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Homology groups of translation planes and flocks of quadratic cones, II; j-planes

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Abstract. The set of j-planes with spreads in $PG(3, K)$, for K a field admitting a quadratic field extension K^+ is shown to be equivalent to the set of all det K^+-monomial partial flocks of a quadratic cone. Using this connection, when K is $GF(2^r)$, the set of j-planes is determined and $j = 0, 1$, or 2 and correspond to the linear, Walker/Betten or Payne conical flocks, respectively. When K is the field of real numbers, the set of j-planes is completely determined and j is any real number $> -\frac{1}{2}$.

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1 Introduction

The 'geometry' of flocks of quadratic cones has now reached into many diverse areas of incidence geometry. For example, it is known that if there is a translation plane of order q^2 with spread in $PG(3, q)$ that admits a Baer group of order q (fixes a Baer subplane pointwise) there is a corresponding flock of a quadratic cone. In Johnson [2] it was shown that the $q - 1$ orbits of length q of the Baer group on the components of the spread define reguli that share the pointwise fixed subspace, which, in turn, defines a partial flock of deficiency one of a quadratic cone. Payne and Thas [7], then show that any deficiency one partial flock may always be extended to a flock of a quadratic cone. This means that the net of degree $q+1$ defined by the components of the Baer subplane is a regulus net and by derivation of this net, there is an associated translation plane with spread in $PG(3, q)$ where the Baer group now becomes an affine elation group. We call such elation groups 'regulus-inducing'.

Hence, translation planes admitting regulus-inducing elation groups are e- quivalent to flocks of quadratic cones. However, using the fundamental analysis of Baker, Ebert and Penttila [1], it has now been shown in Johnson [4] that it is always possible to connect flocks of quadratic cones with translation planes admitting cyclic homology groups.

1 Theorem (Johnson [4]). *The set of translation planes of order q^2 with* *spread in $PG(3,q)$ that admit cyclic affine homology groups of order $q+1$ is equivalent to the set of flocks of a quadratic cone.*

So, there are intrinsic connections and a complete equivalence between the set of translation planes q^2 admitting regulus-inducing elation groups and with the set of translation planes of order q^2 admitting cyclic homology groups of order $q+1$.

An important class of translation planes of order q^2 admitting cyclic homology groups of order $q+1$ are the j-planes that cyclic collineation groups of order q^2-1, within which there is an affine cyclic homology group of order $q+1$. Another equally important subset is the class of planes obtained from a Desarguesian plane by the replacement of a $(q+1)$-nest of reguli. These planes admit a collineation group of order $(q+1)(q^2-1)/2$, that contains two distinct cyclic affine homology groups of order $q+1$.

One of the questions that we are concerned with in this article is how the j-planes are related to flocks of quadratic cones. We consider this in the general case of j-planes in $PG(3,K)$, for K a field that admits a quadratic extension.

In Baker, Ebert and Penttila [1], the connection with a flock is not particularly with a given translation plane but with a set of translation planes and the connection between what are called 'regular hyperbolic fibrations with constant back'.

A 'hyperbolic fibration' is a set $\mathcal{Q}$ of $q-1$ hyperbolic quadrics and two carrying lines L and M such that the union $L \cup M \cup Q$ is a cover of the points of $PG(3,q)$. The term 'regular hyperbolic fibration' is used to describe hyperbolic fibrations such that for each of its $q-1$ quadrics, the induced polarity interchanges L and M. When this occurs, and (x_1,x_2,y_1,y_2) represent points homogeneously, the hyperbolic quadrics have the form $V(x_1^2a_i + x_1x_2b_i + x_2^2c_i + y_1^2e_i + y_1y_2f_i + y_2g_i)$, for $i = 1,2,\ldots,q-1$ (the variety defined by the quadrics). When $(e_i,f_i,g_i) = (e,f,g)$ for all $i = 1,2,\ldots,q-1$. The regular hyperbolic quadric is said to have 'constant back half'.

We recall the principle theorem of Baker, Ebert and Penttila [1].

2 Theorem (Baker, Ebert, Penttila [1])**.**

(1) Let $\mathcal{H}$: $V(x_1^2a_i + x_1x_2b_i + x_2^2c_i + y_1^2e + y_1y_2f + y_2g)$, for $i = 1,2,\ldots,q-1$ be a regular hyperbolic fibration with constant back half.

Consider $PG(3,q)$ as (x_1,x_2,x_3,x_4) and let C denote the quadratic cone with equation $x_1x_2 = x_3^2$.

Define

$$\pi_0 : x_4 = 0,\ \pi_i : x_1a_i + x_2c_i + x_3b_i + x_4 = 0,\ \textit{for}\ 1,2,\ldots,q-1.$$

Then

$$\{\pi_j, j = 0, 1, 2, \ldots, q-1\},$$

is a flock of the quadratic cone C.

(2) *Conversely, if $\mathcal{F}$ is a flock of a quadratic cone, choose a representation as $\{\pi_j, j = 0, 1, 2, \ldots, q-1\}$ above. Then, choosing any convenient constant back half (e, f, g), and defining $\mathcal{H}$ as $V(x_1^2 a_i + x_1 x_2 b_i + x_2^2 c_i + y_1^2 e + y_1 y_2 f + y_2 g)$, for $i = 1, 2, \ldots, q-1$, then $\mathcal{H}$ is a regular hyperbolic fibration with constant back half.*

But all of this works for infinite translation planes as well. In particular, it is possible to construct infinite flocks of quadratic cones and translation planes admitting regulus-inducing elation groups. But here we are interested in how certain homology groups in translation planes produce flocks of quadratic cones in the finite case and also what happens in the infinite case. In particular, we consider regulus-inducing homology groups in $PG(2, K)$, where K is the field of real numbers and show how easy it is to determine such planes from an arbitrary real function that is continuous and non-decreasing on the positive real numbers.

Is also of interest to consider collineation groups isomorphic to the multiplicative group of a field L, since often the group becomes what we call a H-group, where H is a multiplicative endomorphism of $L - \{0\}$. When L is either finite or the field of real numbers, such that H is Lebesque integrable then $H(t) = ct^j$, where j is a real number. In this setting, we obtain a 'j-plane'. In this situation, we may utilize the connection between regulus-inducing homology groups and partial flocks of quadratic cones to show that the associated partial flocks are monomial. When L is isomorphic to $GF(2^r)$, all monomial flocks are known by the work of Penttila and Storme [8]. Hence, this implies that all j-planes are determined. When L is the field of real numbers, again analysis of the associated $\det K^+$-partial flock completely determines all possible j-planes.

2 Homology groups and regular hyperbolic fibrations revisited

In order to understand the method of going back and forth between flocks of quadratic cones and translation planes admitting affine homology groups, we remind the reader of the constructions of Johnson [4] by revisiting the various results.

3 Lemma.

(1) *Let $\mathcal{H}$ be a hyperbolic fibration of $PG(3,q)$ (a covering of the points by a set λ of hyperbolic quadrics union two disjoint carrying lines). For each quadric in λ, choose one of the two reguli (a regulus or its opposite). The union of these reguli and the carrying lines form a spread in $PG(3,q)$.*

(2) *Conversely, any spread in $PG(3,q)$ that is a union of hyperbolic quadrics union two disjoint carrying lines produces a hyperbolic fibration.*

4 Lemma. *Let π be a translation plane with spread in $PG(3,K)$, for K a field isomorphic to $GF(q)$, that admits a cyclic affine homology group H. Let Γ be any H-orbit of components.*

(1) *Then there is a unique Desarguesian spread Σ containing Γ and the axis and coaxis of H.*

(2) *Furthermore, we may represent the coaxis, axis and Γ as follows:*

$$x = 0, y = 0, y = xm; m^{q+1} = 1; m \in K^+$$

where m is in the field K^+, a 2-dimensional quadratic extension of K, so K^+ is isomorphic to $GF(q^2)$.

(3) *A basis may be chosen so that Σ may be coordinatized by K^+ as $\left[\begin{smallmatrix} u & t \\ ft & u+gt \end{smallmatrix}\right]$, for all u, t in K, for suitable constants f and g.*

(4) *If $\{1, e\}$ is a basis for K^+ over K then $e^2 = eg + f$, and $e^\sigma = -e + g$, $e^{\sigma+1} = -f$. Furthermore, $(et + u)^{q+1} = 1$ if and only in matrix form $et + u = \left[\begin{smallmatrix} u & t \\ ft & u+gt \end{smallmatrix}\right]$, such that $u(u + gt) - ft^2 = 1$.*

(5) *The opposite regulus*

$$y = x^q m; m^{q+1} = 1,$$

may be written in the form

$$y = x \begin{bmatrix} 1 & 0 \\ g & -1 \end{bmatrix} \begin{bmatrix} u & t \\ ft & u + gt \end{bmatrix}; u(u + gt) - ft^2 = 1.$$

5 Lemma.

(1) *The spread for π has the following form:*

$$x = 0, y = 0, y = xM_i \begin{bmatrix} u & t \\ ft & u + gft \end{bmatrix}; u(u + gt) - ft^2 = 1$$

and M_i a set of 2×2 matrices over K, where $i \in \rho$, some index set., Let

$$R_i = \left\{ y = xM_iT; T^{q+1} = 1 \right\}, \text{ for } i \in \rho.$$

(2) *Then R_i is a regulus in $PG(3,K)$.*

6 Lemma. *The quadratic form for R_i is*

$$V\left(xM_i\begin{bmatrix}1 & g\\ 0 & -f\end{bmatrix}M_i^t x^t - y\begin{bmatrix}1 & g\\ 0 & -f\end{bmatrix}y^t\right).$$

$xM_i\left[\begin{smallmatrix}1 & g\\ 0 & -f\end{smallmatrix}\right]^t M_i^t x^t$ *is self-transpose and thus equal to* $xM_i\left[\begin{smallmatrix}1 & g\\ 0 & -f\end{smallmatrix}\right]^t M_i^t x^t$.

7 Theorem. *Let π be a translation plane with spread in $PG(3,K)$, for K a field. Assume that π admits an affine homology group H so that some orbit of components is a regulus in $PG(3,K)$.*

(1) *Then π produces a regular hyperbolic fibration with constant back half.*

(2) *Conversely, each translation plane obtained from a regular hyperbolic fibration with constant back half admits an affine homology group H, one orbit of which is a regulus in $PG(2,K)$.*

H is isomorphic to a subgroup of the collineation group of a Pappian spread Σ, coordinatized by a quadratic extension field K^+,

$$H \simeq \left\langle g^{\sigma+1};\, g \in K^+ - \{0)\right\rangle,$$

where σ is the unique involution in $Gal_K K^+$.

(3) *Let $\mathcal{H}$ be a regular hyperbolic fibration with constant back half of $PG(3,K)$. The subgroup of $\Gamma L(4,K)$ that fixes each hyperbolic quadric of a regular hyperbolic fibration $\mathcal{H}$ and acts trivially on the front half is isomorphic to*

$$\left\langle \rho, \left\langle g^{\sigma+1}; g \in K^+ - \{0)\right\rangle\right\rangle,$$

where ρ is defined as follows: If $e^2 = ef + g$, f, g in K and $\langle e, 1\rangle_K = K^+$ then ρ is $\left[\begin{smallmatrix}I & 0\\ 0 & P\end{smallmatrix}\right]$, where

$$P = \begin{bmatrix}1 & 0\\ g & -1\end{bmatrix}.$$

In particular, $\left\langle g^{\sigma+1}; g \in K^+ - \{0)\right\rangle$ fixes each regulus and opposite regulus of each hyperbolic quadric of $\mathcal{H}$ and ρ inverts each regulus and opposite regulus of each hyperbolic quadric.

3 General matrix forms

If we consider the arbitrary case of spreads in $PG(3,K)$, the connection between translation planes admitting appropriate affine homology groups and flocks of quadratic cones is not direct when K is infinite. We recall the pertinent theorem of Johnson [1].

8 Theorem. *A regular hyperbolic fibration with constant back half in $PG(3,K)$, K a field, with carrier lines $x=0, y=0$, may be represented as follows:*

$$V\left(x\begin{bmatrix}\delta & g(\delta)\\ 0 & -f(\delta)\end{bmatrix}x^t - y\begin{bmatrix}1 & g\\ 0 & -f\end{bmatrix}y^t\right)$$

$$\text{for all } \delta \text{ in } \left\{\begin{bmatrix}u & t\\ ft & u+gt\end{bmatrix}^{\sigma+1}; u,t\in K, (u,t)\neq(0,0)\right\},$$

where

$$\left\{\begin{bmatrix}\delta & g(\delta)\\ 0 & -f(\delta)\end{bmatrix}; \delta\in\left\{\begin{bmatrix}u & t\\ ft & u+gt\end{bmatrix}^{\sigma+1}; u,t\in K, (u,t)\neq(0,0)\right\}\right\}\cup\left\{\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\right\}$$

corresponds to a partial flock of a quadratic cone in $PG(3,K)$, and where f and g are functions on $\det K^+$.

9 Theorem. *The correspondence between any spread π in $PG(3,K)$ corresponding to the hyperbolic fibration and the partial flock of a quadratic cone in $PG(3,K)$ is as follows:*

If π is

$$x=0, y=0, y=x\begin{bmatrix}u & t\\ F(u,t) & G(u,t)\end{bmatrix}$$

then the partial flock is given by $\begin{bmatrix}\delta_{u,t} & g(\delta_{u,t})\\ 0 & -f(\delta_{u,t})\end{bmatrix}$ *with*

$$\delta_{u,t} = \det\begin{bmatrix}u & t\\ ft & u+gt\end{bmatrix},$$

$$g(\delta_{u,t}) = g(uG(u,t)+tF(u,t)) + 2(uF(u,t)-tfG(u,t)),$$

$$-f(\delta_{u,t}) = \delta_{F(u,t),G(u,t)},$$

where

$$\delta_{M_i} = \det M_i$$

and

$$\delta_{F(u,t),G(u,t)} = \det\begin{bmatrix}F(u,t) & G(u,t)\\ fG(u,t) & F(u,t)+gG(u,t)\end{bmatrix}\in\det K^+.$$

10 Theorem. *If we have a hyperbolic fibration in $PG(3,K)$, there are corresponding functions given in the previous theorem such that the corresponding functions*

$$\phi_s(t) = s^2t + sg(t) - f(t)$$

are injective for all s in K and for all $t \in \det K^+$.

Indeed, the functions restricted to $\det K^+$ *are surjective on* $\det K^+$.

Furthermore, we obtain.

11 Theorem. *Any partial flock of a quadratic cone in $PG(3,K)$, with defining set λ (i.e., so t ranges over λ and planes of the partial flock are defined via functions in t) equal to* $\det K^+$, *whose associated functions on* $\det K^+$, *as above, are surjective on* $\det K^+$ *(K^+ some quadratic extension of K), produces a regular hyperbolic fibration in $PG(3,K)$ with constant back half.*

4 Real hyperbolic fibrations

Let $\mathcal{R}$ denote the field of real numbers. In Johnson and Liu [5], there is quite a variety of flocks of quadratic cones in $PG(3,\mathcal{R})$.

Let $-f(t)$ denote a continuous, non-decreasing function on the reals such that $f(0) = 0$, such that $\lim_{t\to\pm\infty}(-f(t)) = \pm\infty$. It is shown in Johnson and Liu [5] that the functions ϕ_s are bijective, so that there is an associated flock of a quadratic cone in $PG(3,R)$. Let $-f(1) = -f$. Then

$$\left\{ \begin{bmatrix} u & t \\ ft & u \end{bmatrix}; u,t \in R \right\},$$

is a field, and note that the determinant is $u^2 - t^2f$, which is non-negative, since $-f(1) > 0$. Furthermore, when $t > 0$ if and only if $-f(t) > 0$, so it follows the conditions of the theorem of the previous section are valid. Thus, there is an induced hyperbolic fibration over the reals $\mathcal{R}$

We first observe that if

$$C_1 = \left\{ (v,s); v^2 + -fs^2 = 1 \right\},$$

is an ellipse with center $(0,0)$ in the real affine plane Π and $P = (u,t)$ is any point of Π then the line joining (u,t) and $(0,0)$ nontrivially intersects C_1 in a point (v,s). This means that there is a real number k such that $k(v,s) = (u,t)$. This implies that any spread in $PG(3,\mathcal{R})$, that admits an affine homology group one orbit of which is a regulus then the spread is a union of the axis and coaxis $x = 0, y = 0$ and a union of reguli, where the reguli are defined as follows:

$$D_k; y = x \begin{bmatrix} k & 0 \\ F(k) & G(k) \end{bmatrix} T,$$

such that T is a determinant 1 matrix in the associated field

$$\left\{ \begin{bmatrix} u & t \\ ft & u \end{bmatrix} ; u, t \in \mathcal{R} \right\}.$$

We have chosen the notation so that $F(k,0) = F(k)$ and $G(k,0) = G(k)$. Recall that we have generally:

$$\begin{bmatrix} \delta_{u,t} & g(\delta_{u,t}) \\ 0 & -f(\delta_{u,t}) \end{bmatrix} ; \delta_{u,t} = \det \begin{bmatrix} u & t \\ ft & u + gt \end{bmatrix} ,$$
$$g(\delta_{u,t}) = g(uG(u,t) + tF(u,t)) + 2(uF(u,t) - tfG(u,t)) ,$$
$$-f(\delta_{u,t}) = \delta_{F(u,t),G(u,t)} ,$$

where $\delta_{M_i} = \det M_i$, and

$$\delta_{F(u,t),G(u,t)} = \det \begin{bmatrix} F(u,t) & G(u,t) \\ fG(u,t) & F(u,t) + gG(u,t) \end{bmatrix} .$$

It then follows that $F(u) = 0$ for all u and $G(u)^2(-f) = -f(u^2)$. For convenience, we note that for any u, t, the derivation of the corresponding regulus of a given hyperbolic quadric spread will change $G(u,t)$ to $-G(u,-t)$, since the involution σ is given by the matrix $\left[\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right]$. In other words, we may take as one spread

$$G(u) = \sqrt{f(u^2)/f}.$$

Actually, $\begin{bmatrix} u & 0 \\ 0 & G(u) \end{bmatrix}$ and $\begin{bmatrix} u & 0 \\ 0 & G(u) \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -u & 0 \\ 0 & -G(u) \end{bmatrix}$, are in the same regulus (hence, we require $G(-u) = -G(u)$). That is, we may assume that u is positive. Therefore, we may assume that we also have $\begin{bmatrix} -u & 0 \\ 0 & -\sqrt{f((-u)^2)/f} \end{bmatrix}$ within the spread. Hence, we obtain the following theorem:

12 Theorem. *Let $-f(x)$ be any continuous function on the reals $\mathcal{R}$, such that $f(0) = 0$, which is non-decreasing and* $\lim_{x \to \pm\infty} -f(x) = \pm\infty$.

Then there is a spread of $PG(3, \mathcal{R})$

$$x = 0, y = 0, y = x \begin{bmatrix} u & 0 \\ 0 & \sqrt{f(u^2)/f} \end{bmatrix} T, \ u \geq 0$$

where $f = f(1)$ and

$$T = \left\{ \begin{bmatrix} u & t \\ ft & u \end{bmatrix} ; u^2 - ft^2 = 1 \right\}.$$

Furthermore, the spread produces a hyperbolic fibration.

Let $h(u) = \sqrt{f(u^2)/f}$ for $u > 0$ and assume that $h(-u) = -h(u)$. Then

$$\begin{bmatrix} k & 0 \\ 0 & h(k) \end{bmatrix} \begin{bmatrix} v & s \\ fs & v \end{bmatrix} = \begin{bmatrix} kv & ks \\ h(k)fs & h(k)v \end{bmatrix},$$

where the determinant of $\left[\begin{smallmatrix} u & t \\ ft & u \end{smallmatrix}\right]$ is 1.

In general, the determinant of $\left[\begin{smallmatrix} u & t \\ ft & u \end{smallmatrix}\right]$ is positive, hence, for $k > 0$, there is a given matrix with determinant $k = \delta_{u,t}$. Now assume that H is a multiplicative homomorphism on $\mathcal{R} - \{0\}$. Then

$$G = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & H(\delta_{u,t})^{-1} & 0 & 0 \\ 0 & 0 & u & t \\ 0 & 0 & ft & u \end{bmatrix} ; u, t \in \mathcal{R} - \{0\} \right\}$$

is a group isomorphic to $\mathcal{R}^+$. Moreover, the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & H(\delta_{u,t}) \end{bmatrix} \begin{bmatrix} u & t \\ ft & u \end{bmatrix} = \begin{bmatrix} u & t \\ H(\delta_{u,t})ft & H(\delta_{u,t})u \end{bmatrix},$$

has determinant $\delta_{u,t} H(\delta_{u,t}) = kH(k^2)$. Now let $kv = u$ and $ks = t$, and $h(k) = kH(k^2)$. This means that

$$f(k^2) = fk^2 H(h^2)^2,$$

or rather that

$$f(v) = fvH(v)^2,\ v > 0.$$

In other words, suppose H is a multiplicative endomorphism defined on the positive real numbers. Then there is an associated collineation group G isomorphic to $\mathcal{R} - \{0\}$, which fixes two components $x = 0, y = 0$ and acts transitively on the remaining components of the spread. Conversely, let h be a multiplicative endomorphism on the positive real numbers such that $h(u) > 0$, where $u > 0$.

Now to connect to the notation:

$$x = 0, y = 0, y = x \begin{bmatrix} u & 0 \\ 0 & \sqrt{f(u^2)/f} \end{bmatrix} T,\ u \geq 0,$$

we would require that

$$f(v) = fvH(v)^2,\ v > 0.$$

is continuous on the positive reals, is non-decreasing and has range all positive reals. If H is a function differentiable on the positive reals then $H(v) = v^j$, for some real number. In this case, we consider

$$f(v) = fv^{2j+1} \text{ for } v > 0.$$

To be non-decreasing, we would require that $f(2j+1) > 0$ and to have range all positive reals requires that $f > 0$ and so

$$2j + 1 > 0.$$

As this function may be extended to a continuous function with the property required, we obtained an associated hyperbolic fibration plane.

13 Theorem. *Let h be an endomorphism on the positive real numbers such that $f \cdot h(\sqrt{v})^2$, for $v > 0$ is continuous, strictly increasing and surjective on the positive real numbers. Then, there is an associated flock and partial flock and therefore an associated hyperbolic fibration.*

We generalize this over any field that admits a quadratic extension field K^+ with matrix field

$$\left\{ \begin{bmatrix} u & t \\ ft & u+gt \end{bmatrix}; u, t \in K \right\}.$$

14 Definition. Let K be a field which admits a quadratic extension field K^+. Consider the following group

$$\left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & H(\delta_{u,t})^{-1} & 0 & 0 \\ 0 & 0 & u & t \\ 0 & 0 & ft & u+gt \end{bmatrix};\ x^2 + xg - f \text{ is } K\text{-irreducible},\ u, t \in K,\ \text{not both } 0, \right.$$

$$\left. H \text{ an endomorphism on } \left\{ \delta_{u,t} = \det \begin{bmatrix} u & t \\ ft & u+gt \end{bmatrix}; u, t \in K \right\} \right\}.$$

If

$$\left\{ x = 0, y = 0, y = x \begin{bmatrix} 1 & 0 \\ 0 & H(\delta_{u,t}) \end{bmatrix} \begin{bmatrix} u & t \\ ft & u+gt \end{bmatrix}; u, t \in K \right\}$$

is a spread in $PG(3, K)$, we call this an H-spread and the corresponding translation plane, an H-plane. Clearly, there is an associated affine homology group obtained from post-multiplication of $\left[\begin{smallmatrix} u & t \\ ft & u+gt \end{smallmatrix}\right]$ of determinant 1.

Hence, the spread components other than $x = 0, y = 0$ are

$$y = x \begin{bmatrix} u & t \\ H(\delta_{u,t})ft & H(\delta_{u,t})(u+gt) \end{bmatrix} T$$

where T is the group of field matrices of determinant 1. There is therefore, an associated partial $\det K^+$-partial flock.

5 j-planes

In Johnson, Pomareda and Wilke [6], j-planes are constructed and developed in the finite case. Here we wish to consider this is a more general manner and over any $PG(4, K)$, where K is a field.

15 Definition. Let K be a field and K^+ a quadratic field extension of K represented as follows:

$$K^+ = \left\{ \begin{bmatrix} u & t \\ ft & u+gt \end{bmatrix} ; u, t \in K \right\}.$$

Consider the following group:

$$G_{K^+,j} = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \delta_{u,t}^{-j} & 0 & 0 \\ 0 & 0 & u & t \\ 0 & 0 & ft & u+gt \end{bmatrix} ; \in K,\ (u,t) \neq (0,0) \right\},$$

where j is a fixed integer and $\delta_{u,t} = \det \left[\begin{smallmatrix} u & t \\ ft & u+gt \end{smallmatrix} \right]$.

A 'j-plane' is any translation plane π containing $x = 0, y = 0$ and $y = x$ that admits $G^+_K{,j}$ as a collineation group acting transitively on the components of $\pi - \{x = 0, y = 0\}$.

16 Proposition. *A j-plane produces a regular hyperbolic fibration with constant back half and hence a corresponding* $\det K^+$*-partial flock of a quadratic cone in $PG(3,K)$.*

17 Theorem. *A j-plane with spread set*

$$\left\{ x = 0, y = 0, y = x \begin{bmatrix} 1 & 0 \\ 0 & \delta_{u,t}^{j} \end{bmatrix} \begin{bmatrix} u & t \\ ft & u+gt \end{bmatrix} ; u, t \in K, (u,t) \neq (0,0), K \ \textit{a field} \right\},$$

produces a monomial $\det K^+$ *partial flock of a quadratic cone with monomial functions*

$$f(\delta_{u,t}) = f\delta_{u,t}^{2j+1},\ g(\delta_{u,t}) = g\delta_{u,t}^{j+1}.$$

18 Theorem. *An H-plane with spread set*

$$\left\{ x = 0, y = 0, y = x \begin{bmatrix} 1 & 0 \\ 0 & H(\delta_{u,t}^{j}) \end{bmatrix} \begin{bmatrix} u & t \\ ft & u+gt \end{bmatrix} ; u, t \in K, \right.$$
$$\left. (u,t) \neq (0,0), K \ \textit{a field} \right\},$$

produces a $\det K^+$ *partial flock of a quadratic cone with functions:*

$$\begin{aligned} g(\delta_{u,t}) &= gH(\delta_{u,t})\delta_{u,t} \\ f(\delta_{u,t}) &= fH(\delta_{u,t})^2\delta_{u,t}. \end{aligned}$$

Proof. Let π be a j-plane, so has a spread set of the following form:

$$\left\{x=0,\, y=0,\, y=x\begin{bmatrix}1 & 0\\ 0 & \delta_{u,t}^{j}\end{bmatrix}\begin{bmatrix}u & t\\ ft & u+gt\end{bmatrix};\, u,t\in K,(u,t)\neq(0,0)\right\},$$

where $\delta_{u,t}=\det\left[\begin{smallmatrix}u & t\\ ft & u+gt\end{smallmatrix}\right]$.

Again, we have

$$\begin{bmatrix}\delta_{u,t} & g(\delta_{u,t})\\ 0 & -f(\delta_{u,t})\end{bmatrix};\delta_{u,t}=\det\begin{bmatrix}u & t\\ ft & u+gt\end{bmatrix},$$
$$g(\delta_{u,t})=g(uG(u,t)+tF(u,t))+2(uF(u,t)-tfG(u,t)),$$
$$-f(\delta_{u,t})=\delta_{F(u,t),G(u,t)},$$

where $\delta_{M_i}=\det M_i$, and $\delta_{F(u,t),G(u,t)}=\det\left[\begin{smallmatrix}F(u,t) & G(u,t)\\ fG(u,t) & F(u,t)+gG(u,t)\end{smallmatrix}\right]$.

In this case,

$$F(u,t)=\delta_{u,t}^{j}ft,\ G(u,t)=\delta_{u,t}^{j}(u+gt).$$

Hence,

$$\begin{aligned}g(\delta_{u,t})&=g(uG(u,t)+tF(u,t))+2(uF(u,t)-tfG(u,t))\\&=g(u\delta_{u,t}^{j}(u+gt)+t\delta_{u,t}^{j}ft)+2(u\delta_{u,t}^{j}ft-tf(\delta_{u,t}^{j}(u+gt)))=g\delta_{u,t}^{j+1}.\end{aligned}$$

Also,

$$\begin{aligned}-f(\delta_{u,t})&=\delta_{F(u,t),G(u,t)}=F(u,t)(F(u,t)+gG(u,t))-fG(u,t)^2\\&=\delta_{u,t}^{j}ft(\delta_{u,t}^{j}ft+g(\delta_{u,t}^{j}(u+gt)))-f(\delta_{u,t}^{j}(u+gt))^2\\&=\delta_{u,t}^{2j}f(ft^2+gt(u+gt)-(u^2+2ugt+g^2t^2))\\&=\delta_{u,t}^{2j}f(-(u^2+ugt-ft^2))=-f\delta_{u,t}^{2j+1}.\end{aligned}$$

Hence,

$$f(\delta_{u,t})=f\delta_{u,t}^{2j+1}.$$

This proves the result.

Going through the same argument with $H(\delta_{u,t})$ in place of $\delta_{u,t}^{j}$, we obtain

$$\begin{aligned}g(\delta_{u,t})&=gH(\delta_{u,t})\delta_{u,t}\\-f(\delta_{u,t})&=-fH(\delta_{u,t})^2\delta_{u,t}.\end{aligned}$$

QED

Let $H(u) = u^r$, for $u > 0$, and equal to 0 when $u = 0$. Define $H(-u) = -u^r$, for $u > 0$. Consider putative associated functions

$$\begin{aligned} g(\delta_{u,t}) &= gH(\delta_{u,t})\delta_{u,t} = g\delta_{u,t}^{r+1} \\ f(\delta_{u,t}) &= fH(\delta_{u,t})^2\delta_{u,t} = f\delta_{u,t} \end{aligned}$$

Now we consider the question is what is the $\det K^+$, when K is a field of real numbers? The elements in this set are $u^2 + ugt - ft^2$, where $g^2 + 4f < 0$. Hence, $-f > 0$ so that we know that $u = t = 0$, 0 is in the $\det K^+$ and if $t = 0$, we obtain the positive real numbers and for $t \neq 0$, we see that

$$y = x^2 + xg - f > 0, \text{ for all } x.$$

Hence, $\det K^+$ is the set of positive real numbers when $(u, t) \neq (0, 0)$. So, the question is whether the functions ϕ_s restricted to the positive reals are surjective and injective on the positive real numbers, for any $s \in K$.

6 Classification of the real j-planes

19 Theorem. *A translation plane π is a real j-plane if and only if j is a real number and $j > -1/2$. In all cases, there is a partial monomial flock over the non-negative real numbers. The partial monomial flock may be extended to a monomial flock over the field of real numbers if and only if $(-1)^j$ is defined when g is not zero and if and only if $(-1)^{2j+1}$ is defined when $g = 0$.*

Proof. We know that a real j-plane produces a $\det K^+$ partial monomial flock with the following functions:

$$f(t) = ft^{2j+1}, g(t) = gt^{j+1},$$

where $t \in \det K^+$. Furthermore, we note that

$$\begin{bmatrix} u & t \\ ft & u + gt \end{bmatrix}; u, t \in K,$$

forces

$$u^2 + ugt - ft^2 = 0$$

if and only if $u = t = 0$. When $t = 0$, and K is the field of real numbers, $\det K^+$ contains the non-negative reals. And, since $g^2 + 4f < 0$ (the discriminant must be negative), it follows that $\det K^+$ is the set of all non-negative reals. Thus, consider the functions:

$$\phi_s : \phi_s(t) = s^2t + g(t)s - f(t).$$

For each $s \in K$, ϕ_s must be injective restricted to $t > 0$, and $f(0) = 0$, $g(0) = 0$ (which we require if the original functions are defined only for $t > 0$) and surjective on $\det K^+$.

Therefore, we consider

$$\phi_s(t) = s^2 t + g t^{j+1} s - f t^{2j+1}.$$

We note that

$$\phi_s(t) = t((s + gt^j/2)^2 - (g^2 + 2f)t^{2j}/4) > 0, \ for\ t > 0.$$

The given function is differentiable on $t > 0$, regardless of j, hence, the derivative is

$$s^2 + (j+1)gt^j s - (2j+1)ft^{2j}.$$

We claim that for each s, the derivative is ≥ 0. Furthermore, when $s = 0$, $-ft^{2j+1}$ is clearly injective for $t > 0$. For s non-zero, the derivative at 0 is $s^2 > 0$. Since the derivative function is continuous for $t > 0$, assume that for some positive value of t, the derivative is negative. Then there is a root of

$$s^2 + (j+1)gt^j s - (2j+1)ft^{2j}.$$

But note considering the functions as a polynomial in s, we assert that the discriminant is negative. To see this simply note that

$$((j+1)g)^2 + 4(2j+1)f = (j+1)^2(g + 4f) - 4j^2 < 0$$

since $(g + 4f) < 0$. Hence, the derivative is ≥ 0, which implies that ϕ_s is injective. In order that the function be surjective on $\det K^+$, we see that we must have

$$\lim_{t\to\infty} \phi_s(t) = \infty \text{ and } \lim_{t\to 0^+} \phi_s(t) = 0.$$

We consider the cases: $j > 0$, and $j < 0$ ($j = 0$ produces a Pappian affine plane). First assume that $j > 0$. Then the two required limits are clearly valid, since

$$\phi_s(t) = t((s + gt^j/2)^2 - (g^2 + 2f)t^{2j}/4) > 0, \text{ for } t > 0\,.$$

Now assume that $j < 0$ then $\lim_{t\to\infty} \phi_s(t) = \infty$. Consider

$$\phi_s(t) = s^2 t + t^j(gt - ft^{j+1}).$$

Then we require

$$\lim_{t\to 0^+} t^j(gt - ft^{j+1}) = 0.$$

Note that when $s = 0$, we require that $\lim_{t\to 0+}(-f(t) = -ft^{2j+1}) = 0$. Therefore, $2j + 1 > 0$. Thus, we must also have $\lim_{t\to 0^+} gt^{j+1} = 0$, but if $2j + 1 > 0$ then $j > -1/2$ so that $j + 1 > 1/2$. This completes the proof that a j-plane is obtained if and only if $j > -1/2$. The question remains if we may extend the partial monomial flock to a monomial flock. If so, then the same functions must be used and must be defined on the negative real numbers. Hence, the question is whether $(-1)^{j+1}$ and $(-1)^{2j+1}$ are defined, for g non-zero and whether $(-1)^{2j+1}$ is defined for $g = 0$. In the former case, the necessary and sufficient condition is whether $(-1)^j$ is defined. If g is non-zero and $(-1)^j$ is defined then $(-t)^j = (-1)^j t^j$, for $t > 0$, implies that $(-t)^j$ is defined. It now follows analogously as in the previous argument that the functions ϕ_s are defined for all real elements t and that these functions are bijective on K, implying that there is an associated monomial flock of a quadratic cone. QED

Now assume that we have a real H-plane, so that H is an endomorphism of multiplicative group of non-negative real numbers, which we are assuming is Lebesque integrable. In this setting, we know that $H(t) = t^j$, for j a real number. Hence, we have the following corollary.

20 Corollary. *The Real H-planes are completely determined as j planes for $j > -1/2$, provided H is Lebesque integrable.*

7 Extension of partial $\det K^+$-partial flocks

We have noted that a partial monomial flock may not always be extended to a monomial flock at least when K is the field of real numbers. However, can the partial monomial flock be extended to a flock?

21 Theorem. *Assume that K is the field of real numbers. Every partial $\det K^+$-partial flock may be extended to a flock in non-countably infinitely many ways.*

PROOF. A partial $\det K^+$ exists if and only if the functions $\phi_s(t) = s^2t + g(t)s - f(t)$ are injective and surjective onto the non-negative real numbers, where $t \geq 0$. Take any function $f_1(t)$ when is defined on $(-\infty, 0]$ such that $f_1(0) = 0$ and f_1 is continuous and non-decreasing and $\lim_{t\to -\infty} f_1(t) = -\infty$. Then define

$$g_2(t) = 0 \text{ if } t \leq 0 \text{ and } g_1(t) = g(t) \text{ for } t > 0\,,$$
$$f_2(t) = f_1(t) \text{ for } t \leq 0 \text{ and } f_2(t) = f(t)\,, \text{ for } t > 0\,.$$

Then clearly,

$$\phi_s(t) = s^2t + g_2(t)s - f_2(t)$$

is bijective on the set of real numbers. QED

22 Theorem. *For any conical flock plane defined on the real numbers, let $\mathcal{P}$ and $\mathcal{N}$ denote the components with positive and negative slopes. Then there are non-countably infinitely many replacements of $\mathcal{N}$ producing conical flock planes.*

8 Finite even order

In this section, assume that we have a j-plane of order q^2, where $q = 2^r$. Then there is a corresponding monomial flock of a quadratic cone. These are completely determined in Penttila-Storme, where is it shown that $j = 0, 1, 2$ and correspond to the linear, Betten and Payne flocks respectively. Hence, the associated j-planes are also completely determined. We note that originally the j-planes for $j = 1$ are constructed in Johnson [3] and are due to Kantor as a particular slice of a unitary ovoid (see, in particular, section 3 and (3.6)). Furthermore, for $j = 2$, the planes are constructed in Johnson-Pomareda-Wilke [6].

Hence, we have the following theorem.

23 Theorem. *Let π be a j-plane of even order q^2. Then $j = 0, 1$ or 2 and the plane is one of the following types of planes:*

(1) Desarguesian ($j = 0$ and corresponding to the linear flock),

(2) Slice of a unitary ovoid ($j = 1$ and corresponding to the Betten flock), or

(3) The Johnson-Pomareda-Wilke $j = 2$-plane (corresponding to the Payne flock).

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