wedge \cdots \wedge dt^p \to \infty$ when $|u| \to \infty$ then the Dirichlet problem

$$\Delta u(t) = \nabla F(t, u(t)),$$

$$u\,|_{S_i^-} = u\,|_{S_i^+}, \frac{\partial u}{\partial t}\,|_{S_i^-} = \frac{\partial u}{\partial t}\,|_{S_i^+}, i = 1, \dots, p,$$

has at least a solution which minimizes the action

$$\varphi(u) = \int_{T_0} \left[\frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2 + F(t,u(t))\right] dt^1 \wedge \cdots \wedge dt^p$$

in H_T^1.

Proof. We consider $u = \overline{u} + \widetilde{u}$, where $\overline{u} = \frac{1}{T^1 \dots T^p}\int_{T_0} u(t)\, dt^1 \wedge \cdots \wedge dt^p$. Then

$$\begin{aligned}
\varphi(u) &= \int_{T_0} \left[\frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2 + F(t,u(t))\right] dt^1 \wedge \cdots \wedge dt^p \\
&= \int_{T_0} \left[\frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2 + F(t,\overline{u}) - F(t,\overline{u}) + F(t,u(t))\right] dt^1 \wedge \cdots \wedge dt^p \\
&= \int_{T_0} \left[\frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2 + F(t,\overline{u}(t))\right] dt^1 \wedge \cdots \wedge dt^p \\
&\quad + \int_{T_0}\int_0^1 \left(\nabla_u F(t, \overline{u} + s\widetilde{u}(t)), \widetilde{u}(t)\right) ds \wedge dt^1 \wedge \cdots \wedge dt^p.
\end{aligned}$$

According to property 3) from the hypothesis, we have the inequality

$$\left(\nabla_u F(t, \overline{u} + s\widetilde{u}(t)), \widetilde{u}(t)\right) \le |\nabla_u F(t, \overline{u} + s\widetilde{u}(t))|\,|\widetilde{u}(t)| \le |g(t)|\,|\widetilde{u}(t)|,$$

whence we obtain the relation

$$-|g(t)|\,|\widetilde{u}(t)| \le \left(\nabla_u F(t, \overline{u} + s\widetilde{u}(t)), \widetilde{u}(t)\right)$$

for any $t \in T_0$. By using this inequality we obtain

$$\begin{aligned}\varphi(u) &= \int_{T_0} \frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2 dt^1 \wedge \cdots \wedge dt^p + \int_{T_0} F(t,\overline{u})\, dt^1 \wedge \cdots \wedge dt^p \\ &\quad - \int_{T_0} |g(t)|\, |\widetilde{u}(t)|\, dt^1 \wedge \cdots \wedge dt^p \\ &\geq \int_{T_0} \frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2 dt^1 \wedge \cdots \wedge dt^p + \int_{T_0} F(t,\overline{u})\, dt^1 \wedge \cdots \wedge dt^p \\ &\quad - g_0 \int_{T_0} |\widetilde{u}(t)|\, dt^1 \wedge \cdots \wedge dt^p,\end{aligned}$$

where $g_0 = \max_{t \in T_0} |g(t)|$. According to the multi-time Wirtinger inequality [9], there exists $C_1 > 0$ such that

$$\int_{T_0} |\widetilde{u}(t)|\, dt^1 \wedge \cdots \wedge dt^p \leq C_1 \left(\int_{T_0} \left|\frac{\partial u}{\partial t}(t)\right|^2 dt^1 \wedge \cdots \wedge dt^p\right)^{\frac{1}{2}}.$$

This means

$$\begin{aligned}\varphi(u) \geq \int_{T_0} \frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2 dt^1 \wedge \cdots \wedge dt^p + \int_{T_0} F(t,\overline{u})\, dt^1 \wedge \cdots \wedge dt^p \\ - g_0 C_1 \left(\int_{T_0} \left|\frac{\partial u}{\partial t}(t)\right|^2 dt^1 \wedge \cdots \wedge dt^p\right)^{\frac{1}{2}}.\end{aligned}$$

Of course, if $\|u\| \to \infty$, then, from the relation $\|u\| \leq \|\overline{u}\| + \|\widetilde{u}\|$ it follows that $\|\overline{u}\| \to \infty$ or $\|\widetilde{u}\| \to \infty$. Because $\overline{u}$ is constant in R^n, we have the equalities

$$\begin{aligned}\|\overline{u}\| = \|\overline{u}\|_{W_T^{1,2}} &= \left(\int_{T_0} \left(|\overline{u}|^2 + \left|\frac{\partial \overline{u}}{\partial t}\right|\right)^2 dt^1 \wedge \cdots \wedge dt^p\right)^{\frac{1}{2}} \\ &= \left(\int_{T_0} |\overline{u}|^2\, dt^1 \wedge \cdots \wedge dt^p\right)^{\frac{1}{2}} = |\overline{u}| \left(T^1 \ldots T^p\right)^{\frac{1}{2}}.\end{aligned}$$

This means that if $\|\overline{u}\| \to \infty$, then $|\overline{u}| \to \infty$. Consequently using the hypothesis, we obtain

$$\int_{T_0} F(t,\overline{u})\, dt^1 \wedge \cdots \wedge dt^p \to \infty. \tag{5}$$

Also

$$\|\widetilde{u}\| = \left(\int_{T_0} \left(|\widetilde{u}(t)|^2 + \left|\frac{\partial \widetilde{u}}{\partial t}(t)\right|\right)^2 dt^1 \wedge \cdots \wedge dt^p\right)^{\frac{1}{2}}$$

$$= \left(\int_{T_0} \left(|\widetilde{u}(t)|^2 + \left| \frac{\partial u}{\partial t}(t) \right| \right)^2 dt^1 \wedge \cdots \wedge dt^p \right)^{\frac{1}{2}}.$$

With the Wirtinger inequality we obtain

$$\|\widetilde{u}\| \leq \left(\int_{T_0} \left(C \left| \frac{\partial \widetilde{u}}{\partial t}(t) \right|^2 + \left| \frac{\partial u}{\partial t}(t) \right| \right)^2 dt^1 \wedge \cdots \wedge dt^p \right)^{\frac{1}{2}}$$

$$= (C+1) \left(\int_{T_0} \left| \frac{\partial u}{\partial t}(t) \right|^2 dt^1 \wedge \cdots \wedge dt^p \right)^{\frac{1}{2}}.$$

The condition $\|\widetilde{u}\| \to \infty$ implies

$$\int_{T_0} \left| \frac{\partial u}{\partial t}(t) \right|^2 dt^1 \wedge \cdots \wedge dt^p \to \infty. \tag{6}$$

From the hypothesis and (5) or (6), it follows that if $\|u\| \to \infty$, then $\varphi(u) \to \infty$. So φ is a coercitive application. This means that φ has a minimizing bounded sequence (u_k). The Hilbert space H_T^1 is reflexive. By consequence, the sequence (u_k) (or one of his subsequence) is weakly convergent in H_T^1 with the limit u. Because

$$\varphi_2(u) = \int_{T_0} \delta_{ij} \delta^{\alpha\beta} \frac{\partial u^i}{\partial t^\alpha}(t) \frac{\partial u^j}{\partial t^\beta}(t)\, dt^1 \wedge \cdots \wedge dt^p$$

is a convex functional, it follows that φ_2 is weakly lower semi-continuous, so that the action

$$\varphi(u) = \varphi_1(u) + \varphi_2(u)$$

is weakly lower semi-continuous and $\varphi(u) \leq \underline{\lim}\, \varphi(u_k)$. This means that u is minimum point of φ.

We build the function

$$\Phi : [-1, 1] \to R,$$

$$\Phi(\lambda) = \varphi(u + \lambda v)$$
$$= \int_{T_0} \left[\frac{1}{2} \left| \frac{\partial}{\partial t}(u(t) + \lambda v(t)) \right|^2 + F(t, u(t) + \lambda v(t)) \right] dt^1 \wedge \cdots \wedge dt^p,$$

where $v \in C_T^\infty$. The point $\lambda = 0$ is a critical point of Φ if and only if the point u is a critical point of φ. Consequently

$$0 = \langle \varphi'(u), v \rangle = \int_{T_0} \left[\delta^{\alpha\beta} \delta_{ij} \frac{\partial u^i}{\partial t^\alpha} \frac{\partial v^j}{\partial t^\beta} + \delta_{ij} \nabla^i F(t, u(t))\, v^j(t) \right] dt^1 \wedge \cdots \wedge dt^p,$$

for all $v \in H^1_T$ and hence for all $v \in C^\infty_T$. According to the definition of the weak divergence, i.e.,

$$\int_{T_0} \delta^{\alpha\beta}\delta_{ij}\frac{\partial u^i}{\partial t^\alpha}\frac{\partial v^j}{\partial t^\beta}dt^1\wedge\cdots\wedge dt^p = -\int_{T_0}\delta^{\alpha\beta}\delta_{ij}\frac{\partial^2 u^i}{\partial t^\alpha \partial t^\beta}v^j dt^1\wedge\cdots\wedge dt^p,$$

the Jacobi matrix function $\frac{\partial u}{\partial t}$ has weak divergence (the function u has a weak Laplacian) and

$$\triangle u\,(t) = \nabla F\,(t, u\,(t))$$

a.e. on T_0. Also, the existence of weak derivatives $\frac{\partial u}{\partial t}$ and $\triangle u$ implies that

$$u\mid_{S_i^-} = u\mid_{S_i^+}, \frac{\partial u}{\partial t}\mid_{S_i^-} = \frac{\partial u}{\partial t}\mid_{S_i^+}.$$

QED

5 Remark. If the function u is at least of class C^2, then the definition of the weak divergence of the Jacobian matrix $\dfrac{\partial u}{\partial t}$ (or of the weak Laplacian $\triangle u$) coincides with the classical definition. This fact is obvious if we have in mind the formula of *integration by parts*

$$\int_{T_0}\delta^{\alpha\beta}\delta_{ij}\frac{\partial u^i}{\partial t^\alpha}\frac{\partial v^j}{\partial t^\beta}dt^1\wedge\cdots\wedge dt^p = \int_{T_0}\delta^{\alpha\beta}\delta_{ij}\frac{\partial}{\partial t^\alpha}\left(\frac{\partial u^i}{\partial t^\alpha}v^j\right)dt^1\wedge\cdots\wedge dt^p$$
$$-\int_{T_0}\delta^{\alpha\beta}\delta_{ij}\frac{\partial^2 u^i}{\partial t^\alpha\partial t^\beta}v^j dt^1\wedge\cdots\wedge dt^p.$$

References

[1] L. V. Kantorovici: G. P. Akilov: *Analiză funcţională*, Editura ştiinţifică şi enciclopedică, Bucureşti (1980).

[2] J. Mawhin, M. Willem: *Critical Point Theory and Hamiltonian Systems*, Springer-Verlag (1989).

[3] S. G. Mihlin: *Ecuaţii liniare cu derivate parţiale*, Editura ştiinţifică şi enciclopedică, Bucureşti (1983).

[4] R. E. Showalter: *Hilbert Space Methods for Partial Differential Equations*, Electronic Journal of Differential Equations Monograph **01** (1994).

[5] C. Udrişte: *M. Postolache, Atlas of Magnetic Geometric Dynamics*, Geometry Balkan Press, Bucharest (2001).

[6] C. Udrişte: *From Integral Manifolds and Metrics to Potential Maps*, Conference Michigan State University, April 13-27, 2001; Eleventh Midwest Geometry Conference, Wichita State University, April 27-29, 2001; Atti dell'Accademia Peloritana dei Pericolanti, Classe 1 Scienze Fis. Mat. e Nat., n. 81-82, **A 01006** (2003–2004), 1–14.

[7] C. Udrişte, I. Duca: *Periodical Solutions of Multi-Time Hamilton Equations*, Analele Universitaţii Bucureşti, no. 55 **1** (2005), 178–188.

[8] C. Udrişte, M. Ferrara, D. Opriş: *Economic Geometric Dynamics*, Geometry Balkan Press, Bucharest (2004).

[9] C. Udrişte, A-M. Teleman: *Hamilton Approaches of Fields Theory*, IJMMS, **57** (2004), 3045–3056; ICM Satellite Conference in Algebra and Related Topics, University of Hong-Kong, August 13-18, 2002.

[10] C. Udrişte, M. Neagu: *From PDE Systems and Metrics to Generalized Field Theories*, `http://xxx.lanl.gov/abs/math.DG/0101207`.

[11] C. Udrişte: *Nonclassical Lagrangian Dynamics and Potential Maps*, The Conference in Mathematics in Honour of Professor Radu Roşca on the Occasion of his Ninetieth Birthday, Katholieke University Brussel, Katholieke University Leuven, Belgium, December 11-16, 1999; WSEAS Transactions on Mathematics, no. 7, **1** (2008), 12–18. `http://xxx.lanl.gov/math.DS/0007060`.

[12] C. Udrişte: *Solutions of DEs and PDEs as Potential Maps Using First Order Lagrangians*, Centenial Vrânceanu, Romanian Academy, University of Bucharest, June 30-July 4, (2000); Balkan Journal of Geometry and Its Applications no. 6 **1** 2001, 93–108; `http://xxx.lanl.gov/math.DS/0007061`.