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Simultaneous approximation and interpolation from lattices

Márcia S. Kashimoto
Departamento de Matemática e Computação, Universidade Federal de Itajubá, Caixa postal 50, 37500-903, Itajubá - MG, Brazil
kaxixi@unifei.edu.br

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Abstract. We prove that if a sublattice L of a weighted space is an interpolating family, then simultaneous approximation and interpolation is possible from L.

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1 Introduction and preliminaries

It is well known that sublattices which are dense vector subspaces of weighted spaces have the simultaneous approximation and interpolation property. It can be proved by using Deutsch's theorem [1]. We generalize this result to interpolating sublattices by using the Kakutani-Stone theorem [3].

Throughout this note we assume, unless stated otherwise, that X is a locally compact Hausdorff space. We denote by $C(X)$ the space of all continuous real-valued functions on X.

Let us recall that a subset L of $C(X)$ is called a sublattice if $f, g \in L$ implies $f \wedge g \in L$ and $f \vee g \in L$, where $(f \wedge g)(x) = \inf\{f(x), g(x)\}$ and $(f \vee g)(x) = \sup\{f(x), g(x)\}$ for every $x \in X$.

An upper semicontinuous real-valued function f on X is said to *vanish at infinity* if for every $\varepsilon > 0$, the closed subset $\{ x \in X \mid |f(x)| \geq \varepsilon \}$ is compact.

In what follows, we present the concept of *weighted spaces* as developed by Nachbin [2].

Let V be a set of non-negative upper semicontinuous functions on X. Each element of V is called a *weight*. We assume that V is directed, in the sense that, given $v_1, v_2 \in V$, there exist $\lambda > 0$ and $v \in V$ such that $v_1 \leq \lambda v$ and $v_2 \leq \lambda v$.

The set V is *p*ointwise strictly positive if for every $x \in X$, there is $v \in V$ such that $v(x) > 0$.

The weighted space $CV_\infty(X)$ is the vector subspace of $C(X)$ consisting of all functions f such that vf vanishes at infinity for each $v \in V$. It is endowed with the locally convex topology ω_V generated by the seminorms

$$\begin{array}{rcl} p_v : CV_\infty(X) & \to & \mathbb{R}^+ \\ f & \mapsto & \sup\{\, v(x)|f(x)| \mid x \in X \,\} \end{array}$$

for each $v \in V$.

In the following we present some examples of weighted spaces.

(a) If V consists of the constant function $\mathbf{1}$, defined by $\mathbf{1}(x) = 1$ for all $x \in X$, then $CV_\infty(X)$ is $C_0(X)$, the vector subspace of all functions in $C(X)$ that vanish at infinity. In particular, if X is compact then $CV_\infty(X) = C(X)$. The corresponding weighted topology is the topology of uniform convergence on X.

(b) Let V be the set of characteristic functions of all compact subsets of X. Then the weighted space $CV_\infty(X)$ is $C(X)$ endowed with the compact-open topology.

(c) If $V = \{\, v \in C_0(X) \mid v \geq 0 \,\}$, then $CV_\infty(X)$ is $C_b(X)$, the subspace of $C(X)$ consisting of those functions which are bounded on X. The corresponding weighted topology is the strict topology.

For more information on weighted spaces we refer the reader to [2, 5].

A subset L of $CV_\infty(X)$ is an *interpolating family for* $CV_\infty(X)$ if given any nonempty finite subset $S \subset X$ and any $f \in CV_\infty(X)$, there exists $g \in L$ such that $g(x) = f(x)$ for all $x \in S$.

A subset L of $CV_\infty(X)$ has *property SAI* if for every $f \in CV_\infty(X)$, $v \in V$, $\varepsilon > 0$ and every nonempty finite subset S of X, there exists $g \in L$ such that $p_v(f - g) < \varepsilon$ and $f(x) = g(x)$ for all $x \in S$.

The closure of a subset F of any topological space considered here will be denoted by $\overline{F}$.

2 The result

We need the Kakutani-Stone theorem for sublattices of weighted spaces which can be found in [3, Theorem 2].

1 Lemma. *Let X be a completely regular space and V be a pointwise strictly positive set of weights. Let L be a sublattice of $CV_\infty(X)$ and $f \in CV_\infty(X)$. Then f belongs to the closure of L in $CV_\infty(X)$ if and only if, for any $x, y \in X$ and $\varepsilon > 0$, there exists $g \in L$ such that*

$$|g(x) - f(x)| < \varepsilon,$$
$$|g(y) - f(y)| < \varepsilon.$$

Now we establish the following result whose proof is inspired by that of [4, Theorem 6].

2 Theorem. *Let V be a pointwise strictly positive set of weights and L a sublattice of $CV_\infty(X)$. If L is an interpolating family for $CV_\infty(X)$, then L has property SAI.*

Proof. Let S be a nonempty finite subset of X and $f \in CV_\infty(X)$. Consider the set

$$M := \{ h \in L \mid f(x) = h(x), \text{ for all } x \in S \}.$$

Since L is an interpolating family for $CV_\infty(X)$, for each pair $x, y \in X$ there exists $h_{xy} \in L$ such that $h_{xy}(s) = f(s)$ for all $s \in S \cup \{x, y\}$. Note that $h_{xy} \in M$ and for any $\varepsilon > 0$, we have $|f(x) - h_{xy}(x)| = 0 < \varepsilon$ and $|f(y) - h_{xy}(y)| = 0 < \varepsilon$. Thus, since M is a sublattice of $CV_\infty(X)$, it follows from Lemma 1 that there exists $g \in M$ such that $p_v(f - g) < \varepsilon$. By the definition of M, we conclude that $g \in L$ and $g(x) = f(x)$ for all $x \in S$. Therefore, L has property SAI. QED

3 Remark. Let V be a pointwise strictly positive set of weights. Every ω_V-dense vector subspace of $CV_\infty(X)$ is an interpolating family for $CV_\infty(X)$. Indeed, let $S = \{x_1, \dots, x_n\}$ be a subset of X and G an ω_V-dense vector subspace of $CV_\infty(X)$. Consider the following linear mapping

$$\begin{aligned} T : CV_\infty(X) &\to \mathbb{R}^n \\ f &\mapsto (f(x_1), \dots, f(x_n)). \end{aligned}$$

We claim that T is continuous. To see it, let $\mathbb{R}^n$ endowed with the norm

$$\|x\| = \max \{ |x_i| \mid 1 \le i \le n \}, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

Given $\varepsilon > 0$, let $W(0; \varepsilon) := \{ x \in \mathbb{R}^n \mid \|x\| < \varepsilon \}$. Since V is a pointwise strictly positive set of weights, for each $x_i \in S$ there exists $v_i \in V$ such that $v_i(x_i) > 0$. Choose $0 < \delta_i < \varepsilon v_i(x_i)$, $i = 1, \dots, n$. Consider the following neighborhood of the origin, $U := \bigcap_{i=1}^n B_{v_i,\delta_i}(0)$ where $B_{v_i,\delta_i}(0) = \{ f \in CV_\infty(X) \mid p_{v_i}(f) < \delta_i \}$. If $g \in U$, then $\sup\{ v_i(x)|g(x)| \mid x \in X \} < \delta_i$, $i = 1, \dots, n$. Therefore, $\|(g(x_1), \dots, g(x_n))\| < \varepsilon$. This implies that $T(U) \subset W(0; \varepsilon)$, and so T is continuous at the origin. Moreover, $T(G)$ is closed because it is a vector subspace of $\mathbb{R}^n$. Then by density of G and continuity of T, it follows that

$$T(CV_\infty(X)) = T(\overline{G}) \subset \overline{T(G)} = T(G),$$

Therefore, for any $f \in CV_\infty(X)$, there exists $g \in G$ such that $(f(x_1), \dots, f(x_n)) = (g(x_1), \dots, g(x_n))$.

4 Corollary. *Let V be a pointwise strictly positive set of weights. If L is an ω_V-dense vector sublattice of $CV_\infty(X)$, then L has property SAI.*

PROOF. Since L is an ω_V-dense vector subspace of $CV_\infty(X)$, it follows from Remark 3 that L is an interpolating family for $CV_\infty(X)$. The result follows from Theorem 2. $\boxed{QED}$

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